# New equation for deriving pure state density operators by Weyl correspondence and Wigner operator*

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By virtue of the completeness of Wigner operator and Weyl correspondence we construct a general equation for deriving pure state density operators. Several important examples are considered as the applications of this equation, which shows that our approach is effective and convenient for deducing these entangled state representations.

**Keywords:** density operator, Weyl ordering, Wigner operator, quantum mechanical eigenstate
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## 1. Introduction

Density operator is an important conception in quantum mechanics, quantum optics[1−6] and quantum information.[7−13] Pure state density operator $\rho$ is usually expressed as $\rho = |\Psi\rangle\langle\Psi|$, whose average value is always positive-definite. In this work we shall construct a new equation as follows:

$$\frac{1}{\pi}\int \mathrm{d}p'\,\mathrm{d}q'\delta\left[\alpha - A(q',p')\right]\Delta(q',p') = |\alpha\rangle\langle\alpha|, \quad (1)$$

where $A(q',p')$ is the Weyl correspondence of Weyl ordered operator $\vdots\ A(\hat{Q},\hat{P})\ \vdots$, $|\alpha\rangle$ is the eigenvector of $\vdots\ A(\hat{Q},\hat{P})\ \vdots$ with the eigenvalue $\alpha$, $\Delta(q,p)$ is the Wigner operator. Using this equation, we can effectively and conveniently derive some pure state density operators of continuum variables that may span complete quantum mechanical representation, especially, some pure entangled state density operators. We shall employ the Weyl correspondence and the Wigner operator to realize our goal. This paper is organized as follows. In Section 2 we briefly review the recent development of Wigner operator theory with Weyl ordering. In Section 3, with the help of the completeness of Wigner operator and Weyl correspondence, we construct a new equation in Eq. (1) from the single-mode case to the multi-mode case. Several important examples are considered as the applications of this equation in Section 4.

## 2. Brief review of the Wigner operator

The Wigner operator is introduced as[4,14,15]

$$\Delta(q,p) = \frac{1}{2\pi}\int \left|q-\frac{v}{2}\right\rangle\left\langle q+\frac{v}{2}\right| \mathrm{e}^{-\mathrm{i}pv}\,\mathrm{d}v. \quad (2)$$

By virtue of the technique of integration within an ordered product of operator,[16,17] the integration in Eq. (2) can be performed and the result is an explicit operator[18]

$$\Delta(q,p) = \frac{1}{\pi}{:}\ \mathrm{e}^{-(q-\hat{Q})^2-(p-\hat{P})^2}{:}, \quad (3)$$

where : : denotes normal ordering. The Wigner operator serves as an integral kernel of the Weyl rule[19] which is a quantization scheme connecting classical function $h(q,p)$, where $q$, $p$ are coordinate and momentum, with its quantum correspondence operator $H(\hat{Q},\hat{P})$, here $\hat{Q}$ and $\hat{P}$ are coordinate and momentum operators corresponding to $q$ and $p$, respectively. This quantization scheme has the form

$$\int \mathrm{d}q\mathrm{d}p h(q,p)\Delta(q,p) = H(\hat{Q},\hat{P}). \quad (4)$$

On the other hand, due to Eq. (3), using the properties of the Gaussian integration within normal ordering one can perform the integration over $\mathrm{d}p\mathrm{d}q$ as

$$\int \mathrm{d}p\mathrm{d}q\Delta(q,p) = \frac{1}{\pi}\int \mathrm{d}p\mathrm{d}q{:}\ \mathrm{e}^{-(q-\hat{Q})^2-(p-\hat{P})^2}{:} = 1. \quad (5)$$

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Further, there are

$$\int \mathrm{d}p\Delta(q,p) = \frac{1}{\sqrt{\pi}} \colon \exp[(q-\hat{Q})^2] \colon = |q\rangle\langle q|\,, \quad (6)$$

and

$$\int \mathrm{d}q\Delta(q,p) = |p\rangle\langle p|\,, \quad (7)$$

i.e., the one-side integral of the Wigner operator respectively gives the single-particle' position projection operator and the momentum projection operator. In Ref. [20] we have derived the Weyl ordering form of the Wigner operator

$$\Delta(q,p) = \vdots\, \delta(p-\hat{P})\delta(q-\hat{Q}) \,\vdots\,, \quad (8)$$

where the symbol $\vdots\ \vdots$ denotes Weyl ordering, within which $\hat{P}$ and $\hat{Q}$ are commutative.

# 3. Constructing new equation for deriving pure state density operators

## 3.1. Single-mode case

Our departure point is very fundamental, let operator $\hat{A} = A(\hat{P},\hat{Q})$ obey the eigenvector equation

$$\hat{A}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \quad (9)$$

where $\alpha = \alpha(p,q)$ is the eigenvalue of the operator $\hat{A}$, satisfying orthogonal relation $\langle\alpha'|\,\alpha\rangle = \delta(\alpha-\alpha')$. Using the property of Dirac delta function we have

$$|\alpha\rangle = \int \mathrm{d}\alpha'\delta(\alpha-\alpha')\,|\alpha'\rangle = \int \mathrm{d}\alpha'\delta(\alpha-\hat{A})\,|\alpha'\rangle\,. \quad (10)$$

Obviously, the adjoint operator of Eq. (10) is

$$\langle\alpha| = \int \mathrm{d}\alpha'\,\langle\alpha'|\,\delta(\alpha-\hat{A}). \quad (11)$$

Note $\hat{A} = \hat{A}^{\dagger}$. On the other hand, according to Eqs. (5)–(7) we see

$$\hat{Q} = \hat{Q}\int \mathrm{d}p\mathrm{d}q\Delta(q,p) = \int \mathrm{d}qq\,|q\rangle\langle q| = \int \mathrm{d}q\mathrm{d}pq\Delta(q,p), \quad (12)$$

and

$$\hat{P} = \int \mathrm{d}p\mathrm{d}qp\Delta(q,p). \quad (13)$$

When $A(\hat{P},\hat{Q})$ operator is in Weyl ordering, $A(\hat{P},\hat{Q}) = \vdots\, A(\hat{P},\hat{Q}) \,\vdots$, based on Eq. (8), equations (12) and (13) can be generalized as

$$A(\hat{P},\hat{Q}) = \vdots \int \mathrm{d}p\mathrm{d}qA(p,q)\delta(p-\hat{P})\delta(q-\hat{Q}) \,\vdots = \int \mathrm{d}p\mathrm{d}qA(p,q)\Delta(q,p). \quad (14)$$

Using Eqs. (11) and (14)

$$\begin{aligned}\langle\alpha| &= \int \mathrm{d}\alpha'\,\langle\alpha'|\,\delta(\alpha-\hat{A}) \\ &= \int \mathrm{d}\alpha'\,\langle\alpha'|\int \mathrm{d}p'\mathrm{d}q'\delta\left[\alpha-A(q',p')\right]\Delta(q',p') \\ &= \int \mathrm{d}\alpha'\,\langle\alpha'|\,\vdots\,\delta(\alpha-\hat{A})\,\vdots\,. \end{aligned} \quad (15)$$

Due to

$$\int \mathrm{d}\alpha'\,\langle\alpha'|\,\alpha\rangle\,\langle\alpha| = \int \mathrm{d}\alpha'\delta(\alpha-\alpha')\,\langle\alpha| = \langle\alpha|\,, \quad (16)$$

so from Eq. (15) we infer

$$\int \mathrm{d}p'\,\mathrm{d}q'\delta\left[\alpha-A(q',p')\right]\Delta(q',p') = |\alpha\rangle\langle\alpha|\,, \quad (17)$$

where $|\alpha\rangle\langle\alpha|$ is pure state density operator. This is a new equation for constructing the pure state. To further get the explicit form of $|\alpha\rangle\langle\alpha|$, using Eq. (3) we can rewrite Eq. (17) as

$$\frac{1}{\pi}\int \mathrm{d}p'\,\mathrm{d}q'\delta\left[\alpha-A(q',p')\right] \colon \mathrm{e}^{-(q'-\hat{Q})^2-(p'-\hat{P})^2} \colon = |\alpha\rangle\langle\alpha|\,. \quad (18)$$

We can also see that the eigenvector $|\alpha\rangle$ spans a complete set,

$$\begin{aligned}&\int \mathrm{d}\alpha\,|\alpha\rangle\langle\alpha| \\ &= \int \mathrm{d}\alpha\int \mathrm{d}p'\,\mathrm{d}q'\delta\left[\alpha-A(q',p')\right]\Delta(q',p') \\ &= \int \mathrm{d}p'\,\mathrm{d}q'\Delta(q',p') = 1. \end{aligned} \quad (19)$$

## 3.2. Multi-mode case

Supposing there are $n$ Weyl ordered $n$-mode operators

$$\begin{aligned}\hat{A}_i &\equiv A_i(\hat{P}_1,\hat{Q}_1;\hat{P}_2,\hat{Q}_2;\ldots;\hat{P}_n,\hat{Q}_n) \\ &= \vdots\, A_i(\hat{P}_1,\hat{Q}_1;\hat{P}_2,\hat{Q}_2;\ldots;\hat{P}_n,\hat{Q}_n) \,\vdots\,, \end{aligned} \quad (20)$$

and

$$\prod_{i=1}^{n}\hat{A}_i = \vdots \prod_{i=1}^{n}\hat{A}_i \,\vdots,$$

where $i = 1, 2, \ldots, n$, which can be simultaneously measured, i.e., satisfying the commutation relation $[\hat{A}_i, \hat{A}_j] = 0$, $i, j = 1, \ldots n$. Thus operators $\hat{A}_i$s have their common eigenstate, denoted as $|\alpha_1, \alpha_2, \ldots, \alpha_n\rangle$,

$$\hat{A}_i \left|\alpha_1, \alpha_2, \ldots, \alpha_n\right\rangle = \alpha_i \left|\alpha_1, \alpha_2, \ldots, \alpha_n\right\rangle . \tag{21}$$

By virtue of the property of Dirac function we have

$$\begin{aligned} &\left|\alpha_1, \alpha_2, \ldots, \alpha_n\right\rangle \\ &= \int \prod_{i=1}^{n} \mathrm{d}\alpha_i' \delta(\alpha_i - \alpha_i') \left|\alpha_1', \alpha_2', \ldots, \alpha_n'\right\rangle \\ &= \int \prod_{i=1}^{n} \mathrm{d}\alpha_i' \delta(\alpha_i - \hat{A}_i) \left|\alpha_1', \alpha_2', \ldots, \alpha_n'\right\rangle . \end{aligned} \tag{22}$$

On the other hand, the $n$-mode Wigner operator is

$$\begin{aligned} &\Delta_n(q_1, p_1; \ldots; q_n, p_n) \\ &= \prod_{i=1}^{n} \Delta_i(q_i, p_i) \\ &= \frac{1}{\pi^n} \prod_{i=1}^{n} {:}\ \mathrm{e}^{-(q_i - \hat{Q}_i)^2 - (p_i - \hat{P}_i)^2} {:} . \end{aligned} \tag{23}$$

Do the same procedures as in the one-mode case, we can obtain the pure state density operators

$$\begin{aligned} &\left|\alpha_1, \alpha_2, \ldots, \alpha_n\right\rangle \left\langle\alpha_1, \alpha_2, \ldots, \alpha_n\right| \\ &= \int \prod_{i=1}^{n} \mathrm{d}p_i' \mathrm{d}q_i' \delta\left[\alpha_i - A(q_i', p_i')\right] \Delta_n(q_1', p_1'; \ldots; q_n', p_n') \\ &= \frac{1}{\pi^n} \int \prod_{i=1}^{n} \mathrm{d}p_i' \mathrm{d}q_i' \delta\left[\alpha_i - A(q_i', p_i')\right] \\ &\quad \times {:}\ \exp\left[-(q_i' - \hat{Q}_i)^2 - (p_i' - \hat{P}_i)^2\right] {:} , \end{aligned} \tag{24}$$

and we also have

$$\int \prod_{i=1}^{n} \mathrm{d}\alpha_i \left|\alpha_1, \alpha_2, \ldots, \alpha_n\right\rangle \left\langle\alpha_1, \alpha_2, \ldots, \alpha_n\right| = 1. \tag{25}$$

# 4. Application of Eqs. (17) and (24)

## 4.1. Explicit form of the eigenvector of quadrature $\mu\hat{Q} + \nu\hat{P}$

In the quantum optics theory, $\mu\hat{Q} + \nu\hat{P}$ ($\mu$ and $\nu$ are real numbers) represents all possible linear combinations of quadratures $\hat{Q}$ and $\hat{P}$ of the oscillator field and can be measured by the homodyne measurement just by varying the phase of the local oscillator. Obviously,

$$\mu\hat{Q} + \nu\hat{P} = \vdots\ \mu\hat{Q} + \nu\hat{P}\ \vdots \tag{26}$$

is a Weyl ordered operator, so its classical Weyl correspondence is $\mu q + \nu p$, according to Eqs. (3) and (17) we can construct

$$\begin{aligned} &\int \mathrm{d}p' \mathrm{d}q' \delta\left[q - (\mu q' + \nu p')\right] \Delta(q', p') \\ &= \frac{1}{\pi} \int \mathrm{d}p' \mathrm{d}q' \delta\left[q - (\mu q' + \nu p')\right] {:}\ \mathrm{e}^{-(q' - \hat{Q})^2 - (p' - \hat{P})^2} {:} \\ &= \left|x\right\rangle_{\mu,\nu\mu,\nu} \left\langle x\right| , \end{aligned} \tag{27}$$

where $\left|x\right\rangle_{\mu,\nu}$ is the eigenstate of the quadrature $\mu\hat{Q} + \nu\hat{P}$,

$$(\mu\hat{Q} + \nu\hat{P}) \left|x\right\rangle_{\mu,\nu} = x \left|x\right\rangle_{\mu,\nu} . \tag{28}$$

Carrying out the integration in Eq. (27) we can obtain the explicit form of the state vector

$$\begin{aligned} \left|x\right\rangle_{\mu,\nu\mu,\nu} \left\langle x\right| = {} & [\pi(\mu^2 + \nu^2)]^{-1/2} \exp\left[-\frac{x^2}{2(\mu^2 + \nu^2)} + \frac{\sqrt{2}x}{\mu - \mathrm{i}\nu} a^\dagger - \frac{\mu + \mathrm{i}\nu}{2(\mu - \mathrm{i}\nu)} a^{\dagger 2}\right] \left|0\right\rangle \\ & \times \left\langle 0\right| \exp\left[-\frac{x^2}{2(\mu^2 + \nu^2)} + \frac{\sqrt{2}x}{\mu + \mathrm{i}\nu} a - \frac{\mu - \mathrm{i}\nu}{2(\mu + \mathrm{i}\nu)} a^2\right] , \end{aligned} \tag{29}$$

so

$$\left|x\right\rangle_{\mu,\nu} = [\pi(\mu^2 + \nu^2)]^{-1/4} \exp\left[-\frac{x^2}{2(\mu^2 + \nu^2)} + \frac{\sqrt{2}x}{\mu - \mathrm{i}\nu} a^\dagger - \frac{\mu + \mathrm{i}\nu}{2(\mu - \mathrm{i}\nu)} a^{\dagger 2}\right] \left|0\right\rangle , \tag{30}$$

where we have used $:\ \exp(-a^\dagger a) {:}\ = \left|0\right\rangle \left\langle 0\right|$, $\hat{Q} = (a + a^\dagger)/\sqrt{2}$ and $\hat{P} = (a - a^\dagger)/\mathrm{i}\sqrt{2}$.

## 4.2. Common eigenvectors of two particles' relative position and total momentum

Recently, quantum entanglement and entangled states have been paid much attention to because of their potential uses in quantum computation and quantum communication.[21–23] The bipartite entangled state representation is constructed in Refs. [24]–[26], which is enlightened by Einstein–Podolsky–Rosen's argument that

two particles' relative coordinates $\hat{Q}_1-\hat{Q}_2$ and $\hat{P}_1+\hat{P}_2$ are commutable and can be simultaneously measured. It seems that introducing entangled state representation is inevitable since many entangled problems can only be clearly explained by using it.

Having experienced how to find the above eigenstate $|x\rangle_{\mu,\nu}$, we now search for the common eigenvector $|p,q\rangle$ of $\hat{Q}_2+\hat{Q}_1$ and $\hat{P}_1-\hat{P}_2$, i.e.,

$$\begin{aligned}(\hat{Q}_1+\hat{Q}_2)\,|p,q\rangle &= q\,|p,q\rangle\,,\\ (\hat{P}_1-\hat{P}_2)\,|p,q\rangle &= p\,|p,q\rangle\,.\end{aligned} \tag{31}$$

We now search what is the Weyl ordered form of $(\hat{Q}_2+\hat{Q}_1)(\hat{P}_1-\hat{P}_2)$. Recall that Weyl has given the quantization scheme

$$q^m p^r \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\frac{m!}{l!(m-l)!}\hat{Q}^{m-l}\hat{P}^r\hat{Q}^l, \tag{32}$$

the right-hand side is in Weyl ordering, so

$$\left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\frac{m!}{l!(m-l)!}\hat{Q}^{m-l}\hat{P}^r\hat{Q}^l = \vdots\ \hat{Q}^m\ \hat{P}^r\ \vdots\,, \tag{33}$$

for example,

$$\vdots\ \hat{Q}\hat{P}\ \vdots = \frac{1}{2}\left(\hat{Q}\hat{P}+\hat{P}\hat{Q}\right), \tag{34}$$

so

$$\begin{aligned}&(\hat{Q}_2+\hat{Q}_1)(\hat{P}_1-\hat{P}_2)\\ &= \hat{Q}_1\hat{P}_1-\hat{Q}_2\hat{P}_2+\hat{Q}_2\hat{P}_1-\hat{Q}_1\hat{P}_2,\\ &(\hat{P}_1-\hat{P}_2)(\hat{Q}_2+\hat{Q}_1)\\ &= \hat{P}_1\hat{Q}_1-\hat{P}_2\hat{Q}_2+\hat{Q}_2\hat{P}_1-\hat{Q}_1\hat{P}_2.\end{aligned} \tag{35}$$

Adding these two equations and dividing the sum by 2 and noticing $\left[\hat{Q}_1-\hat{Q}_2,\hat{P}_1+\hat{P}_2\right]=0$, we see

$$\begin{aligned}&(\hat{Q}_2+\hat{Q}_1)(\hat{P}_1-\hat{P}_2)\\ &= \frac{\hat{Q}_1\hat{P}_1+\hat{P}_1\hat{Q}_1}{2}-\frac{\hat{Q}_2\hat{P}_2+\hat{P}_2\hat{Q}_2}{2}+\hat{Q}_2\hat{P}_1-\hat{Q}_1\hat{P}_2\\ &= \vdots\,(\hat{Q}_2+\hat{Q}_1)(\hat{P}_1-\hat{P}_2)\,\vdots\,,\end{aligned} \tag{36}$$

i.e., $(\hat{Q}_2+\hat{Q}_1)(\hat{P}_1-\hat{P}_2)$ itself is Weyl ordered, so its classical Weyl correspondence is $(q_1+q_2)(p_1-p_2)$, we can use Eqs. (24) and (23) to construct the equation

$$\begin{aligned}&\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\,\mathrm{d}q_2'\delta\left[q-(q_1'+q_2')\right]\\ &\times\delta\left[p-(p_1'-p_2')\right]\Delta_2(q_1',p_1';q_2',p_2')\\ &= \frac{1}{\pi^2}\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\,\mathrm{d}q_2'\delta\left[q-(q_1'+q_2')\right]\\ &\times\delta\left[p-(p_1'-p_2')\right]:\prod_{i=1}^{2}\mathrm{e}^{-(q_i'-\hat{Q})^2-(p_i'-\hat{P})^2}:\\ &= |p,q\rangle\langle p,q|\,.\end{aligned} \tag{37}$$

After carrying out the integration in Eq. (37) and decomposing the result we find

$$\begin{aligned}|p,q\rangle &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{p^2+q^2}{4}-a_1^\dagger a_2^\dagger+\frac{q+\mathrm{i}p}{\sqrt{2}}a_1^\dagger\right.\\ &\left.+\frac{q-\mathrm{i}p}{\sqrt{2}}a_2^\dagger\right)|00\rangle\,,\end{aligned} \tag{38}$$

here we have used : $\exp(-a_1^\dagger a_1-a_2^\dagger a_2):\ =|00\rangle\langle 00|$, $\hat{Q}_j=(a_j+a_j^\dagger)/\sqrt{2}$ and $\hat{P}_j=(a_j+a_j^\dagger)/\mathrm{i}\sqrt{2}, j=1,2$. Setting $\xi=(q+\mathrm{i}p)/\sqrt{2}$, it is obtained that

$$\begin{aligned}|p,q\rangle &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{|\xi|^2}{2}+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right)|00\rangle\\ &\equiv |\xi\rangle\,,\end{aligned} \tag{39}$$

this is an entangled state, which was proposed before.[18]

### 4.3. Common eigenstates of two particles' center-of-mass coordinates and mass-weighted relative momentum

In this subsection we go a step further to consider the common eigenstate of the center-of-mass coordinate $\hat{Q}_{\mathrm{c.m.}}\equiv\mu_1\hat{Q}_1+\mu_2\hat{Q}_2$ and the mass-weighted relative momentum $\hat{P}_r\equiv\mu_2\hat{P}_1-\mu_1\hat{P}_2$, where

$$\mu_1=\frac{m_1}{m_1+m_2},\quad \mu_2=\frac{m_2}{m_1+m_2} \tag{40}$$

are the reduced masses, and $m_1$ ($m_2$) is the mass of the first (second) particle. In solving dynamical problems it is frequently useful to convert from individual particle coordinates $\hat{Q}_1$ and $\hat{Q}_2$ to centre-of-mass and relative coordinates (correspondingly, the individual particle momenta transform to total momentum and mass-weighted relative momentum)

$$\hat{Q}_{\mathrm{c.m.}}=\mu_1\hat{Q}_1+\mu_2\hat{Q}_2,\qquad \hat{Q}_r=\hat{Q}_2-\hat{Q}_1, \tag{41}$$

and

$$\hat{P}_{\mathrm{c.m.}}=\hat{P}_1+\hat{P}_2,\qquad \hat{P}_r=\mu_2\hat{P}_1-\mu_1\hat{P}_2. \tag{42}$$

Since $[\hat{Q}_{\mathrm{c.m.}},\hat{P}_r]=0$, it is meaningful to construct the simultaneous eigenstates of $\hat{Q}_{\mathrm{c.m.}}$ and $\hat{P}_r$ in terms of conventional creation and annihilation operators. Let $|\chi,p\rangle$ be the common eigenvecter of the operators $\hat{Q}_{\mathrm{c.m.}}$ and $\hat{P}_r$ obeying the eigenstate equations

$$\hat{Q}_{\mathrm{c.m.}}\,|\chi,p\rangle=\chi\,|\chi,p\rangle\,,\quad \hat{P}_r\,|\chi,p\rangle=p\,|\chi,p\rangle\,. \tag{43}$$

Due to

$$\vdots\,(\mu_1\hat{Q}_1+\mu_2\hat{Q}_2)(\mu_2\hat{P}_1-\mu_1\hat{P}_2)\,\vdots$$
$$=(\mu_1\hat{Q}_1+\mu_2\hat{Q}_2)(\mu_2\hat{P}_1-\mu_1\hat{P}_2), \quad (44)$$

its classical Weyl correspondence is $(\mu_1 q_1+\mu_2 q_2)(\mu_2 p_1-\mu_1 p_2)$. With the aid of Eqs. (24) and (23) we now construct the following integration

$$\begin{aligned}&\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\mathrm{d}q_2'\delta\left[\chi-(\mu_1 q_1'+\mu_2 q_2')\right]\\&\times\delta\left[p-(\mu_2 p_1'-\mu_1 p_2')\right]\Delta_2(q_1',p_1';q_2',p_2')\\&=\frac{1}{\pi^2}\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\mathrm{d}q_2'\delta\left[\chi-(\mu_1 q_1'+\mu_2 q_2')\right]\\&\times\delta\left[\,p-(\mu_2 p_1'-\mu_1 p_2')\right]:\prod_{i=1}^{2}\mathrm{e}^{-(q_i'-\hat{Q})^2-(p_i'-\hat{P})^2}:\\&=|\chi,p\rangle\langle\chi,p|\,. \quad (45)\end{aligned}$$

After performing the integration, we find

$$\begin{aligned}&|\chi,p\rangle\\&=\sqrt{\frac{\mu_1\mu_2}{\pi\left(\mu_2^2+\mu_1^2\right)}}\exp\left\{-\frac{1}{2\left(\mu_1^2+\mu_2^2\right)}\left(p^2+\chi^2\right)\right.\\&+\frac{\sqrt{2}}{\mu_1^2+\mu_2^2}\left[(\chi\mu_1+\mathrm{i}p\mu_2)a_1^\dagger+(\chi\mu_2-\mathrm{i}p\mu_1)a_2^\dagger\right]\\&\left.+\frac{(\mu_2-\mu_1)\left(a_1^{\dagger2}-a_2^{\dagger2}\right)}{2\left(\mu_1^2+\mu_2^2\right)}-\frac{2\mu_1\mu_2 a_1^\dagger a_2^\dagger}{\mu_1^2+\mu_2^2}\right\}|00\rangle\,. \quad (46)\end{aligned}$$

### 4.4. Explicit form of the tripartite entangled state

As the last example, we consider a tripartite entangled system by searching for the common eigenvector of compatible operators $\hat{Q}_1-\hat{Q}_2$, $\hat{Q}_1-\hat{Q}_3$ and $\hat{P}_1+\hat{P}_2+\hat{P}_3$, i.e.,

$$\begin{aligned}&(\hat{Q}_1-\hat{Q}_2)\,|p,\chi_1,\chi_2\rangle=\chi_1\,|p,\chi_1,\chi_2\rangle\,,\\&(\hat{Q}_1-\hat{Q}_3)\,|p,\chi_1,\chi_2\rangle=\chi_2\,|p,\chi_1,\chi_2\rangle\,,\\&(\hat{P}_1+\hat{P}_2+\hat{P}_3)\,|p,\chi_1,\chi_2\rangle=p\,|p,\chi_1,\chi_2\rangle\,. \quad (47)\end{aligned}$$

Using the Weyl correspondence

$$q^2p\to\left(\frac{1}{2}\right)^2(\hat{Q}^2\hat{P}+2\hat{Q}\hat{P}\hat{Q}+\hat{P}\hat{Q}^2)=\vdots\;\hat{Q}^2\hat{P}\;\vdots \quad (48)$$

and taking the similar method described above we can deduce

$$\vdots\,(\hat{Q}_1-\hat{Q}_2)(\hat{Q}_1-\hat{Q}_3)(\hat{P}_1+\hat{P}_2+\hat{P}_3)\,\vdots$$
$$=(\hat{Q}_1-\hat{Q}_2)(\hat{Q}_1-\hat{Q}_3)(\hat{P}_1+\hat{P}_2+\hat{P}_3), \quad (49)$$

so its classical Weyl correspondence is $(q_1-q_2)(q_1-q_3)(p_1+p_2+p_3)$. According to Eqs. (24) and (23) we can obtain

$$\begin{aligned}&\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\mathrm{d}q_2'\,\mathrm{d}p_3'\mathrm{d}q_3'\delta\left[\chi_1-(q_1'-q_2')\right]\\&\times\delta\left[\chi_2-(q_1'-q_3')\right]\delta\left[p-(p_1'+p_2'+p_3')\right]\\&\times\Delta_3(q_1',p_1';q_2',p_2';q_3',p_3')\\&=\frac{1}{\pi^3}\int \mathrm{d}p_1'\,\mathrm{d}q_1'\,\mathrm{d}p_2'\,\mathrm{d}q_2'\,\mathrm{d}p_3'\,\mathrm{d}q_3'\delta\left[\chi_1-(q_1'-q_2')\right]\\&\times\delta\left[\chi_2-(q_1'-q_3')\right]\delta\left[p-(p_1'+p_2'+p_3')\right]\\&\times:\prod_{i=1}^{3}\mathrm{e}^{-(q_i'-\hat{Q})^2-(p_i'-\hat{P})^2}:\\&=|p,\chi_2,\chi_3\rangle\langle p,\chi_2,\chi_3|\,. \quad (50)\end{aligned}$$

After integrating a tripartite entangled state is easily derived as

$$\begin{aligned}&|p,\chi_2,\chi_3\rangle\\&=\frac{1}{\sqrt{3}\pi^{3/4}}\exp\left[-\frac{1}{6}p^2-\frac{1}{3}\left(\chi_1^2-\chi_1\chi_2+\chi_2^2\right)\right.\\&+\frac{\sqrt{2}}{3}a_1^\dagger\left(\mathrm{i}p+\chi_1+\chi_2\right)+\frac{\sqrt{2}}{3}a_2^\dagger\left(\mathrm{i}p-2\chi_1+\chi_2\right)\\&+\frac{\sqrt{2}}{3}a_3^\dagger\left(\mathrm{i}p+\chi_1-2\chi_2\right)+\frac{2}{3}\left(a_1^\dagger a_2^\dagger+a_1^\dagger a_3^\dagger+a_2^\dagger a_3^\dagger\right)\\&\left.-\frac{1}{6}\left(a_1^{\dagger2}+a_2^{\dagger2}+a_3^{\dagger2}\right)\right]|000\rangle\,, \quad (51)\end{aligned}$$

where we have used : $\exp(-\sum_{i=1}^{3}a_i^\dagger a_i)$: $=|000\rangle\langle 000|$.

## 5. Conclusions

In summary, with the help of the completeness of Wigner operator and Weyl correspondence, we construct a general equation for deriving pure state density operators and then extend to the multi-mode case. By making use of the new constructed equations, we reconstruct typical quantum mechanical states, such as the eigenstate of quadrature $\mu\hat{Q}+\nu\hat{P}$, bipartite and tripartite continuous variable entangled state, and so on. It is very clear that our approach is effective and convenient for deducing these states and prove their completeness properties. We believe that our new equations may be interesting for some readers and useful for deriving some new quantum mechanical representations.

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