A criterion for quantum teleportation of an arbitrary N-particle state via a $2N$-particle quantum channel

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A criterion for he tquantum teleportation of an arbitrary N-particle state via a $2N$-particle quantum channel is presented by introducing a term of the "judgment operator". Using the criterion, not only the qualitative judgment of the possibility of successful teleportation can be made but also the quantitative calculation of the probability of successful teleportation can be explicitly given. In addition, a new genuine four-qubit entangled state is proposed, which could not belong to the category of previously known states under stochastic local operations and classical communication.

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1. Introduction

Quantum teleportation is a profound example of the quantum information process, where an unknown state can be perfectly transported from the sending station (the sender Alice) to the receiving station (the receiver Bob) by using previously shared entanglement with the help of some classical information. Since the introduction of quantum teleportation protocol by Bennett *et al.*,[1] quantum teleportation has received much attention, both theoretically and experimentally, in recent years due to its important applications in quantum computation and quantum communication. Several experimental implementations[2−4] of teleportation have been demonstrated and many protocols of quantum teleportation via various entangled channels with different measurement bases have also been proposed.[5−23] We noticed that both the measurement basis and the quantum channel made contribution to the probability of successful teleportation. Recently, a quantum teleportation criterion, the so-called "transformation operator", was presented in Ref. [24]. The criterion gave the qualitative judgment of the possibility of successful teleportation with considering only the transport of the unknown arbitrary two-particle state via the four-particle quantum channel with the tensor product of two Bell states as the measurement base. Here, we generalize their results to teleporting the unknown arbitrary N-particle state via the $2N$-particle quantum channel with the generalized measurement base,[25] and propose a criterion for quantum teleportation by introducing a term of the "judgment operator". On the other hand, different from the transformation operator in Ref. [24], the concrete form of the judgment operator is given for the purpose of getting a more explicit teleportation criterion. Therefore, by using our criterion, not only qualitative judgment of possibility of successful teleportation can be made but also the quantitative calculation of the probability of successful teleportation can be explicitly given. Furthermore, a simpler computing method of the judgment operator is proposed. It is shown that the judgment operator just likes a transfer operator,[5] which is based on the invariance of entanglement teleportation under an arbitrary multi-qubit unitary transformation. In addition, a few mistakes of the previous studies[17,22,24] can be pointed out by our criterion, and our criterion is less likely to arouse the comment made by Chen *et al.*[26]

On the other hand, entanglement teleportation transfers not only the amount of entanglement but also its structure (the entangled state itself). And the physic process of teleportation in principle is accompanied by the process of entanglement swapping.[27] Both measurement base and quantum channel play a crucial role in those processes. In other words, there is an important question of entanglement match between measurement base and quantum channel[24] in the teleportation process. To some extent, the match should be reflected in our criterion.

The organization of the rest of this paper is as fol-

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lows. In Section 2, we introduce the term of "judgment operator" to propose a special criterion for quantum teleportation. In Section 3, we give a method of computing the judgment operator and discuss the property of the judgment operator. In Section 4, a general criterion for quantum teleportation is presented. In Section 5, the applicability of the criterion for quantum teleportation is illustrated by some known examples. Finally, in Section 6, some conclusions are drawn and the general criterion is discussed.

2. Judgment operator, special criterion and new entangled state

To start with, we explicitly present the general protocol for teleporting an unknown arbitrary N-particle state via an arbitrary $2N$-particle entangled channel. Suppose that the unknown arbitrary N-particle state teleported by Alice to Bob is in the following form:

$$|\chi\rangle_{12\ldots N}=\sum_{i=0}^{2^N-1}x_i|bit_i\rangle_{12\cdots N}, \tag{1}$$

where $x_1, x_2, \ldots, x_{2^N-1}$ are arbitrary unknown complex numbers, and it has been assumed that the wave function satisfies the normalization condition $\sum_{i=0}^{2^N-1}x_i^2=1$. The character bit_i is a binary string whose value is equal to i in decimal system, and $|bit_i\rangle_{12\cdots N}$ is an N-particle state in the N-dimensional binary form. We also assume that Alice and Bob share an arbitrary $2N$-particle entangled channel, therefore

$$|\phi\rangle_{(N+1)(N+2)\cdots(3N)}=\sum_{i=0}^{2^{2N}-1}a_i|bit_i\rangle_{(N+1)(N+2)\cdots(3N)}, \tag{2}$$

where $a_1, a_2, \ldots, a_{2^{2N}-1}$ are arbitrary known complex numbers, the wave function satisfies the normalization condition $\sum_{i=0}^{2^{2N}-1}a_i^2=1$, and the characters bit_i are defined as the above, but $|bit_i\rangle_{(N+1)(N+2)\cdots(3N)}$ is a $2N$-particle state in the $2N$-dimensional binary form. The particles $(1,2,\ldots,N)$ and $(N+1,N+2,\ldots,N+N)$ are Alice's possession, and the other particles $(2N+1,2N+2,\ldots,2N+N)$ are Bob's possession. The system state of the $3N$ particles can be expressed as

$$|\psi\rangle_{12\ldots(3N)}=|\chi\rangle_{12\ldots N}\otimes|\phi\rangle_{(N+1)(N+2)\cdots(3N)}. \tag{3}$$

Furthermore, suppose that Alice performs a joint measurement on the $2N$ particles $(1,2,\ldots,2N)$. In this section, the joint measurement base is constructed by using a set of 4^N complete orthogonal states as follows:

$$|MES\rangle_{12\cdots(2N)}^{i_{N+1}i_{N+2}\cdots i_{2N}}=\Big(\prod_{j'=1}^{N}\tilde{\sigma}_{j'}^{1}\otimes\prod_{j=N+1}^{2N}\otimes\tilde{\sigma}_{j}^{i_j}\Big)\,|MES\rangle_{12\cdots(2N)}^{\overbrace{11\cdots1}^{N}}, \tag{4}$$

where $\tilde{\sigma}_{j'}^{1}=I_2$ and $\tilde{\sigma}_{j}^{i_j}=I_2,\ \sigma_{2z},\sigma_{2x},-\mathrm{i}\,\sigma_{2y}$, $(i_j=1,2,3,4)$ ($\tilde{\sigma}_{j'}^{1}$ and $\tilde{\sigma}_{j}^{i_j}$ are the unitary rotations to be implemented on the particles j' and j, respectively); matrix I_2 is the (2×2)-dimensional identity matrix; σ_{2z}, σ_{2x} and σ_{2y} are the Pauli matrices; $|MES\rangle_{12\cdots(2N)}^{\overbrace{11\cdots1}^{N}}$ is regarded as the "head" state of the complete orthogonal measurement base corresponding to $i_{N+1}=i_{N+2}=\cdots=i_{2N}=1$; states $|MES\rangle_{12\cdots(2N)}^{i_{N+1}i_{N+2}\cdots i_{2N}}$ is a bipartite maximally entangled state (MES) $|MES\rangle_{12\cdots(2N)}$,[8] which is a pure state of multipartite qubit system and satisfies the following request:[28] the particles $(1,2,\ldots,N)$ and the other particles $(N+1,N+2,\ldots,N+N)$ must be maximally entangled, namely

$$\rho_{12\cdots N}=\rho_{(N+1)(N+2)\cdots(2N)}=I_{2^N}/2^N, \tag{5}$$

where the reduced density matrices are $\rho_{12\cdots N}=\mathrm{tr}_{(N+1)(N+2)\cdots(2N)}(|MES\rangle_{12\cdots(2N)}\langle MES|)$ and $\rho_{(N+1)(N+2)\cdots(2N)}=\mathrm{tr}_{12\cdots N}(|MES\rangle_{12\cdots(2N)}\langle MES|)$, and I_{2^N} is the $(2^N\times2^N)$-dimensional identity matrix. In the point of view of the von Neumann entropy, equation (5) is equivalent to $S(\rho_{12\cdots N})=S(\rho_{(N+1)(N+2)\cdots(2N)})=N$. Based on the bipartite MES to satisfy equation (5), 4^N complete orthogonal states can always be generated through Eq. (4). The complete orthogonal measurement base is constructed by the bipartite MES, so it is called a special criterion for quantum teleportation in this paper.

For brevity, both the description of the teleportation process and the proposal of the criterion for quantum teleportation are stated alternately in the following. If Alice operates a joint measurement, Bob will correspondingly obtain one of the 4^N states as follows:

$$\begin{aligned}\overbrace{11\cdots1}^{N}_{12\cdots(2N)}\langle MES|\psi\rangle_{12\cdots(3N)} &= |\chi'\rangle_{(2N+1)(2N+2)\cdots(3N)}\\ &\equiv \sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)},\end{aligned} \tag{6}$$

and

$$\begin{aligned}{}^{i_{N+1}i_{N+2}\cdots i_{N+N}}_{12\cdots(2N)}\langle MES|\psi\rangle_{12\cdots(3N)} &= |\chi''\rangle_{(2N+1)(2N+2)\cdots(3N)}\\ &\equiv \sigma^{i_{2N+1}i_{2N+2}\cdots i_{2N+N}}_{(2N+1)(2N+2)\cdots(3N)}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}.\end{aligned} \tag{7}$$

Here $\sigma^{i_{2N+1}i_{2N+2}\cdots i_{3N}}_{(2N+1)(2N+2)\cdots(3N)}$ is called the "judgment operator" corresponding to the measurement outcome

$$|MES\rangle^{i_{N+1}i_{N+2}\cdots i_{N+N}}_{12\cdots(2N)}, \quad \text{and} \quad \sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}$$

is regarded as the "head" judgment operator corresponding to the measurement outcome $|MES\rangle^{\overbrace{11\cdots1}^{N}}_{12\cdots(2N)}$. According to Eqs. (5), (6) and (7), the relation between $\sigma^{i_{2N+1}i_{2N+2}\cdots i_{3N}}_{(2N+1)(2N+2)\cdots(3N)}$ and $\sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}$ is easily expressed as

$$\sigma^{i_{2N+1}i_{2N+2}\cdots i_{3N}}_{(2N+1)(2N+2)\cdots(3N)} = \Big(\prod_{j=2N+1}^{3N}\otimes\tilde{\sigma}_j^{i_j}\Big)\sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}, \tag{8}$$

where $\tilde{\sigma}_j^{i_j}$ are defined in a way similar to that in the above. So in such a case, only the consideration of "head" judgment operator is sufficient to give the criterion of the quantum teleportation.

Noticeably, in order to investigate the special criterion, the second parts of Eqs. (6) and (7) are assumed to hold. In other words, in terms of the "judgment operator", the special criterion for quantum teleportation to transport an arbitrary N-particle state can be given. To facilitate the description of the criterion, we first define a diagonal matrix

$$\Lambda_k = \mathrm{diag}(b_1, b_2, \ldots, b_k), \tag{9}$$

where $|b_i| > 0, i = 1, 2, \ldots, k$.

If the "head" judgment operator can be written in the following form:

$$\sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)} = \kappa U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}\Lambda_{2^N}, \tag{10}$$

where $U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}$ is a $(2^N \times 2^N)$-dimensional collective unitary matrix, κ is a coefficient to be determined, and Λ_{2^N} is a diagonal matrix as defined in Eq. (9). In principle, when one analyses the judgment operators, the coefficient κ may be factored and some or other factors may be put out/into the diagonal matrix for convenience, arbitrarily. Therefore, in this section we assume that the factor κ out of diagonal matrix is identically equal to $1/2^N$ for self-consistency of this paper. Due to Eq. (8), the other judgment operator must be written as

$$\begin{aligned}&\sigma^{i_{2N+1}i_{2N+2}\cdots i_{3N}}_{(2N+1)(2N+2)\cdots(3N)}\\ &= \kappa\Big[\Big(\prod_{j=2N+1}^{3N}\otimes\tilde{\sigma}_j^{i_j}\Big)U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}\Big]\Lambda_{2^N}.\end{aligned} \tag{11}$$

One can see that all of judgment operators may be picked up the same coefficient κ and diagonal matrix Λ_{2^N}.

Alice can inform Bob of the joint measurement outcomes via a classical channel. If Bob receives the measurement outcome $|MES\rangle^{i_{N+1}i_{N+2}\cdots i_{N+N}}_{12\cdots(2N)}$, he locally implements the collective reconstructing operator $U' = \Big[\Big(\prod_{j=2N+1}^{3N}\otimes\tilde{\sigma}_j^{i_j}\Big)U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}\Big]^{-1}$. Then the N-particle state in receiving station will ultimately

be evolved into the following form:

$$U'|\chi'\rangle_{(2N+1)(2N+2)\cdots(3N)} = \kappa\Lambda_{2^N}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}. \quad (12)$$

Now we come to discuss the three categories involved in Eqs. (10) and (12). For the first case where the diagonal matrix Λ_{2^N} is the $(2^N\times 2^N)$- dimensional identity matrix, the unknown arbitrary N-particle entangled state is teleported with a possibility of $|\kappa|^2$. We know that Bob can obtain $1/|\kappa|^2$ kinds of successful conditions in all, therefore the total successful possibility reaches unity. This case is the circumstance of deterministic teleportation. For the second case where the diagonal matrix Λ_{2^N} is not the $(2^N \times 2^N)$-dimensional identity matrix, Bob needs to introduce an auxiliary two-state particle A with the initial state $|0\rangle_A$ and performs a collective unitary transformation on particles $(2N+1, 2N+2, \ldots, 3N$ and A. In order for Bob to reconstruct the original state under the basis $\{|\psi_0\rangle_{(2N+1)(2N+2)\cdots(2N+N)}|0\rangle_A$, $|\psi_0\rangle_{(2N+1)(2N+2)\cdots(2N+N)}|1\rangle_A\}$, where

$$|\psi_0\rangle_{(2N+1)(2N+2)\cdots(2N+N)}$$

represents the basis of a $(2^N\times 2^N)$-dimensional Hilbert space, a unitary transformation, i.e., a $(2^{N+1}\times 2^{N+1})$-dimensional matrix, may take the following form:[18]

$$U'' = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}, \quad (13)$$

where A_1 and A_2 are $(2^N \times 2^N)$-dimensional matrixes and they can be expressed as

$$A_1 = \mathrm{diag}(c_1, c_2, \ldots, c_{2^N}) \quad (14)$$

and

$$A_2 = \mathrm{diag}\left(\sqrt{1-c_1^2}, \sqrt{1-c_2^2}, \ldots, \sqrt{1-c_{2^N}^2}\right), \quad (15)$$

respectively, where $c_i = \lambda/b_i$, $i = 1, 2, \ldots, 2^N$ and $\lambda = \min\{b_i\}$. The unitary transformation U'' will transform $\kappa\Lambda_{2^N}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}|0\rangle_A$ into

$$\begin{aligned} &\kappa U''\Lambda_{2^N}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}|0\rangle_A \\ &= \kappa\lambda|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}|0\rangle_A \\ &\quad + |\psi'\rangle_{(2N+1)(2N+2)\cdots(3N)}|1\rangle_A, \end{aligned} \quad (16)$$

where $|\psi'\rangle_{(2N+1)(2N+2)\cdots(3N)}|1\rangle_A$ is a residual state. Then Bob measures the state of an auxiliary particle A. If the measurement result is $|1\rangle_A$, the teleportation has failed. If the measurement result is $|0\rangle_A$, Bob has successfully realized quantum teleportation with a probability of $|\kappa\lambda|^2$. Because Bob can obtain $1/|\kappa|^2$ kinds of successful conditions in all, the total probability of successful teleportation is $|\lambda|^2$. Hence the probability of successful teleportation is less than unity, so this case is the circumstance of probabilistic teleportation. For the final case where the judgment operator cannot be written as the form as shown in Eq. (10), the unknown arbitrary N-particle state cannot be teleported. So this case is the circumstance of impossible teleportation.

In the present paper, the bipartite MES need not require any k-particles and the other $(2N-k)$-particles $(k = 1, \cdots, N)$ to be entangled. So the MES may be reduced into the tensor product of N pairs of Bell states, or may not be reduced into the tensor product of N pairs of the Bell states (such an entangled state was regarded as a genuine multipartite entangled state,[29] such as the cluster state[11] and the genuine four- qubit entangled state[7] which was afterward called χ-type entangled state[30]). In Zha and Ren's (ZR) criterion, the measurement base was constructed by using 16 complete orthogonal states consisting of the tensor product of two pairs of Bell states;[24] in Yeo and Chua's teleportation, 16 χ-type entangled states constructed the measurement base;[7] in Wang *et al*'s teleportation, 16 cluster states formed the measurement base.[11] Here we propose a new genuine four-qubit entangled state:

$$\begin{aligned} \phi\rangle_{3456} = \frac{1}{2}(|0000\rangle + |0101\rangle + \frac{1}{\sqrt{2}}|1010\rangle + \frac{1}{\sqrt{2}}|1011\rangle \\ + \frac{1}{\sqrt{2}}|1110\rangle - \frac{1}{\sqrt{2}}|1111\rangle)_{3456}. \end{aligned} \quad (17)$$

Due to Eq. (4), 16 complete orthogonal states are also generated based on the new genuine four-qubit entangled state above, which may construct the measurement base. Certainly, they are all employed for teleportation and dense coding as well. The new genuine entangled states could not belong to the category of the previously known states under stochastic local operations and classical communication (SLOCC),[31] such as the states mentioned above.

3. Computing method and property of judgment operator

How can we obtain the judgment operator $\sigma^j_{(2N+1)(2N+2)\cdots(3N)}$? Construct a new form of mathematics by using the construction idea, then the problem will be solved in this form. We investigate the left side of Eq. (6) or Eq. (7), and have

$$
\begin{aligned}
{}_{12\cdots(2N)}\langle MES|\psi\rangle_{12\cdots(3N)} &= {}_{12\cdots(2N)}\langle MES|\chi\rangle_{12\cdots N}\otimes|\phi\rangle_{(N+1)(N+2)\cdots(3N)} \\
&= {}_{12\cdots(2N)}\langle MES|\phi\rangle_{(N+1)(N+2)\cdots(3N)}|\chi\rangle_{12\cdots N}. \qquad (18)
\end{aligned}
$$

Comparing it with Eq. (6) or Eq. (7) yields

$$
\sigma_{(2N+1)(2N+2)\cdots(3N)}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}={}_{12\cdots(2N)}\langle MES|\phi\rangle_{(N+1)(N+2)\cdots(3N)}|\chi\rangle_{12\cdots N}. \qquad (19)
$$

And neglecting the difference between subscripts of the state $|\chi\rangle$, we obtain

$$
\sigma_{(2N+1)(2N+2)\cdots(3N)}={}_{12\cdots(2N)}\langle MES|\phi\rangle_{(N+1)(N+2)\cdots(3N)}. \qquad (20)
$$

Here it seems that our derivation of the judgment operator is crude, but it is feasible according to the Lee's research results.[5] Comparing it with Eq. (12) in Ref. [5] and according to Lee's opinion, we could also claim that the judgment operator $\sigma_{(2N+1)(2N+2)\cdots(3N)}$ plays a crucial role in revealing an invariance of entanglement teleportation. However, we already see that Lee considered only the deterministic teleportation, and that $[U']^{-1}$ and Λ_{2^N} (see Eq. (11) in the present paper, where $N=2$) in our criterion play the roles the same as those that $U^{\dagger}_{\alpha\beta}$ and transfer operator $\hat{T}_{\rm BU}$ play in Ref. [5], respectively.

From Eq. (20), it is easy to find that the judgment operator involves information concerning the measurement base and the quantum channel. Therefore the property of the judgment operator depends on both the measurement base and the quantum channel. In order to illustrate that clearly, two examples are given below.

If the "head" state of the measurement base is reducible into the tensor product of N pairs of Bell state, which is shown in the following form:

$$
|MES\rangle^{\overbrace{11\cdots1}^{N}}_{12\cdots(2N)}=\phi^1_{1(N+1)}\otimes\phi^1_{2(N+2)}\otimes\cdots\otimes\phi^1_{N(N+N)}, \qquad (21)
$$

where

$$
\phi^1_{mn}=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{mn},\quad mn=1(N+1),2(N+2),\ldots,N(N+N), \qquad (22)
$$

from Eq. (20), the "head" judgment operator $\sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}$ will be

$$
\sigma^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}=2^{-N/2}\begin{pmatrix} a_0 & a_{2^N} & \cdots & a_{2^N(2^N-1)} \\ a_1 & a_{2^N+1} & \cdots & a_{2^N(2^N-1)+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2^N-1} & a_{2^{N+1}-1} & \cdots & a_{2^{2N}-1} \end{pmatrix}. \qquad (23)
$$

If the measurement base is not reducible into tensor product of N pairs of Bell bases, for simplicity, as an example we take the quantum teleportation of an unknown arbitrary two-particle state via an arbitrary four-particle entangled channel with the so-called χ-type entangled state used as the measurement base. Here the "head" state of the measurement base is shown as the following form:

$$
|MES\rangle^{11}_{1234}=\frac{1}{2^{3|2}}(|0000\rangle+|0011\rangle-|0101\rangle+|0110\rangle+|1001\rangle+|1010\rangle-|1100\rangle+|1111\rangle)_{1234}. \qquad (24)
$$

By using expression of Eq. (20), the "head" judgment operator σ^{11}_{56} is obtained to be

$$
\sigma^{11}_{56}=\frac{1}{2^{3/2}}\begin{pmatrix} a_0+a_{12} & -a_4+a_8 & a_4+a_8 & -a_0+a_{12} \\ a_1+a_{13} & -a_5+a_9 & a_5+a_9 & -a_1+a_{13} \\ a_2+a_{14} & -a_6+a_{10} & a_6+a_{10} & -a_2+a_{14} \\ a_3+a_{15} & -a_7+a_{11} & a_7+a_{11} & -a_3+a_{15} \end{pmatrix}. \qquad (25)
$$

If the measurement base is given, the judgment operator $\sigma^{i_{2N+1}i_{2N+2}\cdots i_{3N}}_{(2N+1)(2N+2)\cdots(3N)}$ is correspondingly determined according to Eq. (20). The judgment operator is a $(2^N \times 2^N)$-dimensional matrix and its elements are made up of fixed functions $f_{mn}(a_i)$ (here the subscripts m and n respectively denote the numbers of rows and columns of the matrix) which are combined linearly by the coefficients $a_i, i = 0, 1, \cdots, 2^{2N-1}$. Obviously, the coefficient a_i shows quantum channel information, and the fixed function f_{mn} indicates the measurement base information.

In the special criterion, the collective unitary operator $U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}$ also depends on the measurement base and quantum channel, which is interesting and will be discussed in the last section.

4. Criterion for quantum teleportation

If all states of the complete orthogonal measurement base are not generated due to Eq. (4), then the measurement base does not have to satisfy the requirement as shown in Eq. (5). This measurement base is defined as the generalized measurement base (GMB). In this section, the joint measurement base is constructed by using the generalized measurement base composed of 4^N complete orthogonal states $|GMB\rangle^j_{12\cdots(2N)}$ for $j = 1, 2, \cdots, 4^N$. Now we investigate such a teleportation scheme so as to present the criterion for quantum teleportation.

If Alice implements the joint measurement above, Bob will correspondingly obtain one of the 4^N states as follows:

$$\begin{aligned} {}^{j}_{12\cdots(2N)}\langle GMB|\psi\rangle_{12\cdots(3N)} &= |\chi'\rangle_{(2N+1)(2N+2)\cdots(3N)} \\ &\equiv \sigma^j_{(2N+1)(2N+2)\cdots(3N)}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}, \end{aligned} \tag{26}$$

where $\sigma^j_{(2N+1)(2N+2)\cdots(3N)}$ is the judgment operator, corresponding to the measurement outcome $|GMB\rangle^j_{12\cdots(2N)}$.

If the judgment operator $\sigma^j_{(2N+1)(2N+2)\cdots(3N)}$ can be expressed as

$$\sigma^j_{(2N+1)(2N+2)\cdots(3N)} = \kappa_j U^j_{(2N+1)(2N+2)\cdots(3N)} \Lambda^j_{2^N}, \tag{27}$$

where the character $U^j_{(2N+1)(2N+2)\cdots(3N)}$ is a $(2^N \times 2^N)$-dimensional collective unitary matrix, coefficient κ_j is a coefficient to be determined, and $\Lambda^j_{2^N}$ is diagonal matrix defined in Section 2. Similarly, for convenience, the coefficient κ_j may always be factored and some or other factors may be put into/out the diagonal matrix $\Lambda^j_{2^N}$ arbitrarily. Corresponding to the measurement outcome $|GMB\rangle^j_{12\cdots(2N)}$, Bob locally implements the unitary operator $U' = [U^j_{(2N+1)(2N+2)\cdots(3N)}]^{-1}$. Then the multi-particle state in the receiving station will ultimately be evolved into

$$U'|\chi'\rangle_{(2N+1)(2N+2)\cdots(3N)} = \kappa_j \Lambda^j_{2^N}|\chi\rangle_{(2N+1)(2N+2)\cdots(3N)}. \tag{28}$$

According to Eqs. (27) and (28), we complete the following discussion of teleportation. When the diagonal matrix $\Lambda^j_{2^N}$ is the $(2^N \times 2^N)$-dimensional identity matrix, the unknown arbitrary N-particle state is teleported with the successful possibility of $|\kappa_j|^2$. When the diagonal matrix $\Lambda^j_{2^N}$ is not the $(2^N \times 2^N)$-dimensional identity matrix, by a method similar to that in Section 2, Bob introduces an auxiliary two-state particle A with the initial state $|0\rangle_A$ and performs a collective unitary transformation on particles $2N+1, 2N+2, \ldots, 3N$ and A. Finally, Bob measures the state of an auxiliary particle A. If the measurement result is $|1\rangle_A$, the teleportation has failed. If the measurement result is $|0\rangle_A$, Bob has successfully realized the quantum teleportation with the probability of $|\kappa_j\lambda_j|^2$, where λ_j is equal to the smallest nonzero element in the diagonal matrix $\Lambda^j_{2^N}$. If the judgment operator cannot be written in the form as shown in Eq. (27), the unknown arbitrary N-particle state cannot be teleported. We notice that it may be unable to teleport the unknown arbitrary state corresponding to some measurement results Alice presents. Therefore the total probability of successful teleportation can be acquired from the following formula:

$$p = \sum_{j>0}^{n'} |\kappa_j|^2 + \sum_{j>n'}^{n} |\kappa_j\lambda_j|^2, \quad 0 \le n' \le n \le 4^N. \tag{29}$$

The judgment operator $\sigma^j_{(2N+1)(2N+2)\cdots(3N)}$ can be

obtained by the following method:

$$\sigma^{j}_{(2N+1)(2N+2)\cdots(3N)} = {}^{j}_{12\cdots(2N)}\langle GMB|\phi\rangle_{(N+1)(N+2)\cdots(3N)}. \tag{30}$$

We can see that the criterion in Section 2 may also be regarded as a very special case of criterion for quantum teleportation. Furthermore, Zha and Ren's criterion is for one of the special cases of criterion in Section 2. For succinctness of the whole text, three kinds of teleportation will be discussed in detail by the known examples in the next section.

5. Applicability of criterion for quantum teleportation

To show the applicability of the criterion, we examine the probability of transportation for some known examples. Three cases will be discussed below.

The first case is the deterministic teleportation. When all of the judgment operators can be written in the form as shown in Eq. (27), all of the corresponding diagonal matrixes Λ^{j}_{2N} are the identity matrixes, and the sum of all the corresponding coefficients $|\kappa_j|$ is equal to unity, the total probability of successful transformation reaches unity. The deterministic teleportation mentioned in Section 2 is for a special case, where all of coefficients $|\kappa_j|$ are the same and equal to $1/4^N$ (it could also be exclusive case, because we do not know whether there exist any other cases, where the coefficients $|\kappa_j|$ are different but the sum of the coefficients reaches unity). We take the first example of the quantum channel given in Ref. [7] as follows:

$$\begin{aligned}|\phi\rangle_{3456} = \frac{1}{2^{3/2}}(&|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle \\ &+ |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)_{3456},\end{aligned} \tag{31}$$

i.e. $a_0 = -a_3 = -a_5 = a_6 = a_9 = a_{10} = a_{12} = a_{15} = 1/2\sqrt{2}$, and the other coefficients vanish. If the "head" state of the measurement base is constructed by the χ-type entangled states as shown in Eq. (24), then the "head" judgment operator is calculated easily from Eq. (25), namely $\sigma^{11}_{56} = 1/4I_4$, and all other judgment operators $\sigma^{i_5 i_6}_{56}$ can also be easily acquired through Eq. (8) or Eq. (11). According to the special criterion, the unknown arbitrary two-particle state can be perfectly teleported.

The second case is the probabilistic teleportation. When not all the judgment operators can be written in the form as shown in Eq. (27) or not all the corresponding diagonal matrixes Λ^{j}_{2N} are the identity matrixes, the total probability of the successful transformation is less than unity. The examples can be seen in Refs. [14]–[17] and [19]–[23].

Especially, when the quantum channel shared between Alice and Bob is not the bipartite MES, though Alice employs a bipartite MES as the measurement base, the probability of successful teleportation is less than unity. Such probabilistic teleportation is implied in Section 2. For instance, the sender Alice employs the joint Bell measurement (the "head" state of the measurement base is shown as Eq. (21) and all other states are obtained through Eq. (5), here $N = 3$), and Alice and Bob share three pairs of non-maximally entangled Bell states as the quantum channel[16]

$$\begin{aligned}|\phi\rangle_{456789} = (a|00\rangle + b|11\rangle)_{47}&(c|00\rangle \\ &+ d|11\rangle)_{58}(e|00\rangle + f|11\rangle)_{69},\end{aligned} \tag{32}$$

where $|a| \le |b|$, $|c| \le |d|$ and $|e| \le |f|$, i.e.

$$\begin{aligned}&a_0 = ace, \quad a_9 = acf, \quad a_{18} = ade, \quad a_{27} = adf, \\ &a_{36} = bce, \quad a_{45} = bcf, \quad a_{54} = bde, \quad a_{63} = bdf.\end{aligned} \tag{33}$$

And other coefficients vanish. Due to Eq. (23), the judgment operator σ^{111}_{789} can be obtained easily. All other judgment operators $\sigma^{i_7 i_8 i_9}_{789}$ can expediently be obtained through Eq. (8) or Eq. (11). According to the special criterion, the probability to successfully teleport the unknown arbitrary three-particle state can be immediately obtained, and it is equal to $8|ace|^2$.

In general, if some states of the complete orthogonal measurement base are not the bipartite MES,[14,17,23] or if the quantum channel is not the one,[13–17,19–22] then the teleportation must not be the deterministic teleportation.

The final case is for the impossible teleportation. If all of the judgment operators cannot be written in the form as shown in Eq. (27), the probability of the successful transformation vanishes. We have investigated the case where the Greenberger–Horne–Zeilinger (GHZ) state or the W state of four particles is employed as the quantum channel respectively. No matter what is used as the measurement base, the judgment operators must always not be written in the form as shown in Eq. (27). Accordingly, the GHZ state and the W state cannot be employed to faithfully teleport unknown arbitrary two-particle state.

In fact, we have investigated all teleportation protocols mentioned in Section 1. As a result, according to our criterion, the probabilities of all successful

teleportation protocols can be calculated easily, and the unitary reconstructing operators can also be forecasted successfully. Moreover, according to our criterion, it is easy to find that there are disadvantages or inconspicuous mistakes in some papers.[17,22,24] For instance, the probability of teleportation protocol provided was erroneous in original paper.[22] We have already communicated with the author of Ref. [22] by email, and the error of the original paper has also been corrected. A few mistakes in Ref. [24] had been pointed out by email before Ref. [26] was published. In addition, the detailed analysis of the literature[17] can be seen below.

In Ref. [17], an explicit teleportation protocol was presented where, by using a non-maximally entangled state as the measurement base, one can use a non-maximally entanglement state $|\phi\rangle_{23} = N(|00\rangle + n|11\rangle)_{23}$ (where $N{=}1/\sqrt{1+|n|^2}$) as a resource to teleport an unknown arbitrary single particle state $|\chi\rangle_1{=}x_0|0\rangle_1 + x_1|1\rangle_1$. The authors provided the successful probability of their protocol as follows:

$$P = \frac{2|n|^2}{(1+|n|^2)^2}. \tag{34}$$

Given $|n| = 1$, the probabilistic teleportation should become the deterministic teleportation. Due to Eq. (34), the successful probability is 1/2. The contradiction is anticipated. However, according to our criterion, the contradiction can obviously appear, but it is easily solved, namely the successful probability of their protocol should be

$$P = \frac{2(|n|^2+|n|^4)}{(1+|n|^2)^2}, \quad |n| \le 1 \tag{35}$$

or

$$P = \frac{2+2|n|^2}{(1+|n|^2)^2}, \quad |n| > 1. \tag{36}$$

The authors of Ref. [17] could have thought that when the measurement outcomes were $|MES\rangle_{12}^1$ or $|MES\rangle_{12}^4$, the receiver Bob could not reconstruct the desired state. In fact, Bob may introduce an auxiliary two-state particle so as to realize the teleportation with a certain probability. The detailed illustration can be seen in Table 1, in which j denotes Alice's measurement outcomes on qubits 1 and 2; σ_3, κ and p respectively denote the judgment operator, the coefficient and the successful probability corresponding to the measurement outcome; and P represents the total teleportation probability.

Table 1. Judgment operators, coefficients and possibilities corresponding to the measurement outcomes by sender Alice.

j	σ_3	κ	$p\,(\|n\| \le 1)$	$p\,(\|n\| > 1)$
$\|GMB\rangle_{12}^1 = N(\|00\rangle + n\|11\rangle)_{12}$	$N^2\begin{pmatrix}1 & 0\\ 0 & n^*n\end{pmatrix}$	N^2	$N^4\|n\|^4$	N^4
$\|GMB\rangle_{12}^2 = N(n^*\|00\rangle - \|11\rangle)_{12}$	$N^2n\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	N^2n	$N^4\|n\|^2$	$N^4\|n\|^2$
$\|GMB\rangle_{12}^3 = N(\|01\rangle + n^*\|10\rangle)_{12}$	$N^2n\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	N^2n	$N^4\|n\|^2$	$N^4\|n\|^2$
$\|GMB\rangle_{12}^4 = N(n\|01\rangle - \|10\rangle)_{12}$	$N^2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}n^*n & 0\\ 0 & 1\end{pmatrix}$	N^2	$N^4\|n\|^4$	N^4
P		$2N^4(\|n\|^2+\|n\|^4)$		$N^4(2+2\|n\|^2)$

6. Conclusion and discussion

We introduced a term of the "judgment operator" to present a criterion for the quantum teleportation of an unknown arbitrary N-particle state via a $2N$-particle entangled channel. We also give a new method of computing the judgment operator and analyse the property of the judgment operator. In our criterion, the probability of successful quantum teleportation is explicitly provided by a formula as shown in Eq. (29). In addition, we proposed a new genuine four-qubit entanglement state.

Note that reference [24] received a recent comment made by Chen *et al.*[26] that the transformation operator is not really suitable for serving as a criterion for the faithful teleportation. The reason which Chen *et al.* gave was that the receiver had to perform an additional operation which was not relevant

to Alice's operations. It is easily found that the additional operation may just be the collective unitary matrix $\left[U^{\overbrace{11\cdots1}^{N}}_{(2N+1)(2N+2)\cdots(3N)}\right]^{-1}$ mentioned in Section 2, which depends on both the measurement base and the quantum channel. While in Chen *et al.*'s opinion, $\prod_{j=2N+1}^{3N}\otimes\tilde{\sigma}_j^{i_j}$ could just be relevant to Alice's operations. In our opinion, whether the additional operation must be performed by the receiver Bob depends on the measurement base and the quantum channel. In fact, here is involved the question about whether both measurement basis and quantum channel are the optimal match[24] for quantum teleportation. When the measurement base and the quantum channel achieve the optimal match of the entanglement structure, Bob need not perform the additional operation. The typical examples may be found in Refs. [7] and [11]. In summary, the generalized measurement base is employed to propose a more general criterion for quantum teleportation, and consideration of the criterion is not limited to the tensor product of the Bell base, which is different from Zha and Ren's criterion. Therefore, our criterion is less likely to arouse the comment, i.e. "the ZR protocol in principle is equivalent to Rigolin's protocol"[6] and its associated comment.[32]

Our criterion indicates that, when the measurement base and quantum channel satisfy the "optimal matching" of entanglement structure, the collectively reconstructing operator in receiving station could be decomposed into some simple unitary operators or even the Pauli-gates; and when they satisfy that of entanglement amount, the probability of successful teleportation may be higher. In other words, when the entanglement structure between the measurement base and the quantum channel is mismatch with them separately, there will be the residua of the entanglement structure, which is focused on the collective unitary matrix U^j (see Eq. (27)); when the entanglement amount is mismatch, there will be the wasting of entanglement amount. In Ref. [14], the "entanglement matching" was demonstrated in a special protocol, which in fact only showed the matching of entanglement amount. Those viewpoints could be suggested by the entanglement swapping.[27] However, with the research progress in entanglement of multiparticle, the question about "entanglement matching" between the measurement base and quantum channel needs to be further studied.

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