# Resonant tunneling and quantum fluctuation in a driven triple-well system*

Lu Wei-Tao(卢伟涛)[a], Wang Shun-Jin(王顺金)[a]†, and Zhang Hua(张 华)[b]

[a] *Department of Physics, Sichuan University, Chengdu* 610064, *China*
[b] *Institute of Applied Physics and Computational Mathematics, Beijing* 100088, *China*

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The influence of parameters such as the strength and frequency of a periodic driving force on the tunneling dynamics is investigated in a symmetric triple-well potential. It is shown that for some special values of the parameters, tunneling could be enhanced considerably or suppressed completely. Quantum fluctuation during the tunneling is discussed as well and the numerical results are presented and analysed by virtue of Floquet formalism.

**Keywords:** triple-well potential, non-resonant tunneling, resonant tunneling, Floquet theorem, quantum fluctuation

**PACC:** 0365, 7335C, 3280B

## 1. Introduction

The control of quantum tunneling by external driving force is an important subject in many areas of physics,[1,2] e.g., atomic and molecular physics, chemical physics, solid-state physics, and Bose–Einstein condensates. Depending on the strength and frequency of the driving force, suppression or enhancement of tunneling can be achieved. Lin and Ballentine found that tunneling enhancement could be obtained in a driven double-well potential for the driving frequencies close to the harmonic frequency at the bottom of each well and with relatively high strength, and this phenomenon is often referred to as chaos-assisted tunneling.[3−5] Classical mechanical analysis of the motion shows that structures in the classical phase space determines its tunneling behaviour. Hanggi and his coworkers discovered an opposite situation that for certain parameter ratios between the strength and frequency of the driving force, tunneling can be suppressed completely and this phenomenon is termed coherent destruction of tunneling (CDT).[6,7] In the latter case, the value of strength is rather small compared to that required in Lin's case.

For about two decades the tunneling dynamics in a driven symmetric or asymmetric double-well system have been studied widely.[1−10] Recently, tunneling dynamics in triple-well potential have attracted substantial attention,[11−17] which has been practically realized in semiconductor superlattices with different compound compositions, and applied in quantum transport and electronic devices related to lasers. The Hamiltonian of unperturbed symmetric triple-well potential could be expressed as: $H(x,p) = p^2/2m + \frac{1}{2}m\omega_0^2x^2(x^2-a^2)^2$, having three equally deep wells (left, central, and right) located at $x = -a, 0, a$ respectively, and the two potential barriers are located at $x = -a/\sqrt{3}, a/\sqrt{3}$ respectively, where $m$ denotes the mass of the particle. We introduce the dimensionless variable for convenience in notation: $(m/\hbar)^{1/4}x \to x$, and thus the Hamiltonian takes the form: $H(x) = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\omega_0^2x^2(x^2-a^2)^2$. Besides the obvious difference in the number of the potential wells, the novel feature of the above potential compared to double-well potential is that the harmonic frequency of the central well ($a^2\omega_0$) is different from that of both the left and right wells ($2a^2\omega_0$). The ground state of the triple-well potential is approximately the ground state of the harmonic oscillator localized at the central well, while the first and second excited states are linear combinations of the ground states of harmonic oscillators localized at the left and right wells, having odd and even parity respectively (as shown in Fig. 1). Therefore, some remarkable characters of quantum tunneling in triple-well potential are expected.

In the present paper, we conduct our investigation to the tunneling dynamics in a time-dependent driving triple-well potential, and a numerical method is adopted due to the reason that no analytical solution exists in such a complicated driven system. It is shown

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†Corresponding author. E-mail: sjwang@home.swjtu.edu.cn
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that in the non-resonant case, tunneling displays a regular monochromatic oscillation and only transits forth and back between the left and right wells. In the resonant case, the tunneling is enhanced rapidly and presents a complex beat pattern, situated in the central well. For special parameter values of driving force the tunneling could be suppressed completely and lead to CDT. Quantum fluctuation during the tunneling is studied as well and it is found that the tunneling process is accompanied by large quantum fluctuation. These characters are also analysed in the light of the Floquet theorem. It should be noted that the parameters are set to be suitable values, so that the quantum tunneling can take place even in the classical forbidden region. The organization of this paper is outlined as follows: Section 2 presents the driven triple-well potential and its feature. In Section 3, we present numerical results on the transition probability and quantum fluctuation for an initially localized wave packet. Section 4 gives the analysis of the results based on Floquet theorem. Summary and conclusions are contained in Section 5.

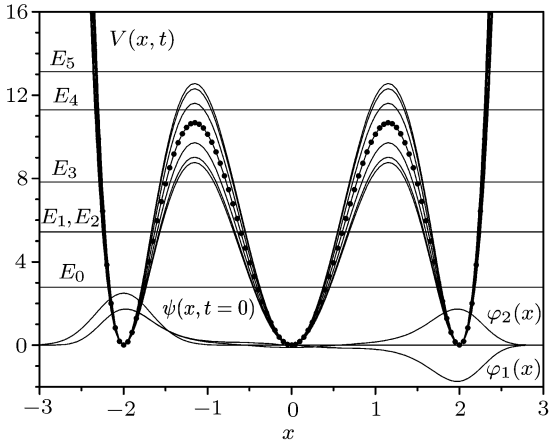



**Fig. 1.** Sketch of the driven triple-well potential $V(x,t) = (\omega_0^2/2 + S\sin(\omega t))x^2(x^2-a^2)^2$, where $S = 0.2$, and the unperturbed potential is denoted by dot curve. The horizontal lines denote some lowest energy levels in absence of driving ($S = 0$): $E_0 = 2.78738$, $E_1 = 5.42904$, $E_2 = 5.44082$, $E_3 = 7.83425$, $E_4 = 11.30215$, $E_5 = 13.12614$, and that $E_1$ and $E_2$ consist of doublet. $\varphi_1(x)$ and $\varphi_2(x)$ are eigenfunctions of $E_1$ and $E_2$ respectively, having opposite parities. The initial state $\psi(x,t=0)$ is located at the bottom of the left well.

## 2. The driven triple-well potential

The dimensionless Hamiltonian of the periodically driven triple-well system under investigation has the form[11]

$$H(x,t) = T(x) + V(x,t), \tag{1}$$

where $T(x) = -\frac{1}{2}\partial_x^2$ and $V(x,t) = (\omega_0^2/2 + S\sin(\omega t))x^2(x^2-a^2)^2$, and that the dimensionless unit of external force strength is $\frac{S}{m} \to S$. The well-depths or barrier heights oscillate periodically without changing the positions of minima and maxima, and the dressed potential is shown in Fig. 1, where the unperturbed potential is displayed by dot curve. The influence of such a driving force on the tunneling dynamics and controllability of wave packet dynamics in a double-well potential was studied numerically by Igarashi and Yamada,[18] while the tunneling feature in the double-well potential with an additive form force has been investigated thoroughly.[1,3,6,7] In concrete, the parameters are set as $\omega_0 = 1.5, a = 2.0$, and several lowest energy levels $E_n$ of the unperturbed system below the barrier are presented (see Fig. 1), where the first excited state $(\varphi_1, E_1)$ and the second excited state $(\varphi_2, E_2)$ having opposite parities are close to each other but not degenerate in energy. In general, the eigenfunction $\varphi_{2n}(x)$ of $E_{2n}$ has even parity, while the eigenfunction $\varphi_{2n-1}(x)$ of $E_{2n-1}$ possesses odd parity ($n$ is integer). The values of frequency $\omega$ and strength $S$ of the driving force in the following numerical experiment are chosen suitably such that the energy of the system is always below the barrier during the time evolution.

Then, we assume a Gaussian wave packet as the initial state: $\psi(x, t=0) = N\exp[-(x+a)^2/2\sigma]$, located at the bottom of the left well (see Fig. 1). $N$ and $\sigma$ are normalization factor and spread of the initial wave packet respectively, and $N = 1.24$, $\sigma = 0.14$. For the solution $\psi(x,t)$ of the time-dependent Schrödinger equation, we use the second-order symplectic integrator by means of fast Fourier transformation[19]

$$\psi(t+\Delta t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{\Delta t}{2}V(t+\Delta t)\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\Delta t T\right) \times \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{\Delta t}{2}V(t)\right)\psi(t) + O(\Delta t^3), \tag{2}$$

the spatial range is $[-4.0,\ 4.0]$ with the spatial mesh width $\Delta x = 8.0/2^7 = 0.0625$, while the time step is $\Delta t = 0.01$. The occupation probabilities in the three wells (left, central, and right) are defined as: $P_L(t) = \int_{-\infty}^{-a/\sqrt{3}} |\psi(x,t)|^2 \mathrm{d}x$, $P_C(t) = \int_{-a/\sqrt{3}}^{a/\sqrt{3}} |\psi(x,t)|^2 \mathrm{d}x$, and $P_R(t) = \int_{a/\sqrt{3}}^{\infty} |\psi(x,t)|^2 \mathrm{d}x$, respectively, which give the probability of finding the wave packet in each well changing with time.

## 3. Numerical results

In the following, the effects of the external strength $S$ and frequency $\omega$ on tunneling dynamics will be studied numerically. Let the initial state $\psi(x, t = 0)$ located in the left well evolve, and it may get delocalized into the central and right wells. Figure 2(a) shows the occupation probabilities $P_L(t)$, $P_C(t)$, and $P_R(t)$ as functions of time in the unperturbed case ($S = 0$). One may find that the wave packet just tunnels between the left and right wells, and there is hardly any probability in the central well, namely it does not occur in the central well since the state of the system tries to have a good quantum number parity. The energy remains conserved and below the top of barrier during tunneling. From the figure it can be seen that the tunneling period (tunneling period used here in its narrow sense is the time of tunneling out of the left well for the wave packet) is about 267, which is approximately equal to the exact tunneling period $\tau = \pi/(E_2 - E_1)$, testifying the validity of the numerical computation. It is worth while also to note that the wave packet will never appear in the left or right well if the initial packet is located in the central well.

Figures 2(b)–2(d) show the occupation probabilities for various parameter values $(S, \omega)$, including non-resonant tunneling and resonant tunneling. In Figs. 2(b) and 2(d), at $\omega = 1.5$ the tunneling just occurs between the left and right wells, displaying a regular sine-oscillation, and its rate increases almost linearly with the strength $S$. When the driving frequency value approaches the intrinsic frequency $(E_n - E_m)$ of the unperturbed system, the tunneling displays some unexpected features. When $\omega = 2.4$ approximately equaling to the fundamental resonance $(E_3 - E_1)$ as shown in Fig. 2(c), the tunneling rate increases rapidly compared to non-resonant tunneling and apparently displays chaotic oscillation. Remarkably, the resonant tunneling arises in the central well and presents a regular oscillation (see Fig. 2(d)). However, in the double-well potential, the fundamental resonance is $(E_3 - E_2)$ instead.[1,6] In the non-resonant case, the time evolution of the wave packet mainly depends on the first excited state $\psi_1(x, t)$ and the second excited state $\psi_2(x, t)$ of the perturbed system $H(x, t)$, and both locate in the left and right wells (the so-called two-state approximation). Therefore, the tunneling presents a regular oscillation and only occurs in the left and right wells. In the resonant case, the evolution of the wave packet consists of not only $\psi_1(x, t)$ and $\psi_2(x, t)$, but also the third assisted state $\psi_3(x, t)$ mostly situated in the central well (the so-called three-state approximation). Thus, the tunneling presents a chaotic oscillation and arises in the central well. Detailed discussion on this will be given later.

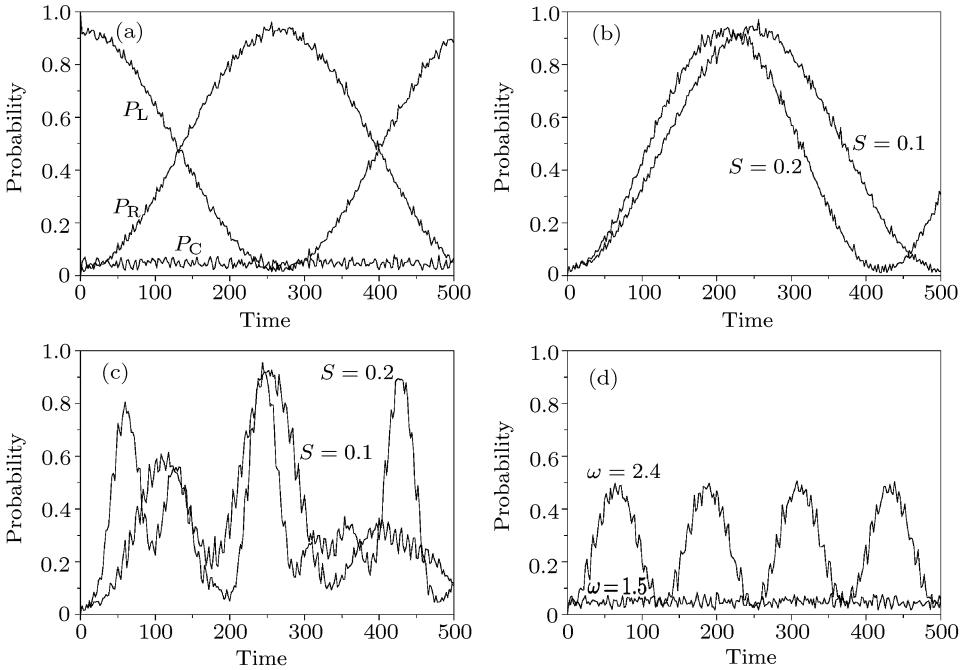



**Fig. 2.** Quantum tunneling in the driven triple-well potential: (a) Occupation probabilities versus time in the absence of driving; (b)–(d) Occupation probabilities versus time in the presence of driving. Occupation probability $P_R(t)$ versus time for various $S$ at (b) $\omega = 1.5$ and (c) $\omega = 2.4$. (d) Occupation probability $P_C(t)$ versus time in non-resonant and resonant cases, and $S = 0.1$.

Figure 3 displays the dependence of tunneling period on $S$ and $\omega$. Generally, tunneling period is seen to decrease reposefully with increase of $S$ for both non-resonant and resonant tunnelings (Fig. 3(a)). This is a behaviour of chaos-assisted tunneling, where its classical phase space becomes more and more chaotic with increase of $S$. From Fig. 3(b), one finds that tunneling period varies slowly as $\omega$ far away from $E_3 - E_1$, but varies rapidly and suddenly as $\omega$ is close to $E_3 - E_1$. There is a notable characteristic that tunneling period changes rapidly from the valley to the peak, which implies the enhancement and suppression of tunneling. The main mechanism for this phenomenon is that $\psi_1(x,t)$ will transit to $\psi_3(x,t)$ in short time at $\omega = 2.4$ and causes rapid tunneling. Of course, in the vicinity of 2.4, the tunneling is determined primarily by $\psi_1(x,t)$ and $\psi_2(x,t)$, and secondarily by $\psi_3(x,t)$, and so the tunneling period could be expressed approximately as a function of the quasienergies $\epsilon_1$ and $\epsilon_2$ as shown in the following. Then for some appropriate values of parameters $(S,\omega)$, the quasienergy difference $|\epsilon_2 - \epsilon_1|$ will be very small, leading to delayed tunneling. Such as $(0.105, 2.45)$ and $(0.218, 2.8)$, it will bring an exact level crossing of $\epsilon_1$ and $\epsilon_2$, namely $\epsilon_1 = \epsilon_2$, which leads to a complete suppression of tunneling. As shown in Figs. 4(a) and 4(b), the wave packet localized in the left well leads to CDT (see Fig. 4(a)), or tunnels between the left and central wells, which manifests the role of $\psi_3(x,t)$ in the tunneling (see Fig. 4(b)).

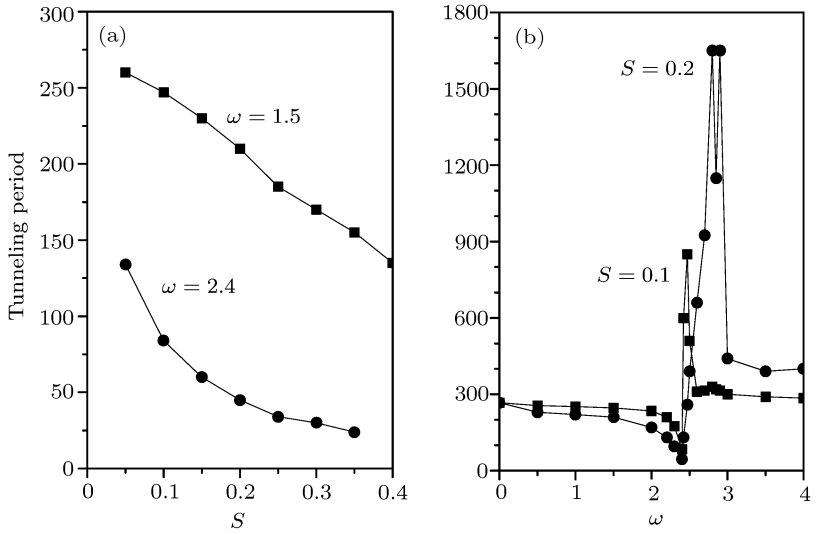



**Fig. 3.** The dependence of tunneling period on $S$ and $\omega$. (a) Tunneling period as a function of $S$ at $\omega = 1.5$ and 2.4. (b) Tunneling period as a function of $\omega$ at $S = 0.1$ and 0.2.

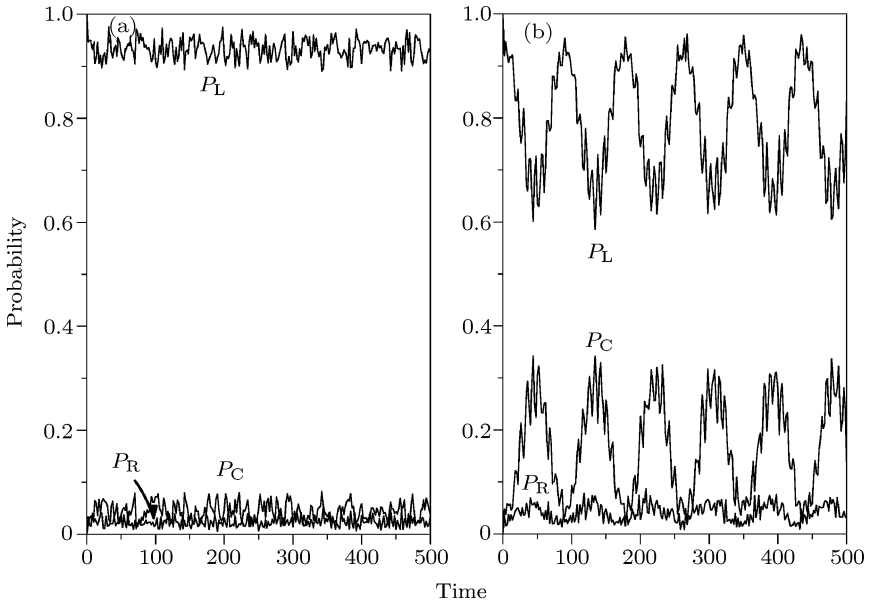



**Fig. 4.** Suppression of quantum tunneling at an exact crossing of the quasienergies $\epsilon_1$ and $\epsilon_2$. At (a) $S = 0.218, \omega = 2.8$ and (b) $S = 0.105, \omega = 2.45$.

The above results imply that different frequency and strength of the two barriers in triple-well could result in different tunneling phenomena, and the non-resonant tunneling is in good agreement with coherent tunneling by adiabatic passage in triple-well optical waveguides.[13] Similar to that in driven double-well system, the tunneling period in this system is determined by the quasienergy splitting $\Delta = \epsilon_2 - \epsilon_1$ of a pair of odd and even Floquet states located in right and left wells. The evolution of tunneling wave-packet could be described as a linear combination of two Floquet states ($\phi_1$, $\phi_2$) in the non-resonant case, or three Floquet states ($\phi_1$, $\phi_2$, $\phi_3$) in the resonant case. The interaction of these states can affect their quasienergies causing either the degeneracy of the two states $\phi_1$ and $\phi_2$ or the increase in quasienergy splitting, which means that the tunneling is suppressed or enhanced respectively. Compared with those in double-well system, the energy resonant tunneling and chaos-assisted tunneling present more complicated phenomena in triple-well system, due to the third state. Moreover, with the increasing of strength, more states will join in the evolution of tunneling wave-packet, and the tunneling will become more complicated.

Now let us discuss quantum fluctuation of the wave packet during non-resonant and resonant tunnelings. Quantum fluctuation can be measured by the uncertainty product: $\Delta q \Delta p \equiv \sqrt{\langle (q - \langle q \rangle)^2 \rangle} \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$. For the initial Gaussian wave packet its initial value reaches the minimum $\Delta q \Delta p = \hbar/2 = 0.5$. Figure 5 shows the uncertainty product for various tunnelings. It is found that in non-resonant case the quantum fluctuation presents a regular oscillation of coherent fluctuation, and the increase of the strength of external force does not destroy the oscillation but enhances the frequency of the fluctuation (Fig. 5(a)). On the other hand, quantum fluctuation displays a chaotic oscillation in resonant case (Fig. 5(b)). In both cases the quantum fluctuation fluctuates around an average level ($\sim 3.0$). As compared with the time evolution of the occupation probability $P_{\rm R}(t)$ in Fig. 2, in the non-resonant case, quantum fluctuation is relatively large in the transition period of tunneling, and reduces to its minimum as the packet entirely tunnels into one well. This is the salient feature of regular coherent tunneling caused by the following physics. In the regular tunneling, the perturbation provides a mechanism to force the system to transit from one stable state to another stable state with different quantum numbers (different parities in our case). And during the transition period, the system is changing its quantum numbers and thus very unstable such that the transition is realized and characterized by large quantum fluctuation. Then as the system enters into one well, it acquires its good quantum numbers again and reaches another stable state characterized by small quantum fluctuation.

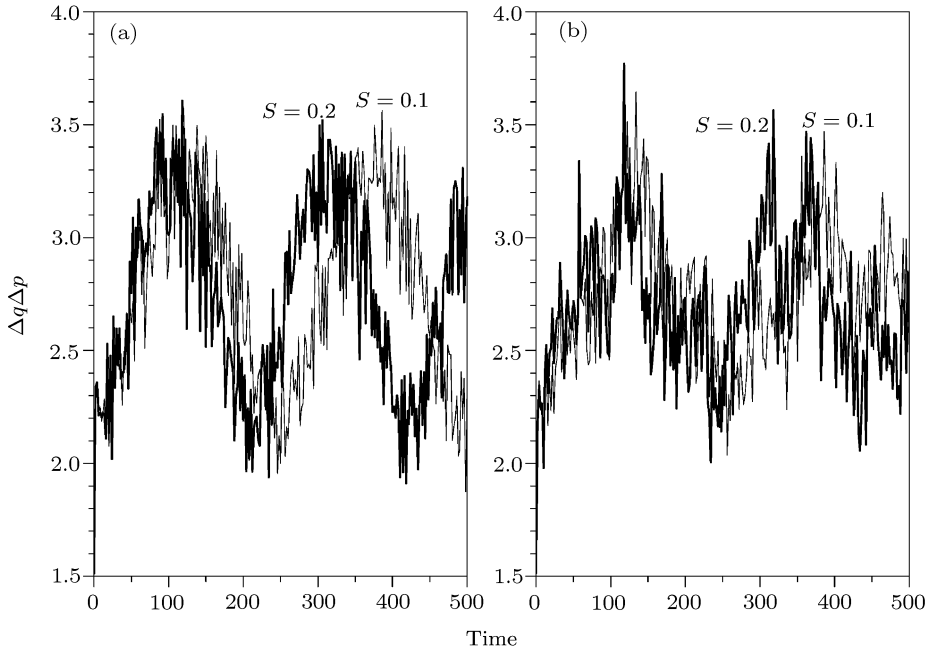



**Fig. 5.** Uncertainty product $\Delta q \Delta p$ versus time for various strength $S$ at (a) $\omega = 1.5$ and (b) $\omega = 2.4$.

However, the resonant chaotic tunneling is quite different where the perturbation destroys all the good quantum numbers and forces the system to tunnel among all possible eigenfunctions with different quantum numbers. Consequently, under the action of the perturbation the system is frequently changing its quantum numbers and always in unstable states, so that the frequent transition is realized and characterized by large chaotic quantum fluctuation.

## 4. Analysis based on Floquet theorem

Here, a physical analysis for the above phenomenon is given based on Floquet theorem.[1,20] Let $\varphi_1(x)$ and $\varphi_2(x)$ denote the eigenfunctions with eigenenergies $E_1$ and $E_2$ of the undriven triple-well system ($S = 0$), which have odd and even parities respectively. The combinations $\varphi_\pm(x) = (\varphi_1(x) \pm \varphi_2(x))/\sqrt{2}$ are functions localized in the left or right well (see Fig. 1). In fact, the initial state in the above computation is approximately equal to one combination of $\varphi_1(x)$ and $\varphi_2(x)$, i.e., $\psi(x, t = 0) \approx \varphi_+$. In the undriven case, the wave packet evolves in time as:

$$\psi(x,t) \approx \frac{1}{\sqrt{2}} \mathrm{e}^{-\mathrm{i}\,E_1 t}(\varphi_1(x) + \varphi_2(x)\mathrm{e}^{-\mathrm{i}(E_2-E_1)t}), \tag{3}$$

and after the tunneling period: $\tau = \pi/(E_2 - E_1)$, the two components of the wave function have acquired a relative phase factor of $-1$, which makes the packet initially localized in the left well ($\varphi_+$) tunnel into the right well ($\varphi_-$). In the driven case, the Hamiltonian is a periodic function of time: $H(x, p, t) = H(x, p, t + T)$, where $T = 2\pi/\omega$. According to the Floquet theorem, there exists a solution of this time-dependent Schrödinger equation with the form: $\psi_n(x, t) = \exp(-\mathrm{i}\,\epsilon_n t)\phi_n(x, t)$, where $\phi_n(x, t)$ is Floquet state obeying $\phi_n(x, t) = \phi_n(x, t + T)$, and $\epsilon_n$ is its quasienergy being a function of $S$ and $\omega$. Thus, under the action of driving force, $\varphi_1(x)$ and $\varphi_2(x)$ will evolve into Floquet states $\phi_1(x, t)$ and $\phi_2(x, t)$, respectively, with similar shape and the same parity as their unperturbed counterparts. Then $\psi(x, t)$ could be expressed as:

$$\psi(x,t) \approx \frac{1}{\sqrt{2}} \mathrm{e}^{-\mathrm{i}\,\epsilon_1 t}(\phi_1(x,t) + \phi_2(x,t)\mathrm{e}^{-\mathrm{i}(\epsilon_2-\epsilon_1)t}), \tag{4}$$

and similar to the unperturbed case the tunneling period is:

$$\tau = \pi/[\epsilon_2(S,\omega) - \epsilon_1(S,\omega)]\,.$$

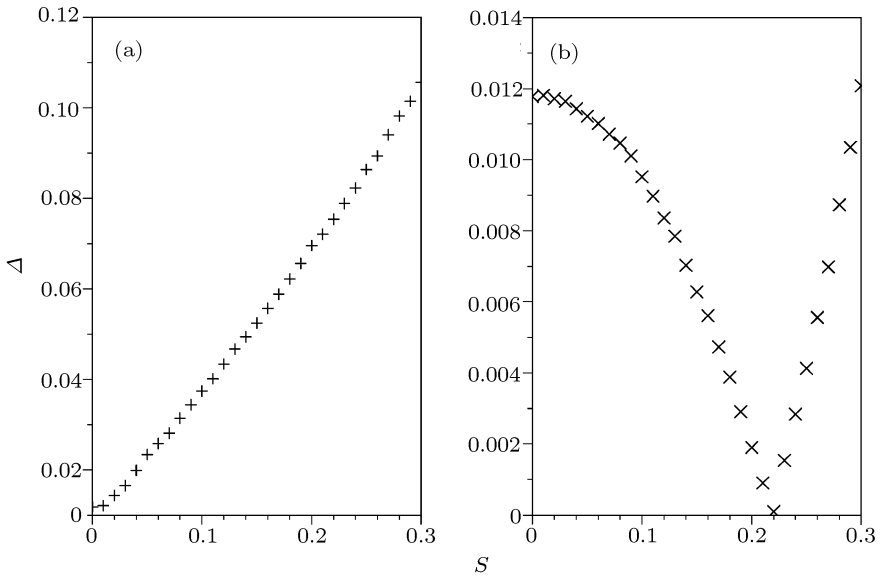



**Fig. 6.** The quasienergy splitting $\Delta = \epsilon_2 - \epsilon_1$ versus external strength $S$, at (a) $\omega = 2.4$ and (b) $\omega = 2.8$.

Therefore, the tunneling dynamics in driven triple-well potential strongly depends on the strength and frequency of the driving force. For instance, at frequency $\omega$ near the fundamental resonance $E_3 - E_1$, the quasienergies $\epsilon_1$ and $\epsilon_3 - \hbar\omega$ exhibiting an exact level crossing at $S = 0$, will become an avoided crossing for $S > 0$, since the corresponding Floquet states have equal parity, and thus the tunneling is expected to be resonant and largely enhanced. As shown in Fig. 6(a), the quasienergy splitting is enhanced as strength increases at $\omega = 2.4$, and the tunneling rate is about one order of magnitude faster than that of ordinary tunneling. Such tunnelings also occur at $\omega = E_4 - E_2$, $E_5 - E_1, \ldots$, where $E_4$ and $E_2$, $E_5$ and $E_1$ hold the same parities respectively. The above case was demonstrated for a two-waveguide optical system by Vorobeichik *et al.*[5] Conversely, CDT occurs approximately for a modulation frequency $\omega$ between two resonant frequencies and a modulation amplitude $S$ corresponding to exact level crossing between the quasienergies $\epsilon_1$ and $\epsilon_2$ associated to the lowest doublet. From Fig. 6(b), it could be found that there is a level crossing between $\epsilon_1$ and $\epsilon_2$ at $\omega = 2.8$, implying that $\phi_1(x,t)$ and $\phi_2(x,t)$ become degeneracy, and tunneling is suppressed completely, as shown in Fig. 4(a). The conditions of CDT in a driven double-well system was investigated clearly by Valle *et al.*[7] and the cause of chaotic oscillation in the resonant tunneling was also studied in Ref. [6]. The probability to stay in the initial state after period of time $nT$ is given by $P = |\langle\psi(nT)|\psi(0)\rangle|^2$, and its expansion in the Floquet basis indicates that chaotic oscillation is mainly composed of a few frequencies associated with transitions among Floquet states at the avoided crossing pertaining to the lowest doublet. Obviously, from Figs. 2 and 4, one may find many small fluctuations during the tunneling and localization of the wave packet. This is due to the "bad chosen" of initial state which is not the exact combination of two lowest excited states, but also contains some higher Floquet states playing a small but non-negligible role in the tunneling.

## 5. Conclusion

We numerically studied the tunneling of a wave packet in a one-dimensional time-dependent driven triple-well potential. It is found that non-resonant tunneling displays a monochromatic oscillation whose behaviour is similar to the unperturbed case, and that the tunneling only takes place between left and right wells. However, resonant tunneling is enhanced rapidly and presents a polychromatic oscillation, and it arises in the central well. For special values of parameters $(S, \omega)$ the wave packet could be localized in one well. Finally, the numerical results are explained based on the Floquet theorem.

Indeed, the reported results have some relation to the stimulated Raman adiabatic pumping, which has been applied to spatial coherence transport in triple-well termed as coherent tunneling via adiabatic passage,[13,16,17] and Opatrny discussed the conditions for the vanishing central-well population in triple-well adiabatic transport.[16] Recently, various robust schemes for the coherent spatial transport of physical particles in a variety of systems have been proposed. The results here can find applications in the coherent transport of atoms in optical traps or electrons in quantum dots, even in other physics fields. They can also serve as a bridge from the double-well system to the multi-well system, to help us understand the tunneling phenomenon profoundly, and more interesting investigations on the triple-well system are expected. This paper presents a preliminary study on the tunneling behaviour in the triple-well potential, and more detailed results will be reported in the near future.

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