Quantization of electromagnetic field in quadratic continuous nonlinear absorptive dielectrics*

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This paper reports that in the quantization of electromagnetic field in the dielectrics, the wave equation with regard to the Green function is analytically solved by a direct integral method for a quadratic continuous nonlinear absorptive dielectric medium. The quantization of the electromagnetic field in such a nonlinear absorptive dielectric is carried out for which the material dielectric function is assumed as a separable variable about the frequency and the space coordinate. The vacuum field fluctuations for different spatial continuous variations of dielectric function are numerically calculated, which shows that the present result is self-consistent.

Keywords: nonlinear absorptive dielectric, noise current operator, Green function

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1. Introduction

With the rapid growth of interest in quantum optical applications or processes that refer to material media,[1,2] the quantization of the electromagnetic field in various dielectrics becomes particularly important. In the early works[3−11] the lossless dielectrics have been considered first, in which the dielectric function is real. In the nondispersive inhomogeneous materials with interfaces the dielectric function is independent of frequency ω,[4,8,9] and the dielectric function is independent of space position r[6] in the dispersive homogeneous materials. For some simple geometries of nondispersive medium, such as an infinite homogeneous medium, a semi-infinite medium with a flat surface, and a plane parallel slab of material the quantization treatments have been also carried out. However, because the loss is unavoidable in any dispersive dielectric media according to Kramers–Kronig relation, it is necessary to develop a proper quantization method for the absorptive dielectrics.

In the last decade, some quantization approaches[12−26] of electromagnetic field in more generally dispersive inhomogeneous materials were presented. Gruner[15] treated an arbitrary dielectric material with a complex permittivity by using a field–field correlate function between different space–time points. Base on the complex dielectric function of the medium, Matloob and his co-workers[18−22] described the loss with Langevin forces in the forms of noise current operators by the Green function method. Subsequently, the quantization of the electromagnetic field in the linear absorptive dielectrics[23−26] has been investigated by introducing a set of auxiliary field operators to replace Maxwell equations, which was also applied to the generation of Cherenkov and transition radiation as well as atomic decay. However, because the equation of the Green function is difficult to solve for a dielectric function that depends on the space coordinate by Fourier transform, the quantization of the field is limited to treating some simple linear absorptive dielectrics[12−28] up to now.

In this paper, we consider that the material dielectric function is dependent on both frequency and space coordinates and the variables are separable regarding the frequency and space coordinate functions. The wave equation with regard to the Green function is analytically solved by a direct integral method[29,30] for quadratic continuous nonlinear absorptive dielectrics. Finally, the quantization of the electromagnetic field in such a nonlinear absorptive dielectric is actualized. To validate the result presented, the vacuum field fluctuations in this dielectric medium are also calculated numerically in comparison with previous results.

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2. Basic equations

In the Green function method of electromagnetic field quantization, the medium is characterized by a dielectric function which is necessarily a complex function of frequency according to the Kramers–Kronig relations. The complex refractive index $n(\omega)$ is related to the material dielectric function $\varepsilon(\omega)$ in the usual way

$$\varepsilon(\omega) = [n(\omega)]^2 = \varepsilon_{\rm r}(\omega) + {\rm i}\,\varepsilon_i(\omega), \tag{1}$$

where the real and imaginary parts of $\varepsilon(\omega)$ are denoted by subscripts, $n(\omega)$ is related to the real refractive index $\eta(\omega)$ and extinction coefficient $\kappa(\omega)$ by

$$n(\omega) = \eta(\omega) + {\rm i}\,\kappa(\omega). \tag{2}$$

As in Refs. [18] and [20], we only consider that electromagnetic waves propagate parallel to the x axis. The field operators satisfy Maxwell's equations in the forms

$$\frac{\partial \hat{E}(x,t)}{\partial x} = -\frac{\partial \hat{B}(x,t)}{\partial t} \tag{3}$$

and

$$-\frac{\partial \hat{B}(x,t)}{\partial x} = \mu_0 \frac{\partial \hat{D}(x,t)}{\partial t} + \mu_0 \hat{J}(x,t), \tag{4}$$

where $\hat{D}(x,t)$ is the electric displacement operator. In the absence of an externally applied current, the transverse operator $\hat{J}(x,t)$ stands for the noise current density operator associated with absorptive nature of the medium.

It is convenient to separate the electromagnetic field operators into the positive and negative frequency components by Fourier transform

$$\hat{E}(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} {\rm d}\omega [\hat{E}^+(x,\omega){\rm e}^{-{\rm i}\,\omega t} + \hat{E}^-(x,\omega){\rm e}^{{\rm i}\,\omega t}] \tag{5}$$

with similar separations and transforms for the displacement and noise current operators. Thus the frequency-domain Maxwell's equations are obtained from Eqs. (3) and (4) as

$$\frac{\partial \hat{E}^+(x,\omega)}{\partial x} = {\rm i}\,\omega \hat{B}^+(x,\omega) \tag{6}$$

and

$$-\frac{\partial \hat{B}^+(x,\omega)}{\partial x} = -\frac{{\rm i}\,\omega\varepsilon(x,\omega)}{c^2}\hat{E}^+(x,\omega) + \mu_0 \hat{J}^+(x,\omega), \tag{7}$$

where the Fourier transform displacement operator is defined as

$$\hat{D}^+(x,\omega) = \varepsilon_0 \varepsilon(x,\omega)\hat{E}^+(x,\omega). \tag{8}$$

The negative frequency components are given by the Hermitian conjugates of the positive frequency operators.

According to the relations

$$\hat{E}^+(x,\omega) = {\rm i}\,\omega \hat{A}^+(x,\omega) \tag{9}$$

and

$$\hat{B}^+(x,\omega) = -\frac{\partial \hat{A}^+(x,\omega)}{\partial x}, \tag{10}$$

the field operators are expressed in terms of a vector potential operator $\hat{A}^+(x,\omega)$. Therefore, equation (7) is changed to

$$-\frac{\partial^2 \hat{A}^+(x,\omega)}{\partial x^2} - \frac{\omega^2 \varepsilon(x,\omega)}{c^2}\hat{A}^+(x,\omega) = \mu_0 \hat{J}^+(x,\omega). \tag{11}$$

Using the Green function methods, we can express the potential operator $\hat{A}^+(x,\omega)$ in the form

$$\hat{A}^+(x,\omega) = \int_{-\infty}^{+\infty} {\rm d}x' G(x,x',\omega)\hat{J}^+(x',\omega), \tag{12}$$

where Green function satisfies the equation

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2\varepsilon(x,\omega)}{c^2}\right) G(x,x',\omega) = \mu_0 \delta(x-x'). \tag{13}$$

Finally, electromagnetic field quantization in absorbing dielectrics can be obtained by solving Green function $G(x,x',\omega)$ in Eq. (13).

Lagrangian mechanics determines the generalized momentum operator which is conjugate to the vector potential operator in the quantized field theory. In Coulomb gauge, the conjugate momentum of vector potential operator $\hat{A}(x,t)$ is $-\varepsilon_0 \hat{E}(x,t)$, and their canonical commutation relation is

$$[\hat{A}(x,t), -\varepsilon_0 \hat{E}(x,t)] = {\rm i}\,\hbar\delta(x-x'). \tag{14}$$

The field commutation relations are closely related to the commutation properties of the noise current operator. Because noise is assumed to be uncorrelated at different frequencies, its commutation relations are

$$[\hat{J}^+(x,\omega), \hat{J}^-(x',\omega')] = 2\varepsilon_0 \hbar \omega^2 \varepsilon_i(x,\omega)\delta(x-x')\delta(\omega-\omega'), \tag{15}$$

$$[\hat{J}^+(x,\omega), \hat{J}^+(x',\omega')] = [\hat{J}^-(x,\omega), \hat{J}^-(x',\omega')] = 0. \tag{16}$$

These relations can be proved in Refs. [18] and [20].

3. Field quantization in an inhomogeneous medium

3.1. Green function

In the second section, we simply introduce the Green function method of electromagnetic field quantization. The key of this method is solving Green function in Eq. (13), but it is very difficult to obtain Green function in inhomogeneous medium. Below, we will choose an approach[29,30] to solve this problem.

We consider that the material dielectric function $\varepsilon(x,\omega)$ has the form with separated variables

$$\varepsilon(x,\omega) = n^2(x,\omega) = \varpi^2(\omega)\sigma^2(x), \tag{17}$$

where

$$\sigma(x) = (A^2 + Bx^2)^{1/2}, \tag{18}$$

A and B are constants which satisfy the constraints. The concrete form of dielectric function discussed below is not considered, so A and B can be regarded as arbitrary constants. It should be noted that the assumption in Eq. (17) is employed for convenience of theoretical study, thus the form of dielectric function is seriously limited, which makes it difficult to relate the function to real models. In this case, the solution of Eq. (13) is obtained in the form

$$G(x,x',\omega) = \frac{\mu_0}{4\pi}\int_{\vartheta}\exp\left[\mathrm{i}\,\frac{\omega\varpi(x)}{c}f(x,x',\xi)\right]g(\xi)\mathrm{d}\xi, \tag{19}$$

where ξ is the complex variable, and ϑ is an integral path moving along 0 to ∞. The necessary condition of Eq. (19) exactly satisfying Eq. (13) is that ξ tends to 0 or ∞ along integral path

$$\exp[\mathrm{i}\,kf(x,x',\xi)]g(\xi) \to 0, \tag{20}$$

meanwhile, the auxiliary functions $f(x,x',\xi)$ and $g(\xi)$ must satisfy equations

$$f_x^2(x,x',\xi) - \sigma^2(x) = \frac{\lambda\partial f(x,x',\xi)}{\partial\xi}, \tag{21}$$

$$f_{xx} = \frac{\lambda\partial g}{g\partial\xi}, \tag{22}$$

where λ is an arbitrary constant, and different choice of it merely leads to a rescaling of the variable ξ. For convenience, in the following sections we only consider $\lambda = -2$.

The approach in Ref. [29] holds that the function $f(x,x',\xi)$ has a biquadratic form

$$f(x,x',\xi) = a + bxx' + \frac{1}{2}p(x-x')^2, \tag{23}$$

where a, b and p are functions of complex variable ξ. Substitution of Eq. (23) into Eqs. (21) and (22) then leads to a set of ordinary differential equations

$$p^2 - B = -\frac{\partial p}{\partial\xi}, \tag{24}$$

$$bp - B = -\frac{\partial b}{\partial\xi}, \tag{25}$$

$$\left(b^2 + 2\frac{\partial b}{\partial\xi} - B\right)x'^2 - A^2 = -2\frac{\partial a}{\partial\xi}, \tag{26}$$

and

$$p = -2\frac{\partial g}{g\partial\xi}. \tag{27}$$

Then the functions a, b and p can be obtained by solving Eqs. (24) and (25), their expressions are

$$a = \frac{1}{2}A^2\xi, \tag{28}$$

$$b = \sqrt{B}\tanh\frac{1}{2}\sqrt{B}\xi, \tag{29}$$

$$p = \sqrt{B}\coth\sqrt{B}\xi, \tag{30}$$

respectively. Through Eqs. (23), (28), (29) and (30), the auxiliary functions $f(x,x',\xi)$ may now be expressed as

$$f = \frac{1}{2}A^2\xi + xx'\sqrt{B}\tanh\frac{1}{2}\sqrt{B}\xi + (x-x')^2\frac{1}{2}\sqrt{B}\coth\sqrt{B}\xi, \tag{31}$$

meanwhile, equations (27) and (30) can be combined in the form

$$g = G(\sqrt{B}/\sinh\sqrt{B}\xi)^{1/2}, \tag{32}$$

where G is an arbitrary constant which will be set to equal $G = [2\pi c\,\mathrm{i}/\omega\varpi]^{1/2}$. Finally, from Eqs. (19), (31) and (32), we get the Green function

$$G(x,x',\omega) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{8\pi\omega\varpi}\right]^{1/2}\int_{\vartheta}\exp\left\{\mathrm{i}\,\frac{\omega\varpi}{c}\left\{\frac{1}{2}A^2\xi + \frac{1}{4}\sqrt{B}\left[(x+x')^2\tanh\frac{1}{2}\sqrt{B}\xi + (x-x')^2\coth\frac{1}{2}\sqrt{B}\xi\right]\right\}\right\}(\sqrt{B}/\sinh\sqrt{B}\xi)^{1/2}\mathrm{d}\xi. \tag{33}$$

The next we will verify is that the solution obtained in Eq. (33) can be reduced to that of a homogeneous medium in the limit as B tends to 0. In this case, equations (31) and (32) give

$$f = \frac{1}{2}A^2\xi + \frac{(x-x')^2}{2\xi}, \tag{34}$$

$$g = \left[\frac{2\pi c\,\mathrm{i}}{\omega\varpi}\right]^{1/2}\frac{1}{\sqrt{\xi}}, \tag{35}$$

and therefore equation (33) gives

$$G(x,x',\omega) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{8\pi\omega\varpi}\right]^{1/2}\int_\vartheta \exp\left\{\mathrm{i}\,\frac{\omega\varpi}{c}\left[\frac{1}{2}A^2\xi + \frac{(x-x')^2}{2\xi}\right]\right\}\frac{\mathrm{d}\xi}{\xi^{1/2}}. \tag{36}$$

Substituting $s = \frac{1}{2}(\xi + \frac{1}{\xi})$, it is then found that

$$\begin{aligned} G(x,x',\omega) &= \left[\frac{|x-x'|\mu_0^2 c\,\mathrm{i}}{8\pi\omega\varpi A}\right]^{1/2}\int_1^\infty \exp\left[\mathrm{i}\,\frac{\omega\varpi(\omega)}{c}A(x-x')s\right]\frac{\mathrm{d}s}{\sqrt{s-1}} \\ &= \frac{\mathrm{i}}{2\varepsilon_0 c\omega n}\exp\left(\mathrm{i}\,\frac{\omega}{c}n(\omega)|x-x'|\right), \end{aligned} \tag{37}$$

this is in agreement with Refs. [18] and [20].

3.2. Field quantization

From the Green function obtained above, we can easily get the potential operator $\hat{A}^+(x,\omega)$ in an inhomogeneous medium. It is expressed as

$$\begin{aligned} \hat{A}^+(x,\omega) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{8\pi\omega\varpi}\right]^{1/2}\int_{-\infty}^{+\infty}\int_\vartheta \exp\left\{\mathrm{i}\,\frac{\omega\varpi}{c}\left\{\frac{1}{2}A^2\xi + \frac{1}{4}\sqrt{B}\left[(x+x')^2\tanh\frac{1}{2}\sqrt{B}\xi\right.\right.\right. \\ \left.\left.\left. + (x-x')^2\coth\frac{1}{2}\sqrt{B}\xi\right]\right\}\right\}(\sqrt{B}/\sinh\sqrt{B}\xi)^{1/2}\hat{J}^+(x',\omega)\mathrm{d}x'\mathrm{d}\xi. \end{aligned} \tag{38}$$

Taking into account the Fourier time transform of Eq. (38) and the contribution of the negative frequency component, the complete expression for the potential operator $\hat{A}(x,t)$ is obtained as

$$\begin{aligned} \hat{A}(x,t) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{16\pi^2\omega\varpi}\right]^{1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_\vartheta \left\{\exp\left\{\frac{\varpi}{ct}\left\{\frac{1}{2}A^2\xi + \frac{1}{4}\sqrt{B}\left[(x+x')^2\tanh\frac{1}{2}\sqrt{B}\xi\right.\right.\right.\right. \\ \left.\left.\left.\left. + (x-x')^2\coth\frac{1}{2}\sqrt{B}\xi\right]\right\}\right\}(\sqrt{B}/\sinh\sqrt{B}\xi)^{1/2}\hat{J}^+(x',\omega) + \mathrm{H.c.}\right\}\mathrm{d}\omega\mathrm{d}x'\mathrm{d}\xi, \end{aligned} \tag{39}$$

then the electric field operator is determined by Eq. (39) and the Fourier transform of Eq. (9).

In the Coulomb gauge, the momentum conjugate to the vector potential is $-\varepsilon_0\hat{E}(x,t)$. Equation (39) and its momentum conjugate satisfy the equal-time commutation relation in terms of the Green function[15]

$$\left[\hat{A}(x,t), -\varepsilon_0\hat{E}(x',t)\right] = \mathrm{i}\,\frac{2\varepsilon_0\hbar}{\pi}\int_0^\infty \mathrm{d}\omega\omega\,\mathrm{Im}\,G(x,x',\omega). \tag{40}$$

In the next section, quantized field operator will be applied to derive the electric field fluctuation in the vacuum state of the electromagnetic field.

4. Vacuum field fluctuations

Using the Green function in an inhomogeneous medium, we can obtain the contribution to the electromagnetic field fluctuations from the modes which propagate parallel to the x axis. The power spectrum $S(x,\omega)$ of the field fluctuations is determined by the value of the correlation function at a common spatial position as

$$\langle 0|\hat{E}(x,\omega)\hat{E}(x,\omega')|0\rangle = S(x,\omega)\delta(\omega-\omega'). \tag{41}$$

Due to the explicit form of the Green function in an inhomogeneous medium, the fluctuation–dissipation theorem will be employed to calculate the vacuum field fluctuation in this case. Their relation is

$$S(x,\omega) = 2\hbar\omega^2 \operatorname{Im} G(x,x,\omega). \tag{42}$$

Here, we consider a special case of Eq. (18) in which A will be set to equal 0. Then the equation (33) is changed to

$$G(x,x',\omega) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{8\pi\omega\varpi}\right]^{1/2} \int_{\vartheta} \exp\left\{\mathrm{i}\,\frac{\omega\varpi}{4c}\sqrt{B}\left[(x+x')^2 \tanh\frac{1}{2}\sqrt{B}\xi + (x-x')^2 \coth\frac{1}{2}\sqrt{B}\xi\right]\right\}(\sqrt{B}/\sinh\sqrt{B}\xi)^{1/2}\mathrm{d}\xi. \tag{43}$$

Substituting $\tau = \tanh\frac{1}{2}\sqrt{B}\xi$, we find

$$G(x,x',\omega) = \left[\frac{\mu_0^2 c\,\mathrm{i}}{4\pi\omega\varpi B^{1/2}}\right]^{1/2} \int_0^1 \exp\left\{\mathrm{i}\,\frac{\omega\varpi}{c}\left\{\frac{1}{4}\sqrt{B}\left[(x+x')^2\tau + (x-x')^2\frac{1}{\tau}\right]\right\}\right\}(\tau-\tau^3)^{-1/2}\mathrm{d}\tau. \tag{44}$$

Inserting Eq. (44) into Eq. (42), we obtain

$$S(x,\omega) = 2\hbar\omega^2 \operatorname{Im}\left\{\left[\frac{\mu_0^2 c\,\mathrm{i}}{4\pi\omega\varpi B^{1/2}}\right]^{1/2} \int_0^1 \frac{\exp\left(\mathrm{i}\,\frac{\omega\varpi}{c}\sqrt{B}x^2\tau\right)}{\sqrt{\tau-\tau^3}}\mathrm{d}\tau\right\}. \tag{45}$$

Since equation (45) includes integral part, it is extremely difficult to analyse the property of the vacuum field fluctuations in an inhomogeneous medium directly. Therefore, we will select numerical calculation to deal this issue. For the convenience of analysis, the coefficient of the fluctuation $\dfrac{\sqrt{2}\hbar\pi}{8S\varepsilon_0}\left(\dfrac{\omega}{c}\right)^{3/2}$ is normalized to unit.

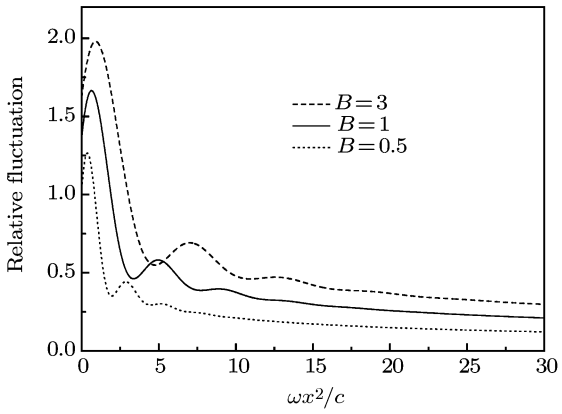


Fig. 1. Spatial variation spectra of the vacuum electric field fluctuations $S(x,\omega)$ for (a) $B = 0.5$, (b) $B = 1$ and (c) $B = 3$, as well as $\varpi = 1.5 + 0.3\,\mathrm{i}$.

Through numerical calculation, we obtain spatial variation of the spectrum $S(x,\omega)$ of vacuum electric field fluctuations. Figure 1 shows that the fluctuations are continuous in the inhomogeneous medium, and the amplitudes of relative fluctuations tend to zero with the increase of $\omega x^2/c$. This is a true behaviour that the electric field propagates in dielectrics whose dielectric functions increase gradually. As shown in Fig. 1, the trend of our result is consistent with previous works.[18,20]

5. Conclusion

In this paper, we amplify and extend the previously developed Green function method[29,30] for electromagnetic field quantization in absorbing dielectrics. In a continuous nonlinear absorptive dielectric, the electromagnetic field is quantized. Besides, the wave equation with regard to the Green function is analytically solved by a direct integral method for quadratic continuous nonlinear absorptive dielectrics. In addition, we also evaluated the vacuum fluctuations of the field in this case, and found numerically that the fluctuations are continuous in the inhomogeneous medium. The results show that the trend of vacuum fluctuations are in agreement with previous works.[18,20]

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