# The existence of generalized synchronisation of three bidirectionally coupled chaotic systems*

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The existence of two types of generalized synchronisation is studied. The model considered here includes three bidirectionally coupled chaotic systems, and two of them denote the driving systems, while the rest stands for the response system. Under certain conditions, the existence of generalised synchronisation can be turned to a problem of compression fixed point in the family of Lipschitz functions. In addition, theoretical proofs are proposed to the exponential attractive property of generalised synchronisation manifold. Numerical simulations validate the theory.

**Keywords:** generalised synchronisation manifold, compression fixed point, exponential attractive property

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## 1. Introduction

Since the pioneer work of Pecora and Carroll,[1] chaos synchronisation has attracted much attention due to its potential applications in various fields.[2−9] At present, several types of chaos synchronisation have been revealed, such as complete synchronisation (CS), phase synchronisation (PS), lag synchronisation (LS), anticipated synchronisation (AS) and generalized synchronisation (GS). GS therein is an interesting and more important topic, which includes many synchronisation phenomena observed in laboratory experiments.[10,11] In 1995, Rulkov *et al.*[12] first described the GS phenomenon and presented the idea of mutual false nearest neighbours to detect the GS. In 1996, Abarbanel *et al.*[13] suggested the auxiliary system method to study GS in driving-response systems, and the theory about this method was given in Ref. [14]. Recently, Hramov *et al.*[15] proposed a modified system approach to study the GS.

Based on the modified system method, we have presented some theoretical results about GS in Refs. [16]–[20], however, the models in these references included only two chaotic systems. In this paper, we will further consider the existence of GS, and the model investigated here consists of three bidirectionally coupled chaotic systems $Y$, $Z$, and $X$. Because this kind of model is close to the dynamical network, which exists everywhere in the real world, the research of the GS of three systems is more valuable. The paper proposes theoretical results about two types of GS, when the modified systems $Y$ and $Z$ are chaotic, and system $X$ collapses to an asymptotically stable equilibrium or asymptotically stable periodic orbits.

## 2. The existence of the GS manifold

Consider the following three bidirectionally coupled chaotic systems:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \boldsymbol{A}_1 y + g(y) + \boldsymbol{K}_1(x-y) + \boldsymbol{K}_1'(z-y), \quad (1)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{A}_2 z + h(z) + \boldsymbol{K}_2(x-z) + \boldsymbol{K}_2'(y-z), \quad (2)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \boldsymbol{A}_3 x + f(x) + \boldsymbol{K}_3(y-x) + \boldsymbol{K}_3'(z-x), \quad (3)$$

where $y \in R_y^n$, $z \in R_z^n$, $x \in R_x^n$, $g(y)$, $h(z)$ and $f(x)$ are smooth vector functions, $\boldsymbol{A}_i$, $\boldsymbol{K}_i$, $\boldsymbol{K}_i'$, $(i = 1, 2, 3)$ are $n \times n$ constant matrices.

**Definition 1**[13] Given two dynamical systems $X$ and $Y$, if there exists a manifold $S = \{(X, Y), X = \Phi(Y)\}$, which includes at least one attractor, then $X$ and $Y$ carry out GS, and $\Phi$ is called the GS map.

For three dynamical systems $X$, $Y$ and $Z$, there are two expressions of GS manifold. One is $X =$

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$\Phi(Y,Z)$, in this case $Y$ and $Z$ are the driving systems, $X$ is the response system; the other is $X=\Phi(Y)$, and $Z=\Psi(Y)$, while $Y$ is the driving system, and $X$ and $Z$ are the response systems.

According to the dynamical property of the modified $Y$ and $Z$ systems, one has

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\boldsymbol{A}_1 y+g(y)-(\boldsymbol{K}_1+\boldsymbol{K}_1')y, \tag{4}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\boldsymbol{A}_2 z+h(z)-(\boldsymbol{K}_2+\boldsymbol{K}_2')z, \tag{5}$$

and the modified $X$ system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\boldsymbol{A}_3 x+f(x)-(\boldsymbol{K}_3+\boldsymbol{K}_3')x. \tag{6}$$

The existence of two kinds of GS is theoretically proved, when the modified $Y$ and $Z$ systems are chaotic, and the modified $X$ system collapses to an asymptotically stable equilibrium or asymptotically stable periodic orbits. Based on the above, the GS manifold here is expressed as $X=\Phi(Y,Z)$.

## 2.1. The first kind of GS

Without loss of generality, suppose the system (6) has an equilibrium point $x_0=\mathbf{0}$ (If $x_0\neq\mathbf{0}$, we will use linear transformation to make $\mathbf{0}$ to be its equilibrium point), and the systems (4) and (5) are chaotic. Then the system (6) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=\boldsymbol{A}_3 x-(\boldsymbol{K}_3+\boldsymbol{K}_3')x+f'(0)x+f(x)-f'(0)x\\&=\boldsymbol{B}x+F(x),\end{aligned}$$

where $\boldsymbol{B}=\boldsymbol{A}_3-\boldsymbol{K}_3-\boldsymbol{K}_3'+f'(0)$, $F(x)=f(x)-f'(0)x$. In addition, all the eigenvalues of matrix $\boldsymbol{B}$ have the negative real part.

Now the system (3) can be denoted as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\boldsymbol{B}x+F(x)+\boldsymbol{K}_3 y+\boldsymbol{K}_3' z. \tag{7}$$

Firstly, we introduce a useful lemma given in Ref. [21].

**Lemma 1**[21] Suppose $\phi(t)$, $\psi(t)$, and $\omega(t)$ are continuous functions defined on $[a,b]$, $\omega(t)>0$, while $\psi(x)$ is a monotonous non-negative and non-decreasing function. If

$$\phi(x)\leq\psi(x)+\int_a^x\omega(s)\phi(s)\,\mathrm{d}s,$$

then

$$\phi(x)\leq\psi(x)\,\mathrm{e}^{\int_a^x\omega(s)\,\mathrm{d}s}.$$

**Lemma 2** Suppose $g(y)$ and $h(z)$ in the systems (1) and (2) satisfy Lipschitz condition, i.e., $|g(y_1)-g(y_2)|\leq L|y_1-y_2|$, $|h(z_1)-h(z_2)|\leq L|z_1-z_2|$, where $L$ is a positive constant, $|\bullet|$ denotes vector norm; $\left\|\mathrm{e}^{(A_1-\boldsymbol{K}_1-\boldsymbol{K}_1')t}\right\|\leq M_1\,\mathrm{e}^{\varsigma t}$, $\left\|\mathrm{e}^{(A_2-\boldsymbol{K}_2-\boldsymbol{K}'_2)t}\right\|\leq M_1\,\mathrm{e}^{\varsigma t}$, where $\varsigma>0$, $t>0$, $\|\bullet\|$ denotes matrix or operator norm.

When $t\leq\tau, y(\tau)=\eta, z(\tau)=\chi$, the solutions $y(t)=y(t;\tau,\eta,\chi,x(.))$, $z(t)=z(t;\tau,\eta,\chi,x(.))$ of the systems (1) and (2) exist for any continuous function $x:(-\infty,\tau]\to U\subset R_x^n$ on $(-\infty,\tau]$, then for any $\eta,\eta',\chi,\chi',x,x'$, the following formulae are satisfied:

$$\begin{aligned}&|y(t;\tau,\eta,\chi,x(.))-y'(t;\tau,\eta',\chi',x'(.))|\\&\leq M_1\,\mathrm{e}^{\varsigma(\tau-t)}|\eta-\eta'|+M_1\int_t^\tau\mathrm{e}^{\varsigma(s-t)}\{L|y-y'|+\|\boldsymbol{K}_1\||x-x'|+\|\boldsymbol{K}_1'\||z-z'|\}\,\mathrm{d}s,\\&|z(t;\tau,\eta,\chi,x(.))-z'(t;\tau,\eta',\chi',x'(.))|\\&\leq M_1\,\mathrm{e}^{\varsigma(\tau-t)}|\eta-\eta'|+M_1\int_t^\tau\mathrm{e}^{\varsigma(s-t)}\{L|z-z'|+\|\boldsymbol{K}_2\||x-x'|+\|\boldsymbol{K}_2'\||y-y'|\}\,\mathrm{d}s.\end{aligned} \tag{8}$$

**Proof** If

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t}&=\boldsymbol{A}_1 y+g(y)+\boldsymbol{K}_1(x-y)+\boldsymbol{K}'_1(z-y),\quad y(\tau)=\eta,\\\frac{\mathrm{d}y'}{\mathrm{d}t}&=\boldsymbol{A}_1 y'+g(y')+\boldsymbol{K}_1(x'-y')+\boldsymbol{K}'_1(z'-y),\quad y'(\tau)=\eta',\end{aligned}$$

then we have

$$\begin{aligned}&\frac{\mathrm{d}(y-y')}{\mathrm{d}t}=(\boldsymbol{A}_1-\boldsymbol{K}_1-\boldsymbol{K}'_1)(y-y')+g(y)-g(y')+\boldsymbol{K}_1(x-x')+\boldsymbol{K}'_1(z-z'),\\&y(\tau)-y'(\tau)=\eta-\eta'.\end{aligned}$$

The solution of the equation can be described as follows:

$$y - y' = \mathrm{e}^{(A_1 - K_1 - K_1')(\tau - t)}(\eta - \eta') + \int_t^\tau \mathrm{e}^{(A - K_1 - K_1')(s-t)}[(g(y) - g(y')) + \boldsymbol{K}_1(x - x') + \boldsymbol{K'}_1(z - z')]\,\mathrm{d}s.$$

Consequently, we have

$$|y - y'| \le M_1\,\mathrm{e}^{\varsigma(\tau - t)}\,|\eta - \eta'| + M_1 \int_t^\tau \mathrm{e}^{\varsigma(s-t)}\{L\,|y - y'| + \|\boldsymbol{K}_1\|\,|x - x'| + \|\boldsymbol{K'}_1\|\,|z - z'|\}\,\mathrm{d}s.$$

Another inequality in formulae (8) will be attained as the same.

Based on the above, we first present the theoretical result about the existence of the first kind of GS manifold as Theorem 1.

**Theorem 1** Suppose the following conditions are satisfied: in the systems (1), (2), and (7) $y \in R_y^n$, $z \in R_z^n$, and $x \in R_x^n$, $F : R_x^n \to R_x^n$, $g : R_y^n \to R_y^n$, $h : R_z^n \to R_z^n$ are sufficiently smooth functions, and

$$|F(x) + \boldsymbol{K}_3 y + \boldsymbol{K'}_3 z - F(x') - \boldsymbol{K}_3 y' - \boldsymbol{K'}_3 z'| \le \lambda[|x - x'| + |y - y'| + |z - z'|], \tag{9}$$

$$|F(x) + \boldsymbol{K}_3 y + \boldsymbol{K'}_3 z| \le N, \tag{10}$$

$$\left|\mathrm{e}^{Bt}\right| \le M\,\mathrm{e}^{-\beta t}, \tag{11}$$

when $t \le \tau, y(\tau) = \eta \in R_y^n$, we have $t \le \tau$, $z(\tau) = \chi \in R_z^n$. Based on Lemma 2, the solutions $y(t) = y(t; \tau, \eta, \chi, x)$, $z(t) = y(t; \tau, \eta, \chi, x)$ of the systems (1) and (2) exist on $(-\infty, \tau)$. For any $\eta, \eta', \chi, \chi'$, the inequality (8) holds. Additionally,

$$|x| \le D, \quad MN/\beta \le D, \quad \lambda M/\beta + \frac{\lambda M(1 + \Delta)M_1(\|\boldsymbol{K}_1\| + \|\boldsymbol{K}_2\|)/\varsigma}{\beta + \varsigma - M_1(L + \|\boldsymbol{K}_1\|\,\Delta + \|\boldsymbol{K}_2\|\,\Delta + \|\boldsymbol{K'}_{12}\|)} < 1,$$

$$\frac{\lambda M(1 + \Delta)M_1}{\beta + \varsigma - M_1(L + \|\boldsymbol{K}_1\|\,\Delta + \|\boldsymbol{K}_2\|\,\Delta + \|\boldsymbol{K}_{12}\|)} \le \Delta, \tag{12}$$

where $\lambda, M, M_1, N, \beta, \varsigma, D, \Delta$ are all non-negative constants, $\|\boldsymbol{K'}_{12}\| = \max\{\|\boldsymbol{K'}_1\|, \|\boldsymbol{K'}_2\|\}$. Then there exists a GS manifold among systems (1), (2), and (7):

$$S = \{(x, y, z)\,|x = \sigma(y, z), \quad -\infty < t < \infty, \quad y \in R_y^n, z \in R_z^n\}, \tag{13}$$

which satisfies $|\sigma(y, z)| \le D$, and $|\sigma(y, z) - \sigma(y', z')| \le \Delta(|y - y'| + |z - z'|)$.

**Proof** Suppose $F_{D,\Delta}$ is a family of Lipschitz function satisfying

$$|\sigma(y, z)| \le D, \quad |\sigma(y, z) - \sigma(y', z')| \le \Delta(|y - y'| + |z - z'|). \tag{14}$$

Define the distance $\|\sigma_1 - \sigma_2\| = \mathrm{Sup}_{y \in R_y^n, z \in R_z^n}\,|\sigma_1(y, z) - \sigma_2(y, z)|$. It is easy to verify that $F_{D,\Delta}$ is a complete metric space. Let $t \le \tau$, and $y(\tau) = \eta \in R_y^n$, then the solutions $y(t) = y(t; \tau, \eta, \chi)$ and $z(t) = z(t; \tau, \eta, \chi)$ of the systems (1) and (2) exist on $(-\infty, \tau)$.

The operator $G$ is defined as follows:

$$G(\sigma(\tau, \eta)) = \int_{-\infty}^\tau \mathrm{e}^{B(\tau - s)}[F(\sigma(y(s), z(s))) + \boldsymbol{K}_3 y(s) + \boldsymbol{K'}_3 z(s)]\,\mathrm{d}s,$$

where $y(s) = y(s; \tau, \eta, \chi)$ and $z(s) = z(s; \tau, \eta, \chi)$.

We will prove that the $G$ maps $F_{D,\Delta}$ on $F_{D,\Delta}$, and it is a contraction map:

$$|G(\sigma(\tau, \eta))| \le \int_{-\infty}^\tau \mathrm{e}^{-\beta(\tau - s)} MN\,\mathrm{d}s = \frac{MN}{\beta} \le D.$$

Suppose $\sigma, \sigma'$ are two maps satisfying formulae (14), $\eta, \eta' \in R_y^n$. Let $y(t) = y(t; \tau, \eta, \chi)$, $y'(t) = y(t; \tau, \eta', \chi')$; from expression (8), we have

$$|y - y'| \le M_1\,\mathrm{e}^{\varsigma(\tau - t)}\,|\eta - \eta'| + M_1 \int_t^\tau \mathrm{e}^{\varsigma(s-t)}\{L\,|y - y'| + \|\boldsymbol{K}_1\|\,|x - x'| + \|\boldsymbol{K'}_1\|\,|z - z'|\}\,\mathrm{d}s$$

$$
\begin{aligned}
&\leq M_1\,\mathrm{e}^{\varsigma(\tau-t)}\,|\eta-\eta'|+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-t)}\{L\,|y_1-y_2|+\|\boldsymbol{K}_1\|\,|\sigma(y,z)-\sigma'(y',z')|+\|\boldsymbol{K}'_1\|\,|z-z'|\}\,\mathrm{d}s\\
&\leq M_1\,\mathrm{e}^{\varsigma(\tau-t)}\,|\eta-\eta'|+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-t)}\{L\,|y_1-y_2|+\|\boldsymbol{K}_1\|\,[\|\sigma-\sigma'\|\\
&\quad +\Delta(|y-y'|+|z-z'|)]+\|\boldsymbol{K}'_1\|\,|z-z'|\}\,\mathrm{d}s\\
&\leq M_1\,\mathrm{e}^{\varsigma(\tau-t)}\,|\eta-\eta'|+M_1\,\|\boldsymbol{K}_1\|\,\|\sigma-\sigma'\|\,\mathrm{e}^{\varsigma(\tau-t)}/\varsigma+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-t)}\{L\,|y_1-y_2|\\
&\quad +\|\boldsymbol{K}_1\|\,\Delta(|y-y'|+|z-z'|)+\|\boldsymbol{K}'_1\|\,\|\,|z-z'|\}\,\mathrm{d}s.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\mathrm{e}^{\varsigma(t-\tau)}\,|y-y'| \leq\ & M_1\,|\eta-\eta'|+M_1\,\|\boldsymbol{K}_1\|\,\|\sigma-\sigma'\|\,/\varsigma+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-\tau)}\{L\,|y-y'|\\
&+\|\boldsymbol{K}_1\|\,\Delta(|y-y'|+|z-z'|)+\|\boldsymbol{K}'_1\|\,|z-z'|\}\,\mathrm{d}s. \qquad (15)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
\mathrm{e}^{\varsigma(t-\tau)}\,|z-z'| \leq\ & M_1\,|\chi-\chi'|+M_1\,\|\boldsymbol{K}_2\|\,\|\sigma-\sigma'\|\,/\varsigma+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-\tau)}\{L\,|z-z'|\\
&+\|\boldsymbol{K}_2\|\,\Delta(|y-y'|+|z-z'|)+\|\boldsymbol{K}'_2\|\,|y-y'|\}\,\mathrm{d}s. \qquad (16)
\end{aligned}
$$

Adding formula (15) to formula (16) yields

$$
\begin{aligned}
\mathrm{e}^{\varsigma(t-\tau)}(|y-y'|+|z-z'|) \leq\ & M_1(|\eta-\eta'|+|\chi-\chi'|)+M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\,\|\sigma-\sigma'\|\,/\varsigma\\
&+M_1\int_t^\tau \mathrm{e}^{\varsigma(s-\tau)}\{L(|y-y'|+|z-z'|)+(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\Delta(|y-y'|+|z-z'|)\\
&+\|\boldsymbol{K}'_{12}\|\,|z-z'|+\|\boldsymbol{K}'_{12}\|\,|y-y'|\}\,\mathrm{d}s,
\end{aligned}
$$

where $\|\boldsymbol{K}'_{12}\|=\max\{\|\boldsymbol{K}'_1\|,\|\boldsymbol{K}'_2\|\}$.

Based on Lemma 1, the following formulae are satisfied

$$
\begin{aligned}
&\mathrm{e}^{\varsigma(t-\tau)}(|y-y'|+|z-z'|)\\
&\leq \{M_1(|\eta-\eta'|+|\chi-\chi'|)+M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\,\|\sigma-\sigma'\|\,/\varsigma\}\,\mathrm{e}^{[M_1(L+\|\boldsymbol{K}_1\|\Delta+\|\boldsymbol{K}_2\|\Delta+\|\boldsymbol{K}'_{12}\|](\tau-t)},\\
&|y-y'|+|z-z'|\\
&\leq \{M_1(|\eta-\eta'|+|\chi-\chi'|)+M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\,\|\sigma-\sigma'\|\,/\varsigma\}\,\mathrm{e}^{[M_1(L+\|\boldsymbol{K}_1\|\Delta+\|\boldsymbol{K}_2\|\Delta+\|\boldsymbol{K}'_{12}\|)-\varsigma](\tau-t)}.
\end{aligned}
$$

Consequently, the following expression can be obtained

$$
\begin{aligned}
&|G(\sigma(\tau,\eta))-G(\sigma'(\tau,\eta'))|\\
&=\left|\int_{-\infty}^\tau \mathrm{e}^{B(\tau-s)}[F(\sigma(y(s),z(s)))+\boldsymbol{K}_3y(s)+\boldsymbol{K}_3z(s)-F(\sigma'(y'(s),z'(s)))-\boldsymbol{K}_3y'(s)-\boldsymbol{K}'_3z(s)]\,\mathrm{d}s\right|\\
&\leq\left|\int_{-\infty}^\tau M\lambda\,\mathrm{e}^{-\beta(\tau-s)}[|\sigma(y(s),z(s))-\sigma'(y'(s),z'(s))|+|y(s)-y'(s)|+|z(s)-z'(s)|]\,\mathrm{d}s\right|\\
&\leq\lambda M\int_{-\infty}^\tau \mathrm{e}^{-\beta(\tau-s)}[\|\sigma-\sigma'\|+(1+\Delta)(|y(s)-y'(s)|+|z(s)-z'(s)|]\,\mathrm{d}s\\
&\leq\frac{\lambda M}{\beta}\,\|\sigma-\sigma'\|+\lambda M(1+\Delta)\int_{-\infty}^\tau \mathrm{e}^{-\beta(\tau-s)}\{M_1(|\eta-\eta'|+|\chi-\chi'|)+M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\,\|\sigma-\sigma'\|\,/\varsigma\}\\
&\quad\times\ \mathrm{e}^{[M_1(L+\|\boldsymbol{K}_1\|\Delta+\|\boldsymbol{K}_2\|\Delta+\|\boldsymbol{K}'_{12}\|)-\varsigma](\tau-s)}\,\mathrm{d}s\\
&\leq\frac{\lambda M}{\beta}\,\|\sigma-\sigma'\|+\frac{\lambda M(1+\Delta)}{\beta+\varsigma-M_1(L+\|\boldsymbol{K}_1\|\,\Delta+\|\boldsymbol{K}_2\|\,\Delta+\|\boldsymbol{K}'_{12}\|)}\{M_1(|\eta-\eta'|+|\chi-\chi'|)\\
&\quad+M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)\,\|\sigma-\sigma'\|\,/\varsigma.
\end{aligned}
$$

Then the operator $G$ is a Lipschitz contraction map on $F_{D,\Delta}$, and has a fixed point, i.e., there exists a GS manifold $S=\{(\sigma(y,z),y,z)\}$.

In addition, the manifold $S$ is invariant. Suppose $(x_0, y_0, z_0) \in S,\ x_0 = \sigma(y_0, z_0)$, for equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \boldsymbol{A}_1 y + g(y) + \boldsymbol{K}_1[\sigma(y,z) - y] + \boldsymbol{K'}_1[z - y],\ y(t_0) = y_0,$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{A}_2 z + h(z) + \boldsymbol{K}_2[\sigma(y,z) - z] + \boldsymbol{K'}_2[y - z],\ z(t_0) = z_0.$$

Let $x(t) = \sigma(y(t), z(t))$; there exists the unique solution of Eq. (7)

$$x_{\mathrm{b}}(t) = \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[F(\sigma(y(s), z(s))) + \boldsymbol{K}_2 y(s) + \boldsymbol{K'}_2 z(s)]\,\mathrm{d}s.$$

Moreover, $x_{\mathrm{b}}(t)$ is bounded when $t \to \infty$. It means that $(x(t), y(t), z(t)) \in S$, then $S$ is invariant.

**Theorem 2** Under the conditions given in Theorem 1, GS manifold is exponentially attractive. Especially, when $x(t),\ y(t)$, and $z(t)$ are the solutions of the systems (1), (2) and (7) respectively, then

$$|x(t) - \sigma(y(t), z(t))| \le M\,|x(t_0) - \sigma(y(t_0), z(t_0))|\ \mathrm{e}^{(-\beta + M\lambda)(t-t_0)},$$

where $M\lambda < \beta$.

**Proof** The solution of Eq. (7) can be written as

$$x(t) = \mathrm{e}^{B(t-t_0)} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[F(x) + \boldsymbol{K}_3 y + \boldsymbol{K'}_3 z]\,\mathrm{d}s,$$

and on $S$,

$$\sigma(y,z) = \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[F(\sigma(y(s), z(s))) + \boldsymbol{K}_3 y(s) + \boldsymbol{K'}_3 z(s)]\,\mathrm{d}s,$$

then

$$\begin{aligned}\xi(t) = x(t) - \sigma(y,z) =\ & \mathrm{e}^{B(t-t_0)} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[F(x) + \boldsymbol{K}_3 y + \boldsymbol{K'}_3 z]\,\mathrm{d}s \\ & - \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[F(\sigma(y(s), z(s))) + \boldsymbol{K}_3 y(s) + \boldsymbol{K'}_3 z(s)]\,\mathrm{d}s, \\ \xi(t) - \mathrm{e}^{B(t-t_0)}\xi(t_0) =\ & \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[F(x(s)) - F(\sigma(y(s), z(s)))]\,\mathrm{d}s.\end{aligned}$$

Consequently,

$$\begin{aligned}&\xi(t) = x(t) - \sigma(y(t), z(t)) = \mathrm{e}^{B(t-t_0)}\,|\xi(t_0)| + \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[F(x(s)) - F(\sigma(y(s), z(s)))]\,\mathrm{d}s, \\ &|\xi(t)| \le \mathrm{e}^{-\beta(t-t_0)} M\,|\xi(t_0)| + \lambda M \int_{t_0}^{t} \mathrm{e}^{-\beta(t-s)}\,|x(s) - \sigma(y(s), z(s))|\,\mathrm{d}s, \\ &\mathrm{e}^{\beta(t-t_0)}\,|\xi(t)| \le M\,|\xi(t_0)| + \lambda M \int_{t_0}^{t} \mathrm{e}^{-\beta(t_0-s)}\,|\xi(s)|\,\mathrm{d}s.\end{aligned}$$

Based on Lemma 1, we have $\mathrm{e}^{\beta(t-t_0)}\,|\xi(t)| \le M\,|\xi(t_0)|\ \mathrm{e}^{M\lambda(t-t_0)}$, then $|\xi(t)| \le M\,|\xi(t_0)|\ \mathrm{e}^{(-\beta + M\lambda)(t-t_0)}$.

## 2.2. The second kind of GS

In this subsection, consider that the modified response system (6) has an asymptotically stable periodic solution $\bar{x}(t) = \bar{x}(t+T),\ T > 0$; and the systems (4) and (5) are chaotic. We introduce another two lemmas in Ref. [22] to prove the existence of this kind of GS manifold.

**Lemma 3**[22] For a linear periodic differential equation described as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \boldsymbol{A}(t)x, \tag{17}$$

where $\boldsymbol{A}(t)$ is a continuous function, $\boldsymbol{A}(t+T)=\boldsymbol{A}(t)$, $T>0$. Then there exists a continuous function $\boldsymbol{Z}(t)$ with period $T$, and equation (17) can be transformed into Eq. (18) using $w(t)=\boldsymbol{Z}(t)x$,

$$\frac{\mathrm{d}w}{\mathrm{d}t}=\boldsymbol{B}w, \tag{18}$$

where $\boldsymbol{B}$ is a constant matrix.

**Lemma 4**[22] Suppose the real parts of the eigenvalues of the matrix $\boldsymbol{B}$ in the equality (18) are $\beta_1,\beta_2,...,\beta_n$; the eigenvalues of $(\boldsymbol{A}(t)+\boldsymbol{A}^*(t))/2$ are $\alpha_1(t),\alpha_2(t),...,\alpha_n(t)$; then there exists a unitary matrix $\boldsymbol{S}(t)=(S_{i,j}(t))$ such that

$$\beta_i=\frac{1}{T}\int_0^T\sum_{j=1}^n|S_{i,j}(t)|^2\alpha_{i,j}(t)\,\mathrm{d}t,\quad i=1,2,\ldots,n.$$

Letting $e(t)=x(t)-\bar{x}(t)$, we have

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t}&=\boldsymbol{A}_3x+f(x)-\boldsymbol{K}_3x-\boldsymbol{K'}_3x-[\boldsymbol{A}_3\bar{x}+f(\bar{x})-\boldsymbol{K}_3\bar{x}-\boldsymbol{K'}\bar{x}_3]+\boldsymbol{K}_3y+\boldsymbol{K'}_3z\\&=\boldsymbol{A}_3e-\boldsymbol{K}_3e-\boldsymbol{K'}_3e+Df(\bar{x})e+f(x)-f(\bar{x})-Df(\bar{x})e+\boldsymbol{K}_3y+\boldsymbol{K'}_3z\\&=\boldsymbol{A}(t)e+F(x,\bar{x},e)+\boldsymbol{K}_3y+\boldsymbol{K'}_3z,\end{aligned} \tag{19}$$

where $\boldsymbol{A}(t)=\boldsymbol{A}-\boldsymbol{K}_3-\boldsymbol{K'}_3+Df(\bar{x})$, $F(x,\bar{x},e)=f(x)-f(\bar{x})-Df(\bar{x})e$.

By using Lemma 3 and $e(t)=\boldsymbol{Z}^{-1}(t)w(t)$, equation (19) is transformed into

$$\frac{\mathrm{d}w}{\mathrm{d}t}=\boldsymbol{B}w+\boldsymbol{Z}(t)F(\boldsymbol{Z}^{-1}(t)w(t))+\boldsymbol{Z}(t)(\boldsymbol{K}_3y+\boldsymbol{K'}_3z). \tag{20}$$

The corresponding GS manifold $x(t)=\bar{x}(t)+\sigma(y(t),z(t))$ near $\bar{x}(t)$ is transformed into

$$e(t)=\boldsymbol{Z}^{-1}(t)w(t)=\sigma(y(t),z(t)),\quad w(t)=\boldsymbol{Z}(t)\sigma(y(t),\ z(t))=\Sigma(y(t),z(t)).$$

Obviously, the existence of the first kind of GS manifold between the systems (1) and (2), and the system described by Eq. (21) means the existence of the second kind of GS manifold among the systems (1), (2), and (3),

$$\frac{\mathrm{d}w}{\mathrm{d}t}=\boldsymbol{B}w+\boldsymbol{Z}(t)F(\boldsymbol{Z}^{-1}(t)w(t))+\boldsymbol{Z}(t)\boldsymbol{K}_3y+\boldsymbol{Z}(t)\boldsymbol{K'}_3z. \tag{21}$$

We derive the following theorem similar to Theorems 1 and 2.

**Theorem 3** Suppose the following conditions are satisfied: in systems (1), (2), and (21) $y\in R_y^n$, $z\in R_z^n$ and $w\in R_w^n$, $F:R_x^n\to R_x^n$, $g:R_y^n\to R_y^n$, $h:R_z^n\to R_z^n$ are sufficiently smooth functions, and

$$\begin{aligned}&\left|\boldsymbol{Z}F(\boldsymbol{Z}^{-1}(t)w(t))+\boldsymbol{Z}K_3y+\boldsymbol{Z}K'_3z-\boldsymbol{Z}F(\boldsymbol{Z}^{-1}(t)w'(t))-\boldsymbol{Z}K_3y'-\boldsymbol{Z}K'_3z'\right|\\\le\ &\lambda[|w-w'|+|y-y'|+|z-z'|],\\&\left|\boldsymbol{Z}F(\boldsymbol{Z}^{-1}(t)w(t))+\boldsymbol{Z}K_3y+\boldsymbol{Z}K'_3z\right|\le N,\\&\left|\mathrm{e}^{Bt}\right|\le M\,\mathrm{e}^{-\beta t},\end{aligned}$$

when $t\le\tau,y(\tau)=\eta\in R_y^n$; $t\le\tau,z(\tau)=\chi\in R_z^n$. Based on Lemma 2, the solutions $y(t)=y(t;\tau,\eta,\chi,x)$, $z(t)=y(t;\tau,\eta,\chi,x)$ of the systems (1) and (2) exist on $(-\infty,\tau)$. For any $\eta,\eta',\chi,\chi'$, the inequality (8) holds. Additionally,

$$|x|\le D,\quad MN/\beta\le D,\quad \lambda M/\beta+\frac{\lambda M(1+\Delta)M_1(\|\boldsymbol{K}_1\|+\|\boldsymbol{K}_2\|)/\varsigma}{\beta+\varsigma-M_1(L+\|\boldsymbol{K}_1\|\,\Delta+\|\boldsymbol{K}_2\|\,\Delta+\|\boldsymbol{K'}_{12}\|)}<1,\quad\text{and}$$

$$\frac{\lambda M(1+\Delta)M_1}{\beta+\varsigma-M_1(L+\|\boldsymbol{K}_1\|\,\Delta+\|\boldsymbol{K}_2\|\,\Delta+\|\boldsymbol{K}_{12}\|)}\le\Delta,$$

where $\lambda,M,M_1,N,\beta,\varsigma,D,\Delta$ are all non-negative constants. Then there exists a GS manifold among systems (1), (2) and (21):

$$S=\{(w,y,z)\,|w=\Sigma(y,z),-\infty<t<\infty,y\in R_y^n,z\in R_z^n\},$$

satisfying

$$|\Sigma(y,z)| \leq D\,|\Sigma(y,z)-\Sigma(y',z')| \leq \Delta(|y-y'|+|z-z'|).$$

Then there exists a second kind of GS manifold among systems (1), (2) and (3):

$$x(t) = \bar{x}(t) + \sigma(y(t),z(t)) = \bar{x}(t) + \boldsymbol{Z}^{-1}\Sigma(y(t),z(t)).$$

## 3. Numerical simulations

In this section, we take the following unified chaotic system[23] for illustration.

$$\begin{aligned}
\dot{u} &= (25\alpha+10)(v-u),\\
\dot{v} &= (28-35\alpha)u - uw + (29\alpha-1)v,\\
\dot{w} &= uv - \frac{\alpha+8}{3}w.
\end{aligned} \tag{22}$$

When $0 \leq \alpha < 0.8$, the system (22) is a Lorenz system;[24] when $\alpha = 0.8$, it is a Lü system;[25] and when $0.8 < \alpha \leq 1$, it is a Chen system.[26]

We select Lü system and Chen system as the driving systems $Y$ and $Z$, and take the Lorenz system as the response system $X$, which are described as

$Y$ system:

$$\begin{aligned}
\dot{y}_1 &= (25\alpha_2+10)(y_2-y_1) + m_{11}(x_1-y_1) + m_{12}(z_1-y_1),\\
\dot{y}_2 &= (28-35\alpha_2)y_1 - y_1y_3 + (29\alpha_2-1)y_2 + m_{21}(x_2-y_2) + m_{22}(z_2-y_2),\\
\dot{y}_3 &= y_1y_2 - \frac{\alpha_2+8}{3}y_3 + m_{31}(x_3-y_3) + m_{32}(z_3-y_3);
\end{aligned} \tag{23}$$

$Z$ system:

$$\begin{aligned}
\dot{z}_1 &= (25\alpha_3+10)(z_2-z_1) + n_{11}(x_1-z_1) + n_{12}(y_1-z_1),\\
\dot{z}_2 &= (28-35\alpha_3)z_1 - z_1z_3 + (29\alpha_3-1)z_2 + n_{21}(x_2-z_2) + n_{22}(y_2-z_2),\\
\dot{z}_3 &= z_1z_2 - \frac{\alpha_3+8}{3}z_3 + n_{31}(x_3-z_3) + n_{32}(y_3-z_3),
\end{aligned} \tag{24}$$

when $\alpha_2 = 0.8$, $\alpha_3 = 1$, $m_{ij} = n_{ij} = 0$ $(i = 1,2,3; j = 1,2)$, systems $Y$ and $Z$ are chaotic.

$X$ system:

$$\begin{aligned}
\dot{x}_1 &= (25\alpha_1+10)(x_2-x_1) + k_{11}(y_1-x_1) + k_{12}(z_1-x_1),\\
\dot{x}_2 &= (28-35\alpha_1)x_1 - x_1x_3 + (29\alpha_1-1)x_2 + k_{21}(y_2-x_2) + k_{22}(z_2-x_2),\\
\dot{x}_3 &= x_1x_2 - \frac{\alpha_1+8}{3}x_3 + k_{31}(y_3-x_3) + k_{32}(z_3-x_3),
\end{aligned} \tag{25}$$

when $\alpha_1 = 0$, $k_{ij} = 0$ $(i = 1,2,3; j = 1,2)$, system $X$ is also chaotic.

From the systems (23), (24) and (25), we have three modified systems:

The modified $Y$ system:

$$\begin{aligned}
\dot{y}_1 &= (25\alpha_2+10)(y_2-y_1) + m_{11}(-y_1) + m_{12}(-y_1),\\
\dot{y}_2 &= (28-35\alpha_2)y_1 - y_1y_3 + (29\alpha_2-1)y_2 + m_{21}(-y_2) + m_{22}(-y_2),\\
\dot{y}_3 &= y_1y_2 - \frac{\alpha_2+8}{3}y_3 + m_{31}(-y_3) + m_{32}(-y_3).
\end{aligned} \tag{26}$$

The modified $Z$ system:

$$\begin{aligned}
\dot{z}_1 &= (25\alpha_3+10)(z_2-z_1) + n_{11}(-z_1) + n_{12}(-z_1),\\
\dot{z}_2 &= (28-35\alpha_3)z_1 - z_1z_3 + (29\alpha_3-1)z_2 + n_{21}(-z_2) + n_{22}(-z_2),
\end{aligned}$$

$$\dot{z}_3 = z_1 z_2 - \frac{\alpha_3 + 8}{3} z_3 + n_{31}(-z_3) + n_{32}(-z_3). \tag{27}$$

The modified $X$ system:

$$\begin{aligned}
\dot{x}_1 &= (25\alpha_1 + 10)(x_2 - x_1) + k_{11}(-x_1) + k_{12}(-x_1),\\
\dot{x}_2 &= (28 - 35\alpha_1)x_1 - x_1 x_3 + (29\alpha_1 - 1)x_2 + k_{21}(-x_2) + k_{22}(-x_2),\\
\dot{x}_3 &= x_1 x_2 - \frac{\alpha_1 + 8}{3} x_3 + k_{31}(-x_3) + k_{32}(-x_3).
\end{aligned} \tag{28}$$

We use the auxiliary system approach to verify GS among systems (23), (24) and (25); as a result, the auxiliary system of $X$ is introduced

$$\begin{aligned}
\dot{X}_1 &= (25\alpha_1 + 10)(X_2 - X_1) + k_{11}(y_1 - X_1) + k_{12}(z_1 - X_1),\\
\dot{X}_2 &= (28 - 35\alpha_1)X_1 - X_1 X_3 + (29\alpha_1 - 1)X_2 + k_{21}(y_2 - X_2) + k_{22}(z_2 - X_2),\\
\dot{X}_3 &= X_1 X_2 - \frac{\alpha_1 + 8}{3} X_3 + k_{31}(y_3 - X_3) + k_{32}(z_3 - X_3).
\end{aligned} \tag{29}$$

In the following simulations, the initial values of systems (23), (24), (25), and (29) are chosen to be (0.1, 0.2, 0.3), (0.4, 0.5, 0.6), (0.5, 1, 1.5) and (1.1, 1.2, 1.3), respectively.

**The first kind of GS** In this case, the system (28) collapses to an asymptotically stable equilibrium, and the systems (26) and (27) are chaotic.

We select $m_{11} = m_{12} = m_{21} = m_{22} = m_{31} = m_{32} = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = n_{31} = n_{32} = 0.1$, $k_{11} = k_{12} = k_{21} = k_{22} = k_{31} = k_{32} = 0.5$.

Under the above conditions, we find that the modified $X$ system (28) approaches the asymptotically stable equilibrium (–9.2736, –10.2010, 25.8000), while the systems (26) and (27) are still chaotic. The response system (25) and the auxiliary system (29) carry out CS, which is shown in Fig. 1. Based on the auxiliary system method, we can have that the driving systems (23), (24), and the response system (25) realize GS.

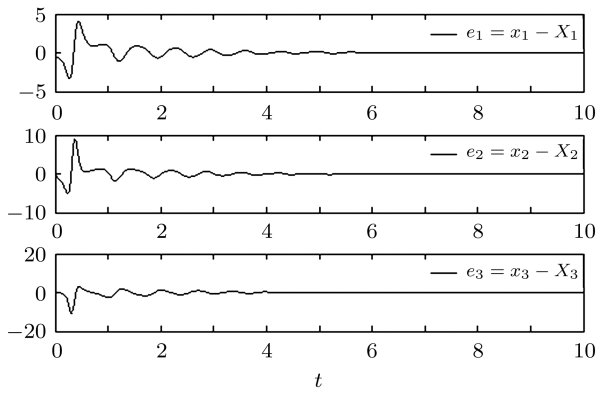



**Fig. 1.** Graphical representation of errors between the systems (25) and (29).

**The second kind of GS** In this case, system (28) collapses to asymptotically stable periodic orbits, and the systems (26) and (27) are chaotic.

We choose $m_{11} = 2$, $m_{12} = 0.05$, $m_{21} = 1.5$, $m_{22} = 0.05$, $m_{31} = 2$, $m_{32} = 0.05$, $n_{11} = 0.74$, $n_{12} = 0.05$, $n_{21} = 1.1$, $n_{22} = 0.05$, $n_{31} = 1$, $n_{32} = 0.05$, $k_{11} = 0.19$, $k_{12} = 0$, $k_{21} = 0.01$, $k_{22} = 0.13$, $k_{31} = 0.24$, $k_{32} = 0.25$. The results are shown in Figs. 2 and 3.

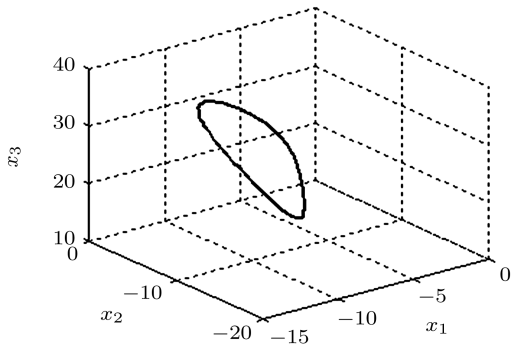



**Fig. 2.** Graphical representation of the periodic orbit of the system (28).

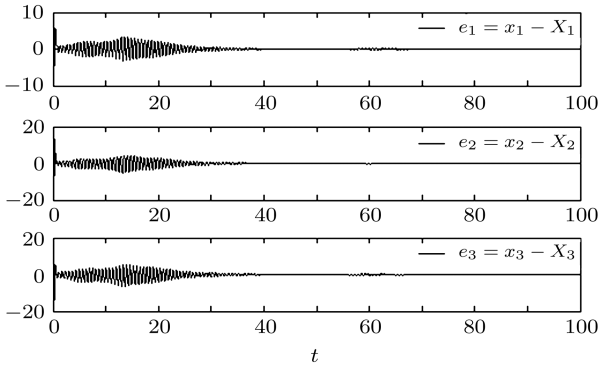



**Fig. 3.** Graphical representation of errors between the systems (25) and (29).

From the above figures, we can see that the response system and the driving systems achieve two kinds of GS. It is worth while pointing out that the approach given in this paper can be easily extended to other complex systems, such as the multi-scroll chaotic systems.[27,28]

## 4. Conclusions and discussions

The mathematical mechanism of the GS of chaotic systems has been unclear for many years. In the earlier works, we presented some theoretical results about the GS manifold of two coupled chaotic systems. This paper further studied the existence of GS manifold of three bidirectionally coupled chaotic systems, and theoretical proofs are proposed to the exponential attractive property of GS manifold. Future work will be focused on the analysis of GS of systems in the complex networks.

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