Phase synchronization on small-world networks with community structure*

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In this paper, we propose a simple model that can generate small-world network with community structure. The network is introduced as a tunable community organization with parameter r, which is directly measured by the ratio of inter- to intra-community connectivity, and a smaller r corresponds to a stronger community structure. The structure properties, including the degree distribution, clustering, the communication efficiency and modularity are also analysed for the network. In addition, by using the Kuramoto model, we investigated the phase synchronization on this network, and found that increasing the fuzziness of community structure will markedly enhance the network synchronizability; however, in an abnormal region ($r \leq 0.001$), the network has even worse synchronizability than the case of isolated communities ($r = 0$). Furthermore, this network exhibits a remarkable synchronization behaviour in topological scales: the oscillators of high densely interconnected communities synchronize more easily, and more rapidly than the whole network.

Keywords: complex networks, small-world, community structure, phase synchronization

PACC: 0520, 0250, 0547

1. Introduction

Many real-life systems are often described by networks.[1,2] Commonly cited examples include the Internet, WWW, metabolic networks, protein networks in the cell, co-author networks, to name a few. The empirical studies have revealed that many of these networks exhibit complex structure properties such as the small-world property characterized by short average path length and high clustering,[3] and the scale-free property characterized by a power-law connectivity distribution.[4] Besides the properties mentioned above, recently, it has been determined that a wide variety of networks also show community structure.[5,6] A community in a network is a set of nodes, where the edges inside are much denser than those connecting the nodes belonging to this set and the rest of the network. Examples of community structures are numerous in biological systems,[7] internet,[8] society[9] and food webs.[10] And now, the investigation of community structures has been one of the most important problems in the field of complex networks and has attracted an exceptional amount of attention.[11−15]

These new discoveries have inspired researchers to develop a variety of models and mechanisms to depict complex networks. A considerable number of models and mechanisms, which can represent processes more realistically taking place in real-life systems, have been proposed.[16,17] However, a large variety of networks may exhibit more than one statistical property in the real world. In particular, a lot of real-life networks have been actually demonstrated to possess the small-world property with clear community. It is common in society that people are divided into different groups by their sex, age, interests and so forth, and each group can be considered as a community. The social network is also an obvious small-world network simultaneously. Recently, Pan *et al.*[18] proposed a simple modular network model where the small-world property arises directly as a result of the modular configuration of such

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networks. The analytical investigations of this network model show that, under a certain condition (see Subsection 2.2), it it more like a random network with community structures than a small-world network. As a matter of fact, some similar network models with the modular network can also be found in earlier papers. For instance, Kanovsky *et al.* in Ref. [19] presented an extension of Watts and Strogatz small-world model for validating their proposed approach (i.e. a very fast iterated cluster recognition algorithm) for community detection in complex networks. And Huang *et al.* in Ref. [20] presented a random clustered network model for investigating an abnormal synchronization phenomenon of community networks, which is almost identical to the modular network model in Ref. [18] in reality. Following these network models, we introduce one possible model with unifying ideas from the two fields of small-world and community structure to address the ubiquity of coexistence of the small-world property with community structure in many real-life systems.

In a different scenario, one of the most significant aims of studies on complex networks is to understand the effects of network structure on the dynamical processes,[20−28] among which the synchronization of coupled oscillators is one of the most popular ways. Generally, the process of convergence to full synchronization exhibits temporally varying patterns which are intrinsically related to the underlying connection topology of complex networks.[21,22] In particular, it has been shown that the connecting patterns of communities or hierarchical organizations have great effects on the synchronization process.[20,23−28] In the synchronization process, the oscillators of high densely interconnected groups synchronize more easily and more rapidly than those with sparse connections. In this paper, based on the proposed small-world network with community structure, we shall investigate a fundamental connection between the community structure and dynamical behaviour occurring on such the networks with the help of Kuramoto model.[29]

This paper is organized as follows. In Section 2, the network construction, its structure properties and corresponding analyses are introduced. In Section 3, we will give the investigation of synchronization properties of Kuramoto oscillators on the proposed networks. And the conclusion remarks are drawn in Section 4.

2. Small-world network with community structure

2.1. The model and structure properties

The simple network proposed in this paper starts from m isolated ring lattices with $N_c = N/m$ nodes in which every node is connected to its first $2z$ neighbours (z on either side), and there are no connections among different lattices at the earliest or original state. Here, each ring lattice can be regarded as a community. Then connect every pair of nodes at random. The connection probability between nodes belonging to the same community is p_1, and that for those belonging to different communities is p_2. Thus, one of the parameters defining the model is the ratio of inter- to intra-community connectivity $p_2/p_1 = r \in [0, 1]$. For $r \to 0$, the network is fragmented into isolated regular clusters, while as $r \to 1$, the network approaches a homogeneous or random network approximately. Clearly, a smaller r corresponds to sparser external edges thus a stronger community structure. Hence, we can generate a simple small-world network with community structure, which is quite distinct from the models related to the WS network proposed by Watts and Strogatz[3] (see Fig. 1). To characterize the structure properties of the networks, we investigate its degree distribution, clustering coefficient, communication efficiency and modularity, respectively.

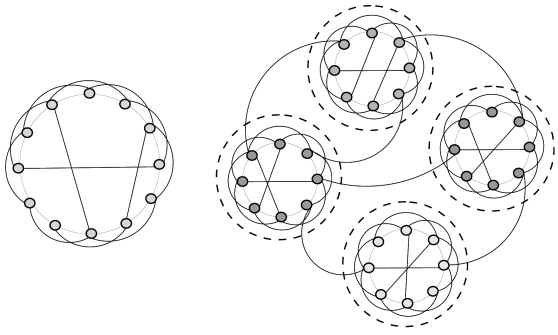


Fig. 1. Schematic diagrams of WS network (left) and the proposed network (right), with communities indicated by broken circles.

Distribution function $P(k)$ gives the probability that a randomly selected node has k edges.[1,2] For the proposed network, because the edges are uniformly added at random besides the inherent edges in the original ring lattices, the nodes should have the homologous degree approximately, close to the average degree $\langle k \rangle$, and it is not hard to achieve the average

degree

$$\langle k\rangle = 2z + \frac{p_1}{m}[(N-m) + rN(m-1)]. \quad (1)$$

Considering the influence of original ring lattices, in fact, $P(k)$ is equivalent to the probability that a randomly selected node has exactly $k-2z$ added edges besides the inherent $2z$ edges. Here, all the $k-2z$ added edges form a random network. Thus with a good approximation the degree distribution of this network is an approximate Poisson distribution with a peak at $P(\langle k\rangle)$

$$P(k) \approx \frac{\lambda^{k-2z}}{(k-2z)!}\mathrm{e}^{-\lambda}, \quad (2)$$

where $\lambda = \langle k\rangle - 2z = (p_1/m)[(N-m) + rN(m-1)]$. And in Fig. 2, the degree distribution is shown for the network with the given different parameters both practically and theoretically. As can be seen the practical and theoretical distributions are comparatively consistent, and exhibit a Poisson distribution approximately.

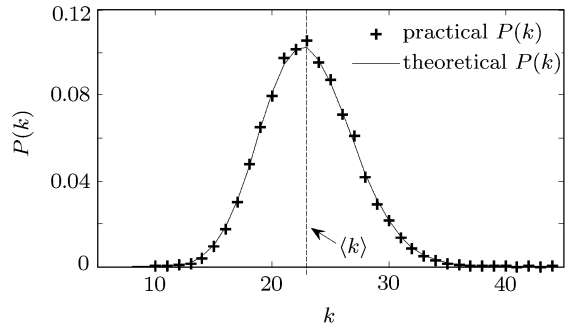


Fig. 2. Degree distribution $P(k)$ of the network in the statistically stationary state with $N = 1024$, $z = 4$, $m = 16$, $p_1 = 0.1$ and $r = 0.1$.

We also quantify the clustering within local neighbourhoods by measuring the clustering coefficient $C = (1/N)\sum_i 2E_i/k_i(k_i-1)$,[3] where k_i and E_i are the degree and the number of links between the neighbours of node i, respectively. From another point of view, the clustering coefficient C_i can also denote the probability that the neighbours of node i are also neighbour nodes. With such the interpretation, the analytical expression of clustering coefficient can be derived exactly for the proposed network:

$$C = p_1' \left[\eta_1^2 + (m-1)\,\eta_2^2\right] + p_2\,(m-1) \times \left[2\eta_1\eta_2 + (m-2)\,\eta_2^2\right], \quad (3)$$

where, taking the inherent $2z$ edges into account, p_1' is a revisory connection probability between nodes belonging to the same community. Due to the clustering coefficient of original regular lattices is $(3z-3)/(4z-2)$,[1,2] p_1' can be shown as

$$\left(2z\left(\frac{3z-3}{4z-2}\right) + p_1^2\left(\frac{N}{m}-1\right)\right)\Big/\left(2z + p_1\left(\frac{N}{m}-1\right)\right).$$

In addition,

$$\eta_1 = \left(2z + p_1\left(\frac{N}{m}-1\right)\right)\Big/\langle k\rangle$$

and

$$\eta_2 = p_2\left(N - \frac{N}{m}\right)\Big/\langle k\rangle$$

are the probabilities that a node has a neighbour in the same or different community, respectively. Moreover, it is worth while to point out that the parts of $p_1'\left[\eta_1^2 + (m-1)\,\eta_2^2\right]$ and $p_2\,(m-1)\left[2\eta_1\eta_2 + (m-2)\,\eta_2^2\right]$ in Eq. (3) can be regarded as the different contributing factors for clustering coefficient when node's neighbours are in the same community or different community, respectively.

On the other hand, the most important property for a small-world network is the average path length, which means the minimum number of edges connecting a pair of nodes, averaged over all pairs of nodes, and generally shown as $L = 2/[N(N-1)]\sum_{i\geq j} d_{ij}$. However, this definition is problematic in some networks that have more than one component.[2] In such the cases, there exist node pairs that have no connecting path. Conventionally one assigns infinite geodesic distance to such pairs, but then the value of L also becomes infinite. To avoid this problem, an alternative and perhaps more satisfactory approach is to define L to be the "harmonic mean" geodesic distance between all pairs, i.e., the reciprocal of the average of the reciprocals:[2]

$$E \equiv L^{-1} = \frac{2}{N(N-1)}\sum_{i\geq j}\frac{1}{d_{ij}}, \quad (4)$$

where d_{ij} is the shortest distance between nodes i and j. Generally, this definition can also be a measure of communication efficiency E, measuring how efficiently information is exchanged over the network, which gives a clear physical meaning to the concept of small-world.[30] In terms of the definition, the small-world behaviour can also be characterized by the high communication efficiency and high clustering. Here, the communication efficiency E of the proposed network is shown in Fig. 3 as a function of parameter r. We can see that the small-world behaviour results from the increase of the communication efficiency E caused by the introduction of a few random connections (see Subsection 2.2).

Next, we characterize the community structure of the network using a measure of modularity. The modularity itself was originally proposed as a way of answering thus the question: a network with strong community structure will have high modularity, and hence the value of modularity can be used as a quality function for community structure.[31] For a given partition of the nodes of a network into communities, modularity is[31]

$$Q = \sum_{i=1}^{m} \left[e_{ii} - \left(\sum_{j=1}^{m} e_{ij} \right)^2 \right], \tag{5}$$

where m is the number of communities, e_{ij} is the fraction of edges in the network that connects nodes in community i to those in community j, and e_{ii} is the fraction of edges that connects two nodes inside the community i. The largest modularity obtained from all possible partitions of the network is denoted by $Q_{\max}$.

An example is shown for a WS network defined on a ring of N nodes (each connected to $2z$ nearest neighbours) where a fraction p of the links has been rewired,[3] modularity can be obtained

$$Q \approx (1-p)\left[\frac{2zn - mz(z+1)}{2zn} - \sum_{i=1}^{m} \left(\sum_{j=1}^{m} e_{ij} \right)^2 \right]. \tag{6}$$

Because $\sum\limits_{i=1}^{m} \left(\sum\limits_{j=1}^{m} e_{ij} \right) = 1$, if $\sum\limits_{j=1}^{m} e_{1j} = \sum\limits_{j=1}^{m} e_{2j} = \cdots = \sum\limits_{j=1}^{m} e_{mj}$, $\sum\limits_{i=1}^{m} \left(\sum\limits_{j=1}^{m} e_{ij} \right)^2$ can reach the minimum. And so

$$Q \approx (1-p)\left(\frac{2N - zm - m}{2N} - \frac{1}{m} \right). \tag{7}$$

Making $\partial Q/\partial m = 0$, $m^* = \sqrt{2N/(z+1)}$, and then $Q_{\max}$ of the given WS network can be obtained

$$Q_{\max} \approx (1-p)\left[1 - \sqrt{\frac{2(z+1)}{N}} \right]. \tag{8}$$

Here, for our proposed network, unlike the WS network, the communities are pre-defined and Q does not need to be maximized with respect to different choices for partitioning the network. Due to the concept of modularity (shown as Eq. (5)) and the useful expressions above, it is not hard to achieve the largest modularity of the network exactly, and shown as

$$Q_{\max} \approx \frac{(m-1)\left[2z + \dfrac{p_1}{m}(N - m - rN) \right]}{2zm + p_1 \left[N(1 - r + rm) - m \right]}. \tag{9}$$

2.2. Model analyses

So far, we have discussed exclusively the corresponding characteristic properties of the network analytically and numerically. Here, the variations of the different properties with r for the proposed network are also analysed, and shown in Fig. 3. For $r = 0$ (the isolated regular clusters), we expect the network to possess the features of the low communication efficiency E on a global scale and high clustering on a local scale. The situation is inverted for $r = 1$ (the random network). In test, we can see that the small-world behaviour appears for intermediate values of r. It results from the increase of the communication efficiency E caused by the introduction of only a few random edges (short cuts), which on the other side do not affect clustering coefficient. With the increasing of r, i.e. the increasing of fuzziness of the community structure, the proposed network has a fast increase of $E(r)$, while $C(r)$ and $Q(r)$ remain almost constant at its original value, indicating that the network exhibits both small-world property and clear community structure fully, which is just the main goal of this model.

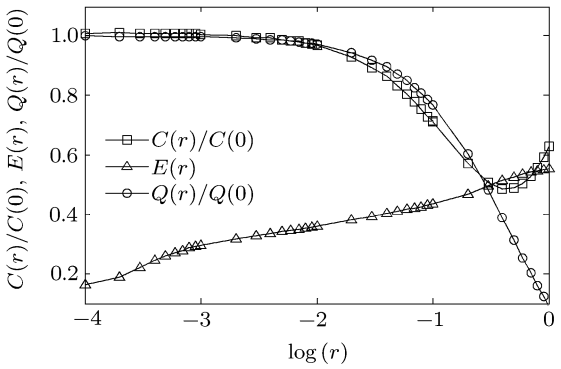


Fig. 3. Clustering coefficient C, communication efficiency E and modularity Q for the proposed network ($N = 512$, $z = 3$, $m = 4$, $p_1 = 0.1$) as a function of r. And clustering coefficient C and modularity Q have been normalized by the values $C(0)$ and $Q(0)$ for the isolated regular clusters (r=0). Note that the data are averaged over 50 random realizations.

As mentioned above, some similar network models can also be found in some earlier papers.[18–20] By comparison, the extension of Watts and Strogatz small-world model in Ref. [19] randomized the ring lattices by rewiring edges, while our model randomized the isolated ring lattices by adding edges, which is similar to some extent. Further, in our paper the small-world network with community structure is not only presented simply, but the detailed topological and dynamics properties of the network are also discussed ex-

clusively. Another similar network model is the modular network model in Ref. [18], in which N nodes are classified into m groups, and each group has N/m nodes, and a pair of nodes is connected with probability p_1 in a group, and nodes belonging to different groups are connected with probability p_2. For our network model, however, every node in each isolated ring lattice is connected to its first $2z$ neighbours (z on either side) at the earliest or original state, which is different from the modular network model in Ref. [18]. For the lack of the inherent edges at the original state, when the connection probabilities are rather small, the modular network more likes a random network with community structures than a small-world network to some extent, and this is a result which one does not look for. Certainly, with larger connection probabilities, the modular network model should be similar to our network model with similar characteristic properties.

3. Phase synchronization on small-world networks with community structure

In this paper, we use the coupled phase oscillators, Kuramoto model,[29] to analyse the dynamical behaviour occurring on the proposed small-world network with community structure. And we write the set of equations of motion governing the dynamics of the N oscillator system ($i = 1, 2, \ldots, N$):

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \omega_i + \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin(\varphi_j - \varphi_i), \qquad (10)$$

where φ_i and ω_i are phase variables and intrinsic frequencies of node i, c is the coupling strength, and $[a_{ij}]$ is the adjacency matrix ($a_{ij} = 1$ if nodes i and j are connected). Initially, φ_i and ω_i are randomly and uniformly distributed in the intervals [0, 2π) and [-0.5, 0.5], respectively.

To measure the synchronized states of the oscillators, an order parameter $\boldsymbol{M}$ is introduced:

$$\boldsymbol{M} = \left\{ \left| \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\,\varphi_j} \right| \right\}, \qquad (11)$$

where $\{\cdot\}$ denotes the time averaging. N is the number of nodes taken into account. Clearly, $\boldsymbol{M}$ is of order $1/\sqrt{N}$ if the oscillators are completely uncoupled (c=0), and will approach 1 if they are all in the synchronized states. In our simulation, the dynamical equations integrate using the Runge–Kutta method with step size 0.05 to integrate numerically Eq. (10). The order parameters are averaged over 2000 time steps, excluding the former 2000 time steps to allow for the relaxation to a steady state.

Here, based on the proposed network, a small-world network with 3 communities is taken into account, and each community contains $N_c = 500$ nodes, thus there are totally $N = 1500$ nodes in the network, and the other interrelated parameters are $z = 30$ and $p_1 = 0.1$. The network is divided into 3 communities clearly, which are labeled as A, B and C, respectively.

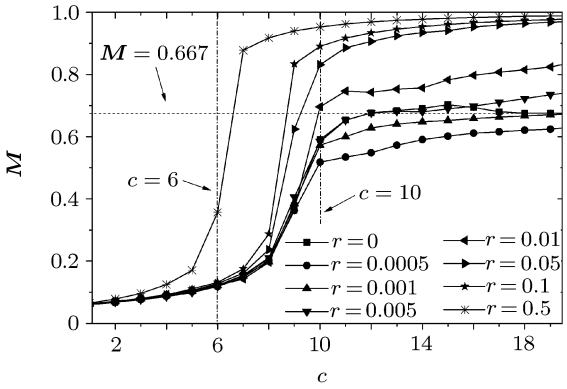


Fig. 4. Order parameter $\boldsymbol{M}$ vs. coupling strength c for different cases of parameter r. All the data are obtained by averaging over 30 realizations.

To get the synchronization properties of the network, we firstly investigate how the order parameters $\boldsymbol{M}$ change with the coupling strength c. Figure 4 shows the simulation results. For all the different r, the corresponding order parameters increase conformably with the increasing of coupling strength c. And for a given r, parameter $\boldsymbol{M}$ will increase sharply after a certain critical coupling strength c that ranged from 6 to 10 (the smaller r, i.e., the stronger community structure, commonly corresponds to the greater critical coupling strength). In addition, for all the cases with $r > 0.001$, the strong community structure (i.e. a smaller r) will hinder global synchronization. For sufficiently large r (see the cases of $r = 0.5$ in Fig. 4), the network can easily get synchronized with a weak coupling strength; however, for sufficiently small r (see the cases of $r = 0.005$ in Fig. 4), the network cannot approach a nearly completely synchronized state even with a very strong coupling strength.

On the other hand, consider a network including m isolated (unconnected) communities ($r = 0$),

and each community itself can approach a nearly completely synchronized state. In this case, the order parameter $\boldsymbol{M}$ of the whole network should be

$$\boldsymbol{M}(m) = \frac{1}{m(2\pi)^m} \int_0^{2\pi} \mathrm{d}\phi_1 \int_0^{2\pi} \mathrm{d}\phi_2 \cdots \int_0^{2\pi} \mathrm{d}\phi_m \chi, \tag{12}$$

where $\chi = \sqrt{\left(\sum_i \sin\varphi_i\right)^2 + \left(\sum_i \cos\varphi_i\right)^2}$. Thus the numerical result $\boldsymbol{M}(3) \approx 0.667$ can be obtained. In Fig. 4, the numerical result $\boldsymbol{M} = 0.667$ is represented by a horizontal dash line, and the corresponding curve for $r = 0$ shows that $\boldsymbol{M}$ slightly fluctuates around 0.667 for sufficiently large c. This phenomenon is in accordance with the above numerical result. In addition, we found an abnormal region $r \leq 0.001$, in which the network has even worse synchronizability than the isolated (unconnected) case with $r = 0$ (see the cases of $r = 0.0005$ and $r = 0.001$ in Fig. 4). Within this region, each isolated community cannot get harmonized with other communities through its few external edges. On the contrary, the input signals containing by these edges disturb the synchronizing process of this community.

Figure 5 shows the order parameter $\boldsymbol{M}$ vs. parameter r for different coupling strength c. As mentioned above, a strong community structure (i.e. a smaller r) hinders global synchronization. For each given coupling strength c, the corresponding parameters increase conformably with the increasing of fuzziness of community structure. However, when $r \leq 0.001$, order parameter is always smaller than 0.667 even for very strong coupling strength c.

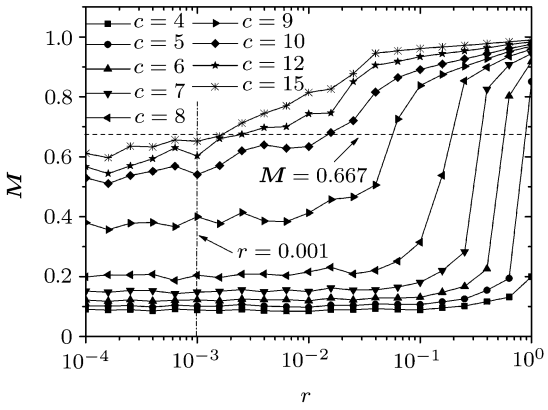


Fig. 5. Order parameter $\boldsymbol{M}$ vs. parameter r for different coupling strength c. All the data are averaged over 30 realizations.

To further understand the underlying mechanism of synchronization on community networks, we investigate the partial synchronization within an individual community. For the i-th community, the corresponding order parameter $\boldsymbol{M}_i$ is defined as

$$M_i = \left\{ \left| \frac{1}{N_c} \sum_j \mathrm{e}^{\mathrm{i}\,\varphi_j} \right| \right\}, \tag{13}$$

where the sum goes over all the nodes belonging to the i-th community.

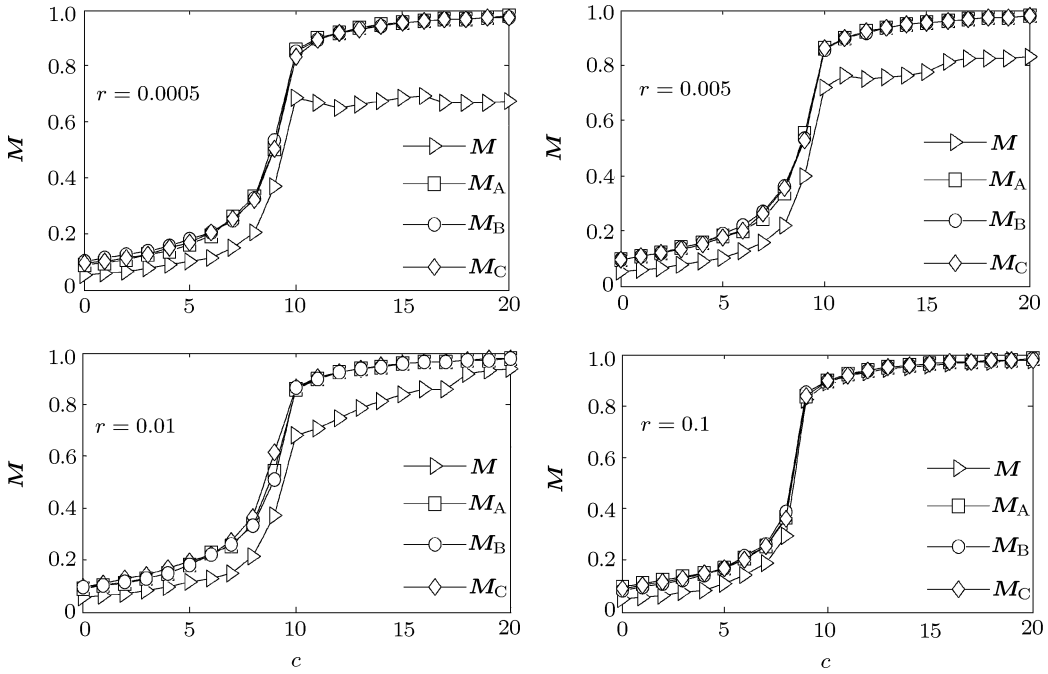


Fig. 6. The order parameter $\boldsymbol{M}$ vs. coupling strength c with different parameters r for the whole network and each community. Square, circle and diamond curves represent the order parameters of community A, B and C, respectively. And the black triangle curve represents the order parameter of the whole network. The data were points averaged over 30 realizations.

In the further investigations, we firstly investigate how the order parameters change with the coupling strength c in two topological scales, an individual community and the whole network. Figure 6 shows the simulation results where four plots correspond to the cases of $r = 0.0005$, $r = 0.005$, $r = 0.01$, and $r = 0.1$, respectively. The change of r from 0.0005 to 0.1 implies that the community structure becomes weaker and weaker. For very strong community structure (the cases of $r = 0.0005$ and $r = 0.005$ in Fig. 6), with the increasing of coupling strength c, the order parameter $\boldsymbol{M}_i$ for each community soon approaches 1, indicating the synchronized state within the community; however, the $\boldsymbol{M}$ for the whole network is much less than that for individual community. In the case of $r = 0.01$, when the coupling strength is not too large ($c = 10$), the oscillators in the same community will synchronize soon, and with the further increasing of coupling strength ($c = 20$), the whole network can also approach the nearly completely synchronized state. For sufficiently large r (the case of $r = 0.1$ in Fig. 6), the whole network can approach the nearly completely synchronized state almost as easily as the individual community. This investigation shows a remarkable relationship between topological scales: the high densely interconnected communities synchronize more easily than the whole network.

In addition, we investigate how the temporal behaviours of order parameters for the whole network and individual community change with time evolution in order to reveal the relationship between topological scales and dynamic time scales. Figure 7 shows the simulation results, and the four plots correspond to the cases of $r = 0.0005$, $r = 0.005$, $r = 0.05$, and $r = 0.1$, respectively, corresponding to weaker and weaker community structure. For the different cases, all the individual communities can easily get synchronized state independently with very short time evolution, while the whole network will not. In the case of $r = 0.0005$ and $r = 0.005$ (strong community structure), the order parameter $\boldsymbol{M}$ for the whole network is much less than $\boldsymbol{M}_i$ for each community, and remarkably below 1 even in the long time limit, indicating the external edges among the communities cannot harmonize all the communities, especially in the abnormal region $r \leq 0.001$ (see the case of $r = 0.0005$ in Fig. 7). When $r = 0.05$, the external edges among communities increase, and then each community itself not only can approach a nearly synchronized state, but also can harmonize with other communities in some extent; therefore, $\boldsymbol{M}$ will continuously increase till a completely synchronized state after $\boldsymbol{M}_i$ gets steady. For sufficiently large r (see the case of $r = 0.1$ in Fig. 7), the whole network can approach the nearly completely synchronized state almost as quickly as the individual community.

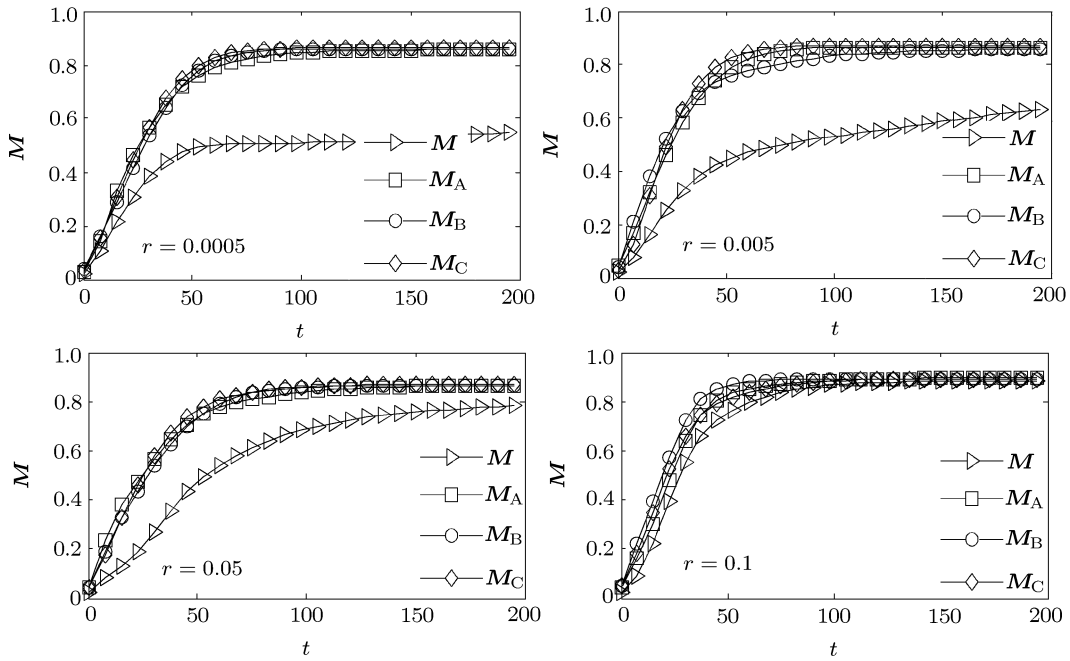


Fig. 7. The order parameter $\boldsymbol{M}$ vs. time evolution with different cases of parameter r for the whole network and each community. Square, circle and diamond curves represent the order parameters of community A, B and C, respectively. And the triangle curve represents the order parameter of the whole network. The data points were averaged over 30 realizations, and the coupling strength $c = 10$ is fixed.

This investigation reveals a remarkable relationship between topological scales and dynamic time scales: starting from random initial conditions, those highly interconnected nodes forming local clusters will synchronize first and then, in a sequential process, larger and larger spatial structures also will do it up to the final state where the whole nodes should be in the same synchronized states, indicating that the oscillators of the high densely interconnected communities synchronize more rapidly than the whole network with sparse connections.

4. Conclusions and discussions

Many real-life networks have been actually demonstrated to possess the small-world property with clear community. A felicitous network model seems to be very significant for representing processes more realistically taking place in real-life systems. In this paper, a simple small-world network with community structure is proposed first. The small-world property can arise directly as a result of the community organization of the network, and the community organization is tunable by a parameter r, which is directly measured by the ratio of inter to intra-community connectivity, and a smaller r corresponds to a stronger community structure. Then, we investigated the proposed network both analytically and numerically. In particular, the properties, including the degree distribution, clustering, the communication efficiency and modularity are analysed respectively. The obtained results show that the network has clear community structure, and simultaneously possesses a clear small-world behaviour.

In addition, based on the proposed small-world network with community structure, we investigated a fundamental relationship between community structure and dynamical behaviour occurring on community networks with the help of the Kuramoto model, and found that increasing the fuzziness of community structure will sharply enhance the network synchronizability, indicating that a strong community structure (a smaller r) will hinder global synchronization. However, in an abnormal region ($r \leq 0.001$), the network has even worse synchronizability than the case of isolated communities ($r = 0$). Furthermore, we investigated the synchronization in two scales, an individual community and the whole network, and found that the corresponding dynamical behaviour reveals a remarkable topological scale: the oscillators of high densely interconnected communities synchronize more easily (i.e. with the weaker coupling strength), and more rapidly (i.e. with the shorter time evolution) than the whole network. It is also worth while pointing out that this synchronization behaviour is highly coincident with that on the other community networks.[20,23,27,28] The result is not only of theoretical interest, but also significant in practice if one wants to enhance the synchronizability of a small-world network with clear communities. Moreover, the conclusions can also be helpful for the detection or identification of community in complex networks, and indeed the observation of dynamical processes has been proposed as one possible method of community detection.[24,32]

Acknowledgments

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