Phase diffusion of a two-component Bose–Einstein condensates: exact and short-time solutions for arbitrary coherent spin state*

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We investigate phase diffusion of a two-component Bose–Einstein condensates prepared initially in arbitrary coherent spin state $|\theta_0, \phi_0\rangle$. Analytical expression of the phase-diffusion time is presented for $\theta_0 \neq \pi/2$ case. In comparison with the symmetrical case (i.e., $\theta_0 = \pi/2$), we find that the diffusion process becomes slow due to the reduced atom number variance.

Keywords: Bose–Einstein condensates, phase diffusion, coherent spin states

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Phase diffusion of Bose–Einstein condensates (BECs),[1−7] aroused from atom–atom interactions destroys phase coherence, and thus sets a limit to the applications of the condensates in high-precision measurement and quantum information processing. The phase diffusion is observable by measuring fringe visibility in atomic interference experiments.[8−12] For a two-component BECs prepared initially in a coherent spin state (CSS) $|\theta_0, \phi_0\rangle$ with $\theta_0 = \pi/2$ [see Ref. [13], or Eq. (2)], it has been shown that the single-particle coherence (or fringe visibility) decays exponentially with the time scale $\chi t_d = (N/2)^{-1/2}$,[2−5] dependent upon atom number N and the self-interaction strength $\chi \sim (a_{11} + a_{22} - 2a_{12})/2$. Here, a_{ii} and a_{ij} $(i \neq j)$ denote intra- and inter-species s-wave scattering lengths, respectively. In this paper, we generalize previous works by considering arbitrary initial CSS, i.e., $\theta_0 \neq \pi/2$. Exact and short-time solutions of the single-particle coherence are obtained. In comparison with the $\theta_0 = \pi/2$ case, we find that the diffusion becomes slow due to the relatively narrow atom number variance.

Considering a two-component BECs with internal states $|1\rangle$ and $|2\rangle$ confined in a deep potential, we adopt a single-mode approximation (SMA),[14] i.e., keeping the condensed-mode wave function $\Phi_0(r)$ for the two components, so the total system can be well described by a second-quantized Hamiltonian ($\hbar = 1$):

$$\hat{H} = \omega_1\hat{n}_1 + \omega_2\hat{n}_2 + u_{12}\hat{n}_1\hat{n}_2 + \frac{u_{11}}{2}(\hat{b}_1^\dagger)^2(\hat{b}_1)^2 + \frac{u_{22}}{2}(\hat{b}_2^\dagger)^2(\hat{b}_2)^2, \quad (1)$$

where $\hat{b}_i$, $\hat{b}_i^\dagger$, and $\hat{n}_i$ $(= \hat{b}_i^\dagger\hat{b}_i)$ are the annihilation, creation, and number operators for the internal states $i = 1, 2$, respectively. In addition, ω_i are single-particle energies, and

$$u_{ij} = (4\pi a_{ij}/M)\int \mathrm{d}^3 r|\Phi_0(r)|^4$$

are atom–atom interaction strengths. Particle number operator $\hat{N} = \hat{n}_1 + \hat{n}_2$ is a conserved quantity and is set to the c number N. Hamiltonian (1) can be rewritten as $\hat{H} = \delta\hat{J}_z + \chi\hat{J}_z^2$, where the detuning $\delta = \omega_2 - \omega_1 + (u_{22} - u_{11})(N-1)/2$, and the self-interaction strength $\chi = (u_{11}+u_{22}-2u_{12})/2$. The spin operators $\hat{J}_+ = (\hat{J}_-)^\dagger = \hat{b}_2^\dagger\hat{b}_1$ and $\hat{J}_z = (\hat{n}_2 - \hat{n}_1)/2$, obeying SU(2) algebra. The nonlinearity $\chi\hat{J}_z^2$ has been proposed to prepare spin squeezed state[15−17] and also quantum entangled state.[18,19] Note that the model considered here with nonzero δ can be also used to study the BECs in an asymmetric double-well.[20] Squeezing via coupling of the BECs in a double-well potential with a cavity light field has been investigated in Ref. [21].

Besides the spin squeezing, mean-field interaction χJ_z^2 also leads to phase diffusion in the two-component

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BECs,[1−7,12] which can be illuminated schematically by Husimi Q function $Q(\theta,\phi;t)=|\langle\theta,\phi|\Psi(t)\rangle|^2$, where

$$|\theta,\phi\rangle=\exp\{\mathrm{i}\,\theta(\hat{J}_x\sin\phi-\hat{J}_y\cos\phi)\}|j,j\rangle \qquad (2)$$

is arbitrary CSS[13] and $|\Psi(t)\rangle=\mathrm{e}^{-\mathrm{i}\,\hat{H}t}|\Psi(0)\rangle$ is a state vector at any time t. For an initial state $|\Psi(0)\rangle=|\theta_0,\phi_0\rangle$, analytic expression of the Q function reads[13]

$$\begin{aligned}Q(\theta,\phi;0)&=|\langle\theta,\phi|\theta_0,\phi_0\rangle|^2\\&=\left[\frac{1+\cos\Theta}{2}\right]^{2j},\end{aligned} \qquad (3)$$

where $\cos\Theta=\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)$. The Q function can be plotted in three-dimensional phase space (i.e., Bloch sphere),[15,16] or alternatively, in a two-dimensional phase space (ϕ,s_z) via a mapping $s_z=j\cos\theta$.[12,17,22] As shown in Figs. 1(a) and 1(b), the density of $Q(\theta,\phi;0)$ is distributed isotropically, indicating the minimal uncertainty relationship of the initial CSS. Under the government of nonlinear interaction χJ_z^2, the system will evolve into a spin squeezed state[15] with anisotropic distribution of the Q function [see Fig. 1(c)]. As time increases, the spin system is over-squeezed[15,23] and shows a spread of the Q function along the ϕ axis, which simulates an increased relative phase fluctuation (i.e., phase diffusion),[12] as shown in Fig. 1(d).

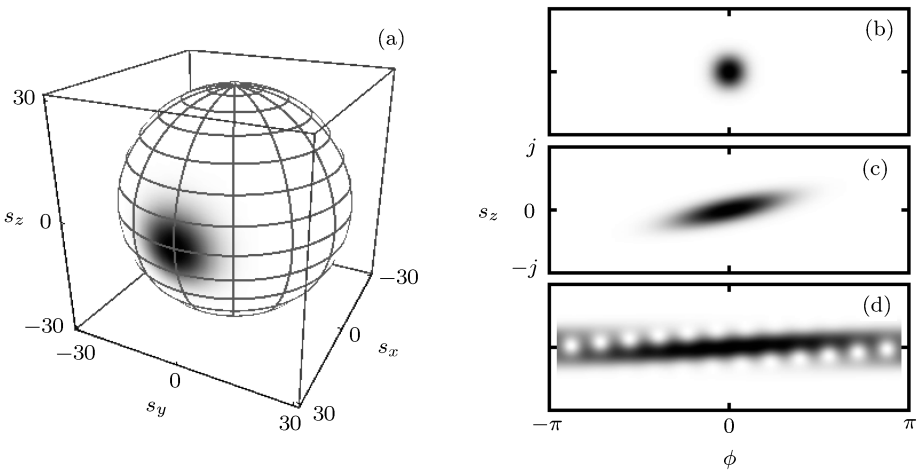


Fig. 1. Husimi Q function $Q(\theta,\phi;t)$ for initial CSS $|\theta_0=\pi/2,\phi_0=0\rangle$ on Bloch sphere (a), with Bloch vector $\boldsymbol{s}=\langle\boldsymbol{J}\rangle=j(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ and $j=N/2=30$. The Q functions in the phase space (ϕ,s_z) for various times: (b) $t=0$, (c) $t=t_s=0.0764\chi^{-1}$, and (d) $t=2t_d=0.2651\chi^{-1}$. Time scale $\chi t_s\simeq 3^{1/6}(2j)^{-2/3}$ is for optimal spin squeezing,[15−17] and $\chi t_d=j^{-1/2}$ is phase-diffusion time.[2−5] The Q function is normalized by its maximum value.

It was shown that the most sensitive states to the diffusion are the symmetrical CSS with $\theta_0=\pi/2$, corresponding to equal populations between the two internal states.[7] The phase-diffusion time scale is $\chi t_d=1/\sqrt{j}$ (with $j=N/2$).[2−5] Here, we generalize it for arbitrary initial CSS

$$|\Psi(0)\rangle=|\theta_0,\phi_0\rangle=\sum_{m=-j}^{j}c_m|j,m\rangle, \qquad (4)$$

where the amplitudes

$$c_m=\left(\frac{(2j)!}{(j+m)!(j-m)!}\right)^{1/2}\cos^{j+m}\left(\frac{\theta_0}{2}\right)\sin^{j-m}\left(\frac{\theta_0}{2}\right)\mathrm{e}^{\mathrm{i}(j-m)\phi_0}. \qquad (5)$$

In single-particle picture, the CSS corresponds to all the atoms occupying a superposed state:[15] $\cos(\theta_0/2)|2\rangle+\mathrm{e}^{\mathrm{i}\,\phi_0}\sin(\theta_0/2)|1\rangle$, where the polar angle θ_0 and the azimuth angle ϕ_0 determine atom population and the relative phase between the two internal states, respectively. In Fig. 2, we plot $|c_m|^2$ as a function of m. One can find that the probability distribution of the CSS can be treated as a Gaussian wave packet [see below Eq. (11)]. At any time t, the spin system evolves into

$$|\Psi(t)\rangle = \sum_{m=-j}^{j} c_m \mathrm{e}^{-\mathrm{i}(\delta m+\chi m^2)t}\,|j,m\rangle\,, \qquad (6)$$

i.e., a superposition of atomic number state $|j,m\rangle = |j-m\rangle_1|j+m\rangle_2$. Due to the presence of atom–atom interaction ($\chi \neq 0$), each number state has different phase evolution rate, which in turn leads to collapse and revival of the Rabi oscillation, a phenomenon that is well known in quantum optics.[14] Similar effect has been investigated in two-component BECs.[24] atom-molecular BECs,[25–27] and exciton emission.[28–30] Phase diffusion of the BECs considered here is in fact collapse of the first-order temporal correlation function:[3,7]

$$g_{12}^{(1)} = \frac{|\rho_{12}^{(1)}|}{\sqrt{\rho_{11}^{(1)}\rho_{22}^{(1)}}} \equiv \frac{|\langle \hat{J}_+\rangle|}{\sqrt{j^2-\langle \hat{J}_z\rangle^2}}, \qquad (7)$$

where $\rho_{kl}^{(1)} = \langle \hat{b}_k^\dagger \hat{b}_l\rangle/N$ with $k,l = 1,2$ are the elements of the single-particle density matrix.[7] The coherence $g_{12}^{(1)}$ is observable in experiments by extracting the visibility of the interference fringes.[8–12] The expectation value $\langle \hat{J}_z\rangle = \langle \Psi(t)|\hat{J}_z|\Psi(t)\rangle$, reads

$$\langle \hat{J}_z\rangle = \sum_{m=-j}^{j} m|c_m|^2 = j\cos\theta_0, \qquad (8)$$

which is a constant for a fixed polar angle of the initial CSS θ_0. This is because of conserved operator $\hat{J}_z$ with respect to the Hamiltonian $\hat{H}$. Similarly, we obtain the expectation value $\langle \hat{J}_z^2\rangle = j^2\cos^2\theta_0 + (j/2)\sin^2\theta_0$, and

$$\begin{aligned}\langle \hat{J}_+\rangle &= \mathrm{e}^{\mathrm{i}(\phi_0+\delta t+\chi t)}\cot\frac{\theta_0}{2}\sum_{m=-j}^{j}(j-m)\,|c_m|^2\,\mathrm{e}^{2\,\mathrm{i}\,m\chi t}\\ &= j\mathrm{e}^{\mathrm{i}(\phi_0+\delta t)}\sin\theta_0\,[\cos\chi t + \mathrm{i}\cos\theta_0\sin\chi t]^{2j-1}\,,\end{aligned} \qquad (9)$$

where we have used the relation: $c_{m+1} = (j-m)^{1/2}(j+m+1)^{-1/2}\cot(\theta_0/2)\mathrm{e}^{-\,\mathrm{i}\,\phi_0}c_m$. Inserting Eq. (8) and Eq. (9) into Eq. (7), we further obtain the exact solution of the coherence

$$g_{12}^{(1)}(t) = \left[1-\sin^2(\theta_0)\sin^2(\chi t)\right]^{j-1/2}, \qquad (10)$$

which shows a decay of the coherence, i.e., phase diffusion. Such a dephasing process depends sensitively on the self-interaction strength χ and the polar angle θ_0 of the initial CSS; while the detuning δ and the azimuth angle ϕ_0 gives vanishing contribution to the coherence.

Time scale of the diffusion is determined by short-time behaviour of the coherence. Following Imamoğlu *et al.*[3] we treat the initial CSS as a Guassian wave-packet with its peak located at $m \simeq \langle \hat{J}_z\rangle = j\cos\theta_0$, namely

$$|c_m|^2 \simeq \frac{1}{[2\pi(\Delta\hat{J}_z)^2]^{1/2}}\exp\left[-\frac{(m-\langle \hat{J}_z\rangle)^2}{2(\Delta\hat{J}_z)^2}\right], \qquad (11)$$

where the width

$$(\Delta\hat{J}_z)^2 = \langle \hat{J}_z^2\rangle - \langle \hat{J}_z\rangle^2 = (j/2)\sin^2\theta_0. \qquad (12)$$

Due to conserved particle number N ($= 2j$), $\hat{n}_1 = j - \hat{J}_z$ and $\hat{n}_2 = j + \hat{J}_z$, and thus atom number variances $(\Delta\hat{n}_1)^2 = (\Delta\hat{n}_2)^2 = (\Delta\hat{J}_z)^2$.[17] In Fig. 2, we check the validity of Eq. (11) by comparing it with the exact result of $|c_m|^2$ [where c_m is given by Eq. (5)], and find both results fit with each other. For symmetric BEC with $\theta_0 = \pi/2$, we have $\langle \hat{J}_z\rangle = 0$ and atom number variance $(\Delta\hat{J}_z)^2 = j/2$; while for $\theta_0 \neq \pi/2$, the variance (width) becomes narrow, as shown in Fig. 2(b). Inserting Eq. (11) into Eq. (9), and replacing the discrete sum over m by an integral, we get

$$\begin{aligned}\langle \hat{J}_+\rangle \simeq{}& \frac{1}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}(\phi_0+\delta t+\chi t)}\mathrm{e}^{2\,\mathrm{i}\langle \hat{J}_z\rangle\chi t}\cot\frac{\theta_0}{2}\\ &\times\int_{\rho_{\min}}^{\rho_{\max}}[j-\langle \hat{J}_z\rangle-\sqrt{2}(\Delta\hat{J}_z)\rho]\\ &\times\exp[-\rho^2+2\,\mathrm{i}\sqrt{2}\chi t(\Delta\hat{J}_z)\rho]\mathrm{d}\rho,\end{aligned} \qquad (13)$$

where we have set

$$\rho = \frac{m-\langle \hat{J}_z\rangle}{\sqrt{2}(\Delta\hat{J}_z)}. \qquad (14)$$

For $\theta_0 \sim \pi/2$ and $j \to \infty$ (large N limit), the integral upper limit $\rho_{\max} = \sqrt{j}\tan(\theta_0/2) \to \infty$ and the lower limit $\rho_{\min} = -\sqrt{j}\cot(\theta_0/2) \to -\infty$. In the short-time limit ($\chi t \ll 1$), $1 - 2\,\mathrm{i}\,\chi t\cos^2(\theta_0/2) \simeq \exp\{-2\,\mathrm{i}\,\chi t\cos^2(\theta_0/2)\}$, and thus from Eq. (13) we obtain

$$\begin{aligned}\langle \hat{J}_+\rangle \simeq{}& j\sin(\theta_0)\exp\{\mathrm{i}[\phi_0+\delta t+\chi t(2j-1)\cos\theta_0]\}\\ &\times\exp[-2\chi^2(\Delta\hat{J}_z)^2t^2],\end{aligned} \qquad (15)$$

where two integrals: $\int_{-\infty}^{\infty}\mathrm{e}^{-\rho^2}\mathrm{e}^{\mathrm{i}\,\lambda\rho}\mathrm{d}\rho = \sqrt{\pi}\mathrm{e}^{-\lambda^2/4}$ and $\int_{-\infty}^{\infty}\rho\,\mathrm{e}^{-\rho^2}\mathrm{e}^{\mathrm{i}\,\lambda\rho}\mathrm{d}\rho = \mathrm{i}\,\lambda\sqrt{\pi}\,\mathrm{e}^{-\lambda^2/4}/2$ have been used in derivation of the above result. Thus, the short-time solution of the coherence reads

$$g_{12}^{(1)}(t) \simeq \exp[-2\chi^2(\Delta\hat{J}_z)^2t^2] \equiv \exp[-(t/t_d)^2] \qquad (16)$$

with characteristic time scale of the phase diffusion

$$\chi t_d = \frac{1}{\sqrt{2}(\Delta\hat{J}_z)} = \frac{1}{\sqrt{j}\sin\theta_0}. \qquad (17)$$

Obviously, the phase-diffusion time $\chi t_d = j^{-1/2}$ for the initial CSS with $\theta_0 = \pi/2$;[3,7,12] while for $\theta_0 \neq \pi/2$, our results show that the diffusion becomes slowly due to an enhanced factor $\sin\theta_0$ in the phase-diffusion rate [see also Fig. 3]. The revival of the coherence occurring later at a time $\chi t_r = \pi$[5,7] is observable in real experiment, such as $t_r = \pi/\chi \simeq 108.7$ microsecond for nonlinearity $\chi \simeq 2\pi \times 4.6$ Hz and particle number $N = 60$.[12]

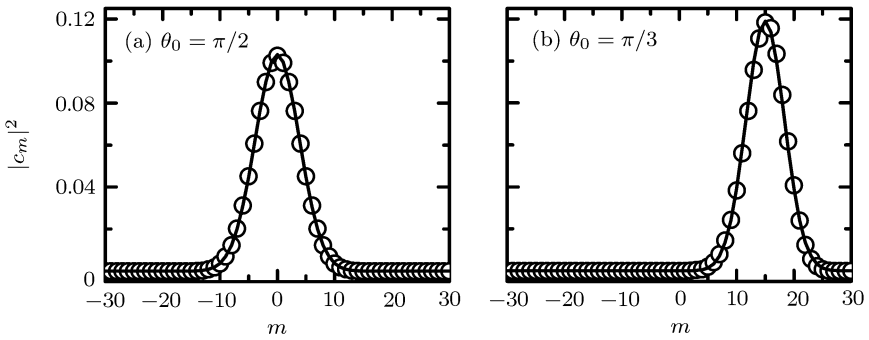


Fig. 2. Probability distribution, $|c_m|^2$ as a function of m for the initial CSS $|\theta_0, \phi_0\rangle$. (a) $\theta_0 = \pi/2$ and (b) $\theta_0 = \pi/3$. The empty circle is given by Eq. (5), and solid line is given by Eq. (11). The parameter $j = N/2 = 30$.

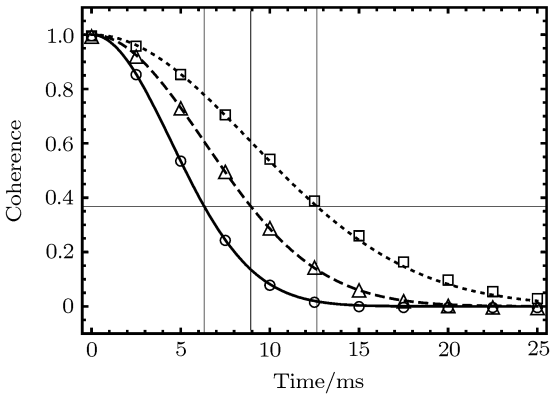


Fig. 3. Time evolution of the first-order temporal coherence $g_{12}^{(1)}(t)$ for various polar angles: $\theta_0 = \pi/6$ (dotted line, squares), $\pi/4$ (dashed line, triangles), and $\pi/2$ (solid line, circles). The empty squares, triangles, and circles are given by Eq. (10) for corresponding θ_0's, while the curves are given by Eq. (16). Horizontal grid line gives the value $1/e$, vertical lines denote the diffusion times t_d for different θ's. Other parameters are $N = 60$, $\chi = 2\pi \times 4.6$ Hz.[12]

Finally, we note that similar results of Eq. (9) and Eq. (15) have been obtained in Ref. [31]. However, the authors focused on the increase of the diffusion time as $\chi \to 0$, but not $\theta_0 \neq \pi/2$. In addition, we emphasize that for negative χ the so-called phase separation may take place due to the dynamic unstability of the BECs.[32−39] Both the phase diffusion and the phase separation reduce the Ramsey fringes' visibility (i.e., the first-order coherence $g_{12}^{(1)}$). However, the latter effect results from dynamics of spatial degree-of-freedom, rather than that of the internal states considered here.

In summary, we have investigated phase diffusion of the two-component BEC for arbitrary initial CSS. We show analytically that for the CSS with $\theta_0 \neq \pi/2$, the diffusion process is suppressed due to the reduced atom number variance $(\Delta\hat{J}_z)^2$ below the standard quantum limit $j/2$ [see also Eqs. (12) and (17)]. Our analytical results are based upon the SMA, which works well to describe the condensates tightly confined in a three-dimensional (3D) trapping potential. For lower dimensional cases,[40], the coherence $g_{12}^{(1)}$ [or the fringe visibility] is expected to decay quicker due to the whole continuum of excitation modes in confined degrees of freedom.[12,41]

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