The feedback control of fractional order unified chaotic system*

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This paper studies the stability of the fractional order unified chaotic system. On the unstable equilibrium points, the "equivalent passivity" method is used to design the nonlinear controller. With the definition of fractional derivatives and integrals, the Lyapunov function is constructed by which it is proved that the controlled fractional order system is stable. With Laplace transform theory, the equivalent integer order state equation from the fractional order nonlinear system is obtained, and the system output can be solved. The simulation results validate the effectiveness of the theory.

Keywords: state feedback control, fractional order, unified chaotic system, Lyapunov function

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1. Introduction

Fractional calculus is the theoretical foundation of fractional order dynamical system. It has been found that the behaviour of many physical systems could be properly described by using fractional order system theory such as electromagnetic waves, dielectric polarization, heat conduction, and so on. Fractional order chaotic systems are also studied by researchers.[1−6] It has been demonstrated that chaos can appear in nonlinear systems of fractional order with an appropriate adjustment of the ranges of the system parameters.

For the fractional order linear system, there are many theories and criterions regarding the observability, controllability and stabilization, including the linear delay system, linear uncertain system and the linear time-variant system.[7,8] However, there are a few results regarding the controllability and stabilization for the fractional order nonlinear system. Li C G presented a synchronization control method by linear feedback for the fractional order Chen chaotic system.[2] Ahmad *et al.* used the recursive "backstepping" method of nonlinear control design to control a chaotic "jerk" model.[4] For the fractional order Duffing chaotic system which has different system parameters and system order, the synchronization control was also designed.[6] Those methods are all based on nonlinear state feedback. Using the nonlinear controller we remove the nonlinear stems of the nonlinear systems, and transfer nonlinear systems into linear systems. The controller is complicated.

The nonlinear equivalent passivity theory of the integer order system can be used to stabilize the nonlinear system by linear state feedback.[9,10] It has the advantage of a simple controller from the nonlinear stems in system improving the transient behaviour, and so on. The researching results show that fractional order systems are, at least, as stable as their integer order counterpart. Using the "nonlinear equivalent passivity" method of the integer order system, we design a linear state feedback controller for the fractional order unified chaotic system in this paper. The main results are presented.

2. Mathematical aspects

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches. The definition of fractional integrals by Grunwald–Letnikov and Riemann–Liouviller is as follows:[5]

$$I_a^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) \mathrm{d}\tau, \qquad (1)$$

where $(a,t) \in R^2$, $a < t$, $0 < \alpha < 1$, $\Gamma(\bullet)$ is Gamma function, $\Gamma(\alpha) = \int_0^{\infty} v^{\alpha-1} \mathrm{e}^{-v} \mathrm{d}v$, and $\Gamma(z+1) = z\Gamma(z)$.

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The definition of fractional derivatives is as follows:

$$D_a^\alpha x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[I_a^\alpha x(t)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-\alpha}x(\tau)\mathrm{d}\tau. \quad (2)$$

3. The passive control of fractional order unified chaotic system

3.1. The linear state feedback controller

The fractional order unified chaotic system is described by[3]

$$\begin{cases} D_a^\alpha x(t) = (25\theta+10)(z-x), \\ D_a^\alpha y(t) = zx - \dfrac{8+\theta}{3}y, \\ D_a^\alpha z(t) = (28-35\theta)x - xy + (29\theta-1)z + u, \end{cases} \quad (3)$$

where x, y, z are state variables, $\theta \in [\,0\ 1\,]$, $0 < \alpha < 1$, and u is the controller.

It is known that systems with memory are typically more stable than their memoryless counterparts. When system order satisfies $0 < \alpha < 1$, fractional order systems can be stabilized by the controllers of its counterpart integer order systems. Because the stabilization area of the integer order system is the subset of the fractional order system,[11] we can use the passive controller of the integer order unified chaotic system to stabilize the fractional order one.

Theorem 1[12] The integer order unified chaotic system can be described as

$$\begin{cases} \dot{x} = (25\theta+10)(y-x), \\ \dot{y} = xz - \dfrac{8+\theta}{3}y, \\ \dot{z} = (28-35\theta)x - xy + (29\theta-1)z + u, \end{cases} \quad (4)$$

where x, y, z are state variables, $\theta \in [\,0\ 1\,]$, and u is the controller.

When there exists constant $k > 0$, and v is exterior input, the controller

$$u = (1-29\theta-k)z + v - (38-10\theta)x \quad (5)$$

is put into the system, and the system (4) can be stabilized on the equilibrium points.

Based on the relationship between the integer order system and fractional order system, we use the passive controller of the integer order system to control the fractional order system.

Theorem 2 For the fractional order unified chaotic system (3), when there exist constant $k > 0$ and controller $u = (1-29\theta-k)z - (38-10\theta)x$, the system can be stabilized.

Proof The reference [12] has presented the designing method of the equivalent passivity controller of the integer order unified chaotic system. We only analyse the stabilization of the fractional order unified chaotic system under control.

Put controller $u = (1-29\theta-k)z - (38-10\theta)x$ into system (3), we have

$$\begin{cases} D_a^\alpha x(t) = (25\theta+10)(z-x), \\ D_a^\alpha y(t) = zx - \dfrac{8+\theta}{3}y, \\ D_a^\alpha z(t) = -(10+25\theta)x - xy - kz. \end{cases} \quad (6)$$

Based on the definition of fractional integrals, the system Lyapunov function is constructed as

$$V = \frac{1}{2}(I_a^\alpha x)^2 + \frac{1}{2}(I_a^\alpha y)^2 + \frac{1}{2}(I_a^\alpha z)^2. \quad (7)$$

From Eqs. (6) and (7), we have

$$\begin{aligned} \dot{V} &= (I_a^\alpha x)\cdot(\dot{I}_a^\alpha x) + (I_a^\alpha y)\cdot(\dot{I}_a^\alpha y) + (I_a^\alpha z)\cdot(\dot{I}_a^\alpha z) \\ &= I_a^\alpha x D_a^\alpha x + I_a^\alpha y D_a^\alpha y + I_a^\alpha z D_a^\alpha z \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau \cdot (25\theta+10)(z-x) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau \cdot \left[xz - \frac{8+\theta}{3}y\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{z(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau \\ &\quad \times (-(10+25\theta)x - xy - kz). \end{aligned} \quad (8)$$

There exists

$$\begin{aligned} \dot{V} &\le \frac{t^\alpha}{\Gamma(\alpha+1)}\left[\|x\|_2 \cdot (25\theta+10)\cdot(\|z\|_2 - \|x\|_2)\right] \\ &\quad \times \frac{t^\alpha}{\Gamma(\alpha+1)}\left[\|y\|_2 \cdot \|x\|_2 \cdot \|z\|_2 - \frac{8+\theta}{3}\|y\|_2^2\right] \\ &\quad \times \frac{t^\alpha}{\Gamma(\alpha+1)}[(-10-25\theta)\|x\|_2\|z\|_2 \\ &\quad - \|x\|_2 \cdot \|y\|_2 \cdot \|z\|_2 - k\|z\|_2^2]. \end{aligned} \quad (9)$$

From expression (9), we get

$$\dot{V} \le \frac{t^\alpha}{\Gamma(\alpha+1)}\left[-\|x\|_2^2 - k\|z\|_2^2 - \frac{8+\theta}{3}\|y\|_2^2\right]. \quad (10)$$

Since k, θ are all constants, and $k > 0$, $\theta \in [0,1]$, we have

$$\dot{V} < 0. \quad (11)$$

The fractional order unified chaotic system can be stabilized on the origin.

3.2. The solving method of fractional order unified chaotic system

The solving method of the fractional order differential equation includes the Laplace transformation method, Flourier transformation method and Adomian decomposed method.

We use the Laplace transformation method to solve the system.

The fractional integrator is approximated by a linear time-invariant transfer function as reported in Ref. [4]. Without loss of generality, we have used in our simulations a fractional integrator of order 0.9. Hence, the approximate transfer function is

$$\frac{1}{s^{0.9}} = \frac{c_1 s^2 + c_2 s + c_3}{s^3 + c_4 s^2 + c_5 s + c_6}, \tag{12}$$

where $c_1 = 1.766$, $c_2 = 38.27$, $c_3 = 4.914$, $c_4 = 36.15$, $c_5 = 7.789$, $c_6 = 0.01$.

The Laplace transformation of system (6) is

$$\begin{cases} X(s) = \dfrac{1}{s^{0.9}}(25\theta + 10)(Z(s) - X(s)), \\ Y(s) = \dfrac{1}{s^{0.9}}(L(xz) - \dfrac{8+\theta}{3}Y(s)), \\ Z(s) = \dfrac{1}{s^{0.9}}(-(25\theta + 10)X(s) - kZ(s) - L(xy)). \end{cases} \tag{13}$$

From Eqs. (12) and (13), using inverse Laplace transformation, we have

$$\begin{aligned} \dddot{x} &= (25\theta + 10)(c_1\ddot{z} + c_2\dot{z} + c_3 z - c_1\ddot{x} - c_2\dot{x} - c_3 x) \\ &\quad - c_4\ddot{x} - c_5\dot{x} - c_6 x, \\ \dddot{y} &= -\frac{8+\theta}{3}(c_1\ddot{y} + c_2\dot{y} + c_3 y) - c_4\ddot{y} - c_5\dot{y} - c_6 y \\ &\quad + c_1\ddot{x}z + 2c_1\dot{x}\dot{z} + c_1 x\ddot{z} + c_2\dot{x}z + c_2 x\dot{z} + c_3 xz, \\ \dddot{z} &= -(25\theta + 10)(c_1\ddot{x} + c_2\dot{x} + c_3 x) \\ &\quad - kc_1\ddot{z} - kc_2\dot{z} - kc_3 z - (c_1\ddot{x}y + 2c_1\dot{x}\dot{y} \\ &\quad + c_1 x\ddot{y} + c_2\dot{x}y + c_2 x\dot{y} + c_3 xy). \end{aligned} \tag{14}$$

Defining $x = x_1$, $\dot{x} = x_2$, $\ddot{x} = x_3$, $y = x_4$, $\dot{y} = x_5$, $\ddot{y} = x_6$, $z = x_7$, $\dot{z} = x_8$, $\ddot{z} = x_9$, we have

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= (25\theta + 10)(c_1 x_9 + c_2 x_8 + c_3 x_7 \\ &\quad - c_1 x_3 - c_2 x_2 - c_3 x_1) - c_4 x_3 - c_5 x_2 - c_6 x_1, \\ \dot{x}_4 &= x_5, \\ \dot{x}_5 &= x_6, \\ \dot{x}_6 &= -\frac{8+\theta}{3}(c_1 x_6 + c_2 x_5 + c_3 x_4) \\ &\quad - c_4 x_6 - c_5 x_5 - c_6 x_4 + c_1 x_3 x_7 + 2c_1 x_2 x_8 \\ &\quad + c_1 x_1 x_9 + c_2 x_2 x_7 + c_2 x_1 x_8 + c_3 x_1 x_7, \\ \dot{x}_7 &= x_8, \\ \dot{x}_8 &= x_9, \\ \dot{x}_9 &= -(10 + 25\theta)(c_1 x_3 + c_2 x_2 + c_3 x_1) \\ &\quad - kc_1 x_9 - kc_2 x_8 - kc_3 x_7 - c_1 x_3 x_4 - 2c_1 x_2 x_5 \\ &\quad - c_1 x_1 x_6 - c_2 x_2 x_4 - c_2 x_1 x_5 - c_3 x_1 x_4. \end{aligned} \tag{15}$$

Through solving the differential equation (15), we can obtain the outputs of the fractional order system.

3.3. Simulation results

Suppose the initial point of the fractional order unified chaotic system is (1, −1, 2) and $k = 2$. Figure 1 cures the plots of the system outputs when $\theta = 0$, $\theta = 0.66$, $\theta = 1$. From Fig. 1, we can find that it takes a very short time for the chaotic system to be stabilized.

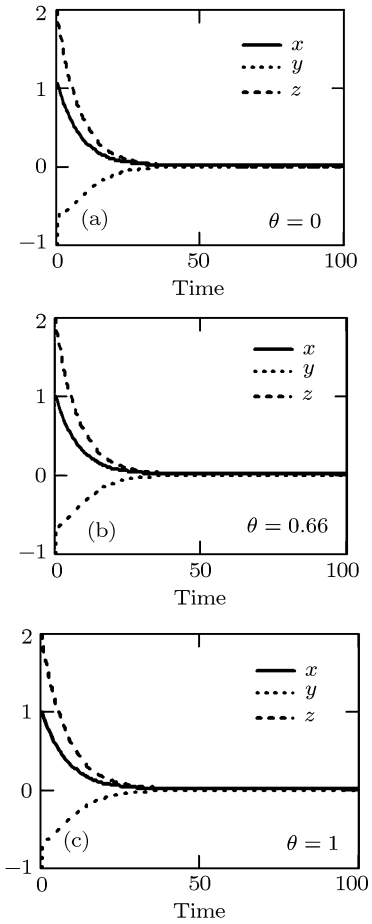


Fig. 1. The outputs of fractional order unified chaotic system when $k = 2$, (a) $\theta = 0$, (b) $\theta = 0.66$, and (c) $\theta = 1$.

Reference [12] gave us some results about the passive control of the integer order unified chaotic system. Compared with the integer order system, it takes longer to stabilize the fractional order system by an equivalent passive controller. Control parameter k can improve the convergence rate in the control of the integer order unified chaotic system, but it does not work well when it is the fractional order unified chaotic system.

4. Conclusion

The nonlinear equivalent passivity theory of the integer order system can be used to stabilize the nonlinear system by linear state feedback. When the system order satisfies $0 < \alpha < 1$, fractional order systems can be stabilized by the controllers of its counterpart integer order systems. Based on the relationship of the two systems, we use the passive controller of the integer order system to control the fractional order system. With the definition of fractional order integrals, the Lyapunov function is constructed by which it is proved that the equivalent passivity fractional order system is stable. However, the controlled area is decreased when we put the controller of the integer order system into the fractional order system. How to improve the stabilization controlled area is an important problem which will be solved in future research.

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