Exact solution of the one-dimensional Klein–Gordon equation with scalar and vector linear potentials in the presence of a minimal length

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Using the momentum space representation, we solve the Klein–Gordon equation in one spatial dimension for the case of mixed scalar and vector linear potentials in the context of deformed quantum mechanics characterized by a finite minimal uncertainty in position. The expressions of bound state energies and the associated wave functions are exactly obtained.

Keywords: Klein–Gordon equation, linear potential, minimal length, exact solution

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1. Introduction

The linear potential is an important mechanical model which has been widely used especially with relativistic wave equations in the investigation of quark confinement.[1,2] When this potential is introduced as the time component of a Lorentz vector no bound states are found[3,4] and only tunneling solutions arise. This is related to the existence of the Klein paradox. On the contrary, when the potential is considered as a Lorentz scalar, confinement becomes possible. Also for a mixture of a scalar-like and a vector-like linear potential, it was established that bound states can occur only when the scalar term is sufficiently strong compared with the electric field.[5−8] All these results are valid for $(3 + 1)$-dimensional space-time and were established within the framework of usual quantum mechanics, where position and momentum operators acting on the Hilbert space of states verify the usual Heisenberg algebra. However, if one takes into account the effects of quantum fluctuations of the gravitational field in order to incorporate gravity into quantum mechanics, an induced minimal length, i.e., a lower bound to the possible resolution of distances, appears. This idea was mainly motivated by string theory[9−11] and loop quantum gravity.[12−14] The minimal length may be described quantum theoretically as a nonzero minimal uncertainty in position measurements. In Refs. [15]–[18] Kempf *et al.* showed that such a minimal length can be naturally obtained from a specific modification of the usual commutation relations between position and momentum operators. An important feature of the resulting uncertainty principle is the intriguing UV/IR mixing which has appeared in several other contexts such as ADS/CFT correspondence,[19] and noncommutative field theory.[20] Physically speaking, the UV/UR mixing means that short distance physics may be probed by long distance physics. This lends an additional justification to the analysis of quantum mechanical problems in the presence of a minimal length.

On the other hand, it has been argued[17,21] that the formalism of minimal length can provide an effective theory for describing non-pointlike particles such as quasi-particles and various collective excitations in solids, or composite particles, such as nucleons, nuclei and molecules.[17] In this case the minimal length may be interpreted as an intrinsic scale characterizing the structure of the considered system and its finite size.

Recently, several physical problems have been reinvestigated within the framework of the minimal length formalism. Among them, we cite the solution of the harmonic oscillator in arbitrary dimensions in momentum space,[15,16,22] the study of the Coulomb potential in one and three dimensions,[23−26] the investigation of the cosmological constant problem and the

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classical limit of the physics with nonzero minimum uncertainties,[27,28] the solution of the Dirac oscillator problem in one dimension[29] and in three dimensions using supersymmetric quantum mechanics.[30] In the present work, we study the effect of a minimal length in relativistic quantum mechanics by considering the Klein–Gordon equation in one spatial dimension in the presence of an electrostatic and a scalar-like linear potentials. The paper is organized as follows. In Section 2, we introduce the main relations of quantum mechanics based on modified Heisenberg algebra. In Section 3, using the momentum space representation, we solve exactly the Klein–Gordon equation in one spatial dimension with an electrostatic and scalar-like linear potentials in the presence of a minimal length. We then deduce the bound state energy spectrum and the associated wave functions. The final section contains our conclusion.

2. Quantum mechanics with generalized Heisenberg algebra

Consider in one-dimensional momentum space the following realization of the position and momentum operators[16]

$$\hat{X} = \mathrm{i}\hbar\left[\left(1+\beta p^2\right)\frac{\partial}{\partial p} + \gamma p\right], \quad \hat{P} = p, \tag{1}$$

where β and γ are arbitrary parameters. These operators satisfy the deformed commutation relation

$$\left[\hat{X}, \hat{P}\right] = \mathrm{i}\hbar\left(1+\beta\hat{P}^2\right). \tag{2}$$

The resulting uncertainty relation reads

$$\left(\Delta X\right)\left(\Delta P\right) \geq \frac{\hbar}{2}\left[1+\beta\left(\Delta p\right)^2\right], \tag{3}$$

which implies a nonzero minimal uncertainty in position

$$\left(\Delta x\right)_{\min} = \hbar\sqrt{\beta}. \tag{4}$$

As mentioned above, the peculiarity of expression (3) is that it exhibits a form of the UV/IR mixing: when ΔP is large (UV), ΔX is proportional to ΔP, and therefore is also large (IR). This phenomenon seems to be necessary in dealing with certain types of new physics[31] being considered recently.

A consequence of equation (4) is that the notion of localized states in the position space disappears since space coordinates can no longer be probed with accuracy more than $(\Delta x)_{\min}$. Hence momentum space is more convenient in order to solve any eigenvalue problem. With the new modified algebra, the identity operator can be expanded as

$$\mathbf{1} = \int \frac{\mathrm{d}p}{\left(1+\beta p^2\right)^{1-\gamma/\beta}} \left|p\right\rangle \left\langle p\right|, \tag{5}$$

where $|p\rangle$ are eigenstates of the momentum operator. Note that the arbitrary constant γ does not affect the observable quantities, it determines only the weight function in the above definition of the identity operator. In order to simplify the calculations, in the following we will set $\gamma = 0$ in the realization (1) of the position operator.

3. The Klein–Gordon equation with mixed scalar and vector linear potentials in the presence of minimal length

In one spatial dimension, the Klein–Gordon equation for a particle of mass m in the presence of a scalar potential $S\left(X\right)$ and a vector potential $V\left(X\right)$ is given by

$$\left\{c^2P^2 + \left[mc^2 + S\left(X\right)\right]^2 - \left[E - V\left(X\right)\right]^2\right\}\Psi = 0. \tag{6}$$

In what follows, we shall examine the case where

$$S\left(X\right) = \lambda X, \quad V\left(X\right) = kX \tag{7}$$

with λ and k being two constants characterizing the strengths of the potentials. Substituting the operators X and P by their momentum space realization (1), we obtain the following differential equation

$$\left[-\hbar^2\left(\lambda^2-k^2\right)\left(1+\beta p^2\right)^2\frac{\partial^2}{\partial p^2} + \left[2\mathrm{i}\hbar\left(\lambda mc^2+kE\right) - 2\hbar^2\left(\lambda^2-k^2\right)\beta p\right]\right.$$
$$\left.\times\left(1+\beta p^2\right)\frac{\partial}{\partial p} + c^2p^2 + m^2c^4 - E^2\right]\Psi\left(p\right) = 0. \tag{8}$$

Introducing the following change of variable

$$\rho = \frac{1}{\sqrt{\beta}} \arctan\left(\sqrt{\beta} p\right), \tag{9}$$

which maps the region $p \in (-\infty, \infty)$ to $\rho \in (-\pi/2\sqrt{\beta}, \pi/2\sqrt{\beta})$, and defining the dimensionless parameters

$$\varepsilon = \frac{m^2c^4 - E^2}{\hbar^2 (\lambda^2 - k^2) \beta}, \qquad \alpha = \frac{\lambda mc^2 + kE}{\hbar (\lambda^2 - k^2) \sqrt{\beta}}, \qquad \delta = \frac{c^2}{\hbar^2 (\lambda^2 - k^2) \beta^2}. \tag{10}$$

Equation (8) simplifies to

$$\left[\frac{\partial^2}{\partial \rho^2} - 2\,\mathrm{i}\,\alpha\sqrt{\beta}\frac{\partial}{\partial \rho} - \beta\delta \tan^2\left(\sqrt{\beta}\rho\right) - \beta\varepsilon\right] \Psi(\rho) = 0. \tag{11}$$

Making now the substitution

$$\Psi(\rho) = \mathrm{e}^{\mathrm{i}\,\alpha\sqrt{\beta}\rho} \varphi(\rho), \tag{12}$$

we obtain the following equation for $\varphi(\rho)$

$$\left[\frac{\partial^2}{\partial \rho^2} - \beta\delta \tan^2\left(\sqrt{\beta}\rho\right) + \beta\left(\alpha^2 - \varepsilon\right)\right] \varphi(\rho) = 0. \tag{13}$$

Next, let us set

$$\varphi(\rho) = \cos^{\nu}\left(\sqrt{\beta}\rho\right) f(\rho), \tag{14}$$

then equation (13) becomes

$$\left[\frac{\partial^2}{\partial \rho^2} - 2\nu\sqrt{\beta}\tan\left(\sqrt{\beta}\rho\right)\frac{\partial}{\partial \rho} + \beta\left[\nu(\nu-1) - \delta\right]\tan^2\left(\sqrt{\beta}\rho\right) + \beta\left(\alpha^2 - \varepsilon - \nu\right)\right] f(\rho) = 0. \tag{15}$$

We choose ν to cancel the tangent squared term:

$$\nu(\nu - 1) - \delta = 0. \tag{16}$$

Since the wave function should be non singular at $\cos\left(\sqrt{\beta}\rho\right) = 0$, we should take

$$\nu = \frac{1}{2} + \sqrt{\frac{1}{4} + \delta}. \tag{17}$$

We will show in a later discussion that the second value of ν does not lead to a physical bound state wave function. Then equation (15) reduces to

$$\left[\frac{\partial^2}{\partial \rho^2} - 2\nu\sqrt{\beta}\tan\left(\sqrt{\beta}\rho\right)\frac{\partial}{\partial \rho} + \beta\left(\alpha^2 - \varepsilon - \nu\right)\right] f(\rho) = 0. \tag{18}$$

At this stage, we introduce another change of variable defined by

$$z = \frac{1}{2}\left[1 - \sin\left(\sqrt{\beta}\rho\right)\right], \tag{19}$$

where the range of the new variable is $-1 < z < 1$. This casts Eq. (18) into the form

$$\left\{z(z-1)\frac{\partial^2}{\partial z^2} + \left[\nu + \frac{1}{2} - (2\nu+1)z\right]\frac{\partial}{\partial z} + \alpha^2 - \varepsilon - \nu\right\} f(z) = 0. \tag{20}$$

This is just a hypergeometric equation, and its solution is the hypergeometric functions

$$f(z) = A\mathrm{F}(a, b; c; z), \tag{21}$$

where A is a normalizing factor and parameters a, b, c are given by

$$a = \nu - \sqrt{\delta + \alpha^2 - \varepsilon}, \qquad b = \nu + \sqrt{\delta + \alpha^2 - \varepsilon}, \qquad c = \nu + \frac{1}{2}.$$

For bound states, we need polynomial solutions to $f(z)$ in order to keep the wave function regular at $z = 0, 1$. Since the hypergeometric function $\mathrm{F}(a, b; c; z)$ reduces to a polynomial of degree n in z when either a or b equals a negative integer, one must impose the restriction

$$\nu - \sqrt{\delta + \alpha^2 - \varepsilon} = -n, \quad n = 0, 1, 2 \ldots \tag{22}$$

from which follows the energy equation:

$$\lambda E_n + kmc^2 = \pm\hbar\sqrt{\beta}\left|\lambda^2 - k^2\right|\left[n^2 + (2n+1)\nu\right]^{1/2}, \quad n = 0, 1, 2 \ldots . \tag{23}$$

Using the expressions of ν and δ, we obtain

$$\lambda E_n + kmc^2 = \pm\hbar\left|\lambda^2 - k^2\right|\left[\beta\left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\frac{\beta^2}{4} + \frac{c^2}{\hbar^2\left(\lambda^2 - k^2\right)}}\right]^{1/2}, \quad n = 0, 1, 2 \ldots . \tag{24}$$

We see that the energy spectrum depends on the deformation parameter β; this is not surprising, since the introduction of the modified Heisenberg algebra affects physical results.

We also note that the energy levels depend on the square of the quantum number n, which is a feature of hard confinement. This can be explained by the fact that the change of variable from p to ρ maps our original problem to a $\tan^2\left(\sqrt{\beta}\rho\right)$ potential problem. For higher energy levels the latter is in essence a square well problem (the boundaries of the well are $\pm\pi/(2\sqrt{\beta})$). This leads to the n^2 dependence of the energy eigenvalues.

Another property of the energy eigenvalues is the constant energy level spacing for large n:

$$\lim_{n\to\infty}\left|\Delta E_n\right| = \hbar\sqrt{\beta}\left|\frac{\lambda^2 - k^2}{\lambda}\right|. \tag{25}$$

Hence our problem can be described for large n by a non-relativistic harmonic oscillator with frequency $\omega = \sqrt{\beta}\left|(\lambda^2 - k^2)/\lambda\right|$.

We end our comments by noting that in the limit $\beta = 0$, the expression of the energy spectrum reduces to

$$\lambda E_n + kmc^2 = \pm\left(\lambda^2 - k^2\right)^{3/4}\sqrt{(2n+1)\hbar c}, \tag{26}$$

which coincides exactly the result obtained in the case where the minimal length is absent.[4]

Let us now write the complete expression of the normalized bound state wave functions. These should satisfy the following normalization condition[16]

$$\int \frac{\mathrm{d}p}{(1+\beta p^2)}\left|\Psi(p)\right|^2 = 1. \tag{27}$$

Expressing $f(z)$ in terms of Gengenbauer's polynomials as

$$f(z) = A\frac{\Gamma(2\nu)}{n!\Gamma(n+2\nu)}C_n^{\nu}(1-2z) \tag{28}$$

and using the following integral property[32]

$$\int_{-1}^{1}\mathrm{d}u\left(1-u^2\right)^{\alpha-1/2}\left[C_n^{\alpha}(u)\right]^2 = \frac{\pi 2^{1-2\alpha}\Gamma(2\alpha+n)}{n!(\alpha+n)\left[\Gamma(\alpha)\right]^2}, \tag{29}$$

we obtain the following expression for the normalized energy wave functions

$$\Psi_n(p) = \frac{2^{\nu}\beta^{1/4}}{\sqrt{2\pi}}\left[\frac{n!(\nu+n)\left[\Gamma(\nu)\right]^2}{\Gamma(2\nu+n)}\right]^{1/2} \times\left(1+\beta p^2\right)^{-\nu/2}C_n^{\nu}\left(\frac{\sqrt{\beta}p}{\sqrt{1+\beta p^2}}\right). \tag{30}$$

Let us examine the necessary condition on parameter ν in order to have a physical bound states. According to Ref. [16] the wave function should be in the domain of p to be physically acceptable. This physically means that it should have a finite uncertainty in momentum

$$\left\langle p^2\right\rangle = \int\frac{\mathrm{d}p}{(1+\beta p^2)}p^2\left|\Psi_n(p)\right|^2 < \infty. \tag{31}$$

It is easy to check that the integrand in Eq. (31) behaves for $p\to\infty$ as $p^{-2\nu}$. Since for $p\to 0$ the integrand behaves as p^2, the convergence of the integral (31) will be ensured provided $\nu > \frac{1}{2}$. The second solution of Eq. (16) does not satisfy this condition therefore it was ignored.

From the previous discussion we can see that the existence of physical bound states remains possible as long as the following constraint is fulfilled

$$(\Delta x)_{\min}^4 > \frac{4c^2\hbar^2}{k^2 - \lambda^2}. \tag{32}$$

This means that confinement solutions can occur even when the electric field is stronger than the scalar potential (which must be non null) provided that inequality (32) is respected. However, this conclusion remains purely mathematical since in real situations condition (32) is far from being realized. Indeed, the

largest upper bound of the minimal length found in the literature is[24,29]

$$(\Delta x)_{\min} \leq 1 \text{ fm}. \tag{33}$$

In the case when condition (32) is not satisfied, parameter ν becomes complex with Re $(\nu) = \frac{1}{2}$; the integrand in Eq. (31) behaves as $1/p$ at momentum infinity, leading to a logarithmic divergence of integral (31). Hence the particles cannot be bounded in this case. Only scattering solutions occur and the whole spectrum is continuous.

4. Conclusion

In this paper, using the momentum space representation, we have solved exactly the Klein–Gordon equation with linear scalar-like and vector-like potentials, in the framework of quantum mechanics with generalized uncertainty relation implying the existence of a minimal length. The bound states energy spectrum and the corresponding normalized wave functions were exactly obtained. In contrast to the usual case the energy eigenvalues show an n^2 dependence, this is due to the fact that our initial problem is mapped via an appropriate change of variable to that of a particle in a well potential whose corresponding energy levels are n^2-dependent.

Given the interesting role of the linear potential in nuclear and hadron physics, we expect that the results exposed in this work may have relevant applications, especially in the quarkonium phenomenology.

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