# Existence and asymptotic estimates of periodic solutions of El Niño mechanism of atmospheric physics*

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This paper is devoted to studying the El Niño mechanism of atmospheric physics. The existence and asymptotic estimates of periodic solutions for its model are obtained by employing the technique of upper and lower solution, and using the continuation theorem of coincidence degree theory.

**Keywords:** nonlinear, time delay, El Niño phenomenon, periodic solution

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## 1. Introduction

Interactions between the ocean and atmosphere contribute to climate fluctuations over a broad spectrum of time scales. Studies of those interactions have so far focused on El Niño Southern Oscillation (ENSO) phenomena that have a period of 3 to 4 years and whose principal signature is in the tropical Pacific.[1,2] The ENSO is an interannual phenomenon involved in the tropical Pacific Ocean–atmosphere interactions. Phenomenon of the ENSO is a very attractive object of study in the international academic circles.[3−6] Feng *et al.*[3] considered an instable model of air–sea oscillator, Wang[5] studied an especial ENSO mechanism. Mo *et al.*,[7−15] Lin W T[16,17] and Lin Y H *et al.*[18−20] also considered some problems in atmospheric physics, ocean weather, dynamic system and so on. Recently, Li have studied the periodic solution to some problems for the ENSO.[21−23]

In this paper, we consider the following delayed oscillator of the ENSO model:[9]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = AT - \epsilon T^3 - BT^3(t-\delta), \tag{1}$$

where $T$ is the sea surface temperature (SST) anomaly in the equatorial eastern Pacific, the model parameters $A$, $B$, $\delta$ and $\epsilon$ are positive parameters, their physical definitions can be seen from Refs. [3], [4] and [16]. Using the variational iteration theory, we can obtain the approximation of the solution for this model by the short-cut calculation in Ref. [9].

Stimulated by the beautiful results obtained in Ref. [9], in the present paper, we continue to study the delayed oscillator of ENSO model (1), the existence and asymptotic estimates of periodic solutions of the corresponding problem are obtained by using the technique of upper and lower solutions, and the continuation theorem of coincidence degree theory.

## 2. The periodic solution of a nonlinear problem

In order to study Eq. (1), we first discuss the following nonlinear problem:

$$T'(t) = f(t, T(t), T(t-\delta)), \tag{2}$$

where $f \in C(\boldsymbol{R}^3, \boldsymbol{R})$, $f(t+\omega, x, y) \equiv f(t, x, y)$ for arbitrary $(x, y) \in \boldsymbol{R}^2$, $\omega > 0$ is a given constant.

First, we give some definitions as follows.

**Definition 1** If $\alpha(t) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $\alpha(t+\omega) \equiv \alpha(t)$, and

$$\alpha'(t) > f(t, \alpha(t), \alpha(t-\delta)), \quad \forall t \in [0, \omega],$$

then $\alpha(t)$ is called a strict $\omega$-periodic lower solution of Eq. (2).

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**Definition 2** If $\beta(t) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $\beta(t+\omega) \equiv \beta(t)$, and

$$\beta'(t) < f(t, \beta(t), \beta(t-\delta)), \quad \forall t \in [0, \omega],$$

then $\beta(t)$ is called a strict $\omega$-periodic upper solution of Eq. (2).

Further, we let $C_\omega = \{T | T \in C(\boldsymbol{R}, \boldsymbol{R}), T(t+\omega) \equiv T(t)\}$ with the norm $|\varphi|_0 = \max\limits_{t\in[0,\omega]} |\varphi(t)|, \forall \varphi \in C_\omega$. Clearly, $C_\omega$ is a Banach space. We also define operator $L$ as

$$L : D(L) \subset C_\omega \longrightarrow C_\omega, \quad LT = T',$$

where $D(L) = \{T | T \in C^1(\boldsymbol{R}, \boldsymbol{R}), T(t+\omega) \equiv T(t)\}$. It is easy to see that Ker $L = \boldsymbol{R}$ and Im $L = \{T | T \in C_\omega, \int_0^\omega T(s)\mathrm{d}s = 0\}$. Thus, $L$ is a Fredholm operator with index zero.[24] Let

$$P : C_\omega \longrightarrow \text{Ker } L, \quad Q : C_\omega \longrightarrow \frac{C_\omega}{\text{Im } L}$$

defined respectively by

$$PT = T(0), \quad QT = \frac{1}{\omega}\int_0^\omega T(s)\mathrm{d}s$$

and let

$$L_P = L|_{D(L)\cap \text{Ker } P} : D(L) \cap \text{Ker } P \longrightarrow \text{Im } L.$$

Then Im $P$ = Ker $L$, Ker $Q$ = Im $L$, and $L_P$ has a unique continuous pseudo-inverse $K_P$ on Im $L$ defined by

$$[K_P y](t) = \int_0^t y(s)\mathrm{d}s. \tag{3}$$

Let $N : C_\omega \longrightarrow C_\omega, [NT](t) = f(t, T(t), T(t-\delta))$. It is easy to check that equation (2) is equivalent to the following operator equation

$$T = MT,$$

where $MT = PT + QNT + K_P(I-Q)NT$. From Eq. (3), one can find that $M$ is compact on any bounded subset of $C_\omega$.

**Theorem 1** Suppose the following conditions hold.

(i) $f(t, x, y)$ is non-increasing with respect to $y$.

(ii) There exists a constant $R_0 > 0$ such that

$$f(t, R_0, R_0) > 0 \text{ and } f(t, -R_0, -R_0) < 0, \quad \forall t \in \boldsymbol{R}.$$

Then $\deg\{I - M, \Omega, \theta\} \neq 0$, where $\Omega = \{T : T \in C_\omega, |T|_0 < R_0\}$.

**Proof** Consider operator equation as follows:

$$LT = \lambda NT, \quad \forall \lambda \in (0, 1]. \tag{4}$$

In what follows, we will prove that

$$LT \neq \lambda NT, \quad \forall \lambda \in (0, 1], \forall T \in \partial\Omega. \tag{5}$$

Otherwise, if $T \in \partial\Omega$ satisfies

$$LT = \lambda NT, \quad \text{for some } \lambda \in (0, 1],$$

then

$$T'(t) = \lambda f(t, T(t), T(t-\delta)). \tag{6}$$

In view of $T \in \partial\Omega$, one can find that $|T|_0 = R_0$. Then there exists a $t^* \in [0, \omega]$ such that $|T(t^*)| = \max\limits_{t\in[0,\omega]} |T(t)| = |T|_0 = R_0$. Without loss of generality, we may assume that

$$T(t^*) = R_0, \tag{7}$$

then $t^*$ is a maximum point of $T(t)$ on $\boldsymbol{R}$. So

$$T'(t^*) = 0. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6), we have

$$f(t^*, R_0, T(t^* - \delta)) = 0. \tag{9}$$

As $T(t^* - \delta) \leq R_0$, it follows from condition (i) that

$$f(t^*, R_0, R_0) \leq f(t^*, R_0, T(t^* - \delta)) = 0,$$

which contradicts to condition (ii). Therefore

$$|T|_0 < R_0.$$

Thus, equation (5) is satisfied.

One can easily verify that the equation $LT = \lambda NT$ is equivalent to the equation of the type

$$T = M(T, \lambda) := PT + QNT + \lambda K_P(I-Q)NT, \tag{10}$$

and $M(T, \lambda)$ is compact on $\overline{\Omega} \times [0, 1]$. Further, we can check that

$$T \neq M(T, \lambda), \quad \forall T \in \partial\Omega, \lambda \in [0, 1].$$

In fact, from the above proof, it is easy to see that

$$T \neq M(T, \lambda), \quad \forall T \in \partial\Omega, \lambda \in (0, 1]. \tag{11}$$

Now we need to prove that $T \neq M(T, 0), \forall T \in \partial\Omega$. If the statement is not true, then

$$M(T, 0) = T, \quad \text{for some } T \in \partial\Omega,$$

i.e.

$$T = QNT + PT,$$

which yields that

$$T \in \text{Ker } L \text{ and } QNT = 0. \tag{12}$$

That is $QN(R_0) = 0$ or $QN(-R_0) = 0$. But by the definition of $Q$ and $N$, we know that $QN(R_0) = \frac{1}{\omega}\int_0^\omega f(t, R_0, R_0)\mathrm{d}t > 0$ and $QN(-R_0) = \frac{1}{\omega}\int_0^\omega f(t, -R_0, -R_0)\mathrm{d}t < 0$, which contradicts to Eq. (12). Thus

$$T \neq M(T, \lambda), \text{ for } \forall (T, \lambda) \in \partial\Omega \times [0, 1],$$

and

$$QNT \neq 0, \ \forall T \in \partial\Omega \cap \mathrm{Ker}\ L.$$

Therefore,

$$\begin{aligned} &\deg\{I - M, \Omega, \theta\} \\ &= d_B\{-QN|_{\mathrm{Ker}\ L\cap\Omega}, \mathrm{Ker}\ L \cap \Omega, 0\}. \end{aligned}$$

As $R_0 QN(R_0) \neq 0$, it follows that

$$\deg\{I - M, \Omega, \theta\} \neq 0.$$

The proof is completed.

**Theorem 2** Suppose the following conditions hold:

(I) $f(t, x, y)$ is non-increasing with respect to $y$;

(II) $\alpha(t), \beta(t)$ are the strict lower and strict upper $\omega$-periodic solution of Eq. (2) with $\alpha(t) < \beta(t), \forall t \in [0, \omega]$.

Then equation (2) exists at least one $\omega$-periodic solution $T(t)$ such that

$$\alpha(t) < T(t) < \beta(t), \quad \forall t \in [0, \omega].$$

**Proof** Take $R_0$ large enough such that

$$f(t, \beta(t), \beta(t-\delta)) + R_0 - \beta(t) > 0, \quad \forall t \in [0, \omega],$$

$$f(t, \alpha(t), \alpha(t-\delta)) - R_0 - \alpha(t) < 0, \quad \forall t \in [0, \omega],$$

and

$$R_0 > \max\{-\min_{t\in[0,\omega]} \alpha(t), \max_{t\in[0,\omega]} \beta(t)\}.$$

Meanwhile, define auxiliary function $F_1(t, x, y)$ as follows:

$$\begin{cases} f(t, x, \beta(t-\delta)), & |x| \leq R_0, \beta(t-\delta) \leq y \leq R_0, \\ f(t, x, y), & |x| \leq R_0, \alpha(t-\delta) < y < \beta(t-\delta), \\ f(t, x, \alpha(t-\delta)), & |x| \leq R_0, -R_0 \leq y \leq \alpha(t-\delta), \end{cases}$$

and define auxiliary function $F_2(t, x, y)$ as follows:

$$\begin{cases} F_1(t, \beta(t), y) + x - \beta(t), & \beta(t) \leq x \leq R_0, \\ F_1(t, x, y), & \alpha(t) < x < \beta(t), \\ F_1(t, \alpha(t), y) + x - \alpha(t), & -R_0 \leq x \leq \alpha(t). \end{cases}$$

Clearly, $F_2 \in C(\boldsymbol{R} \times [-R_0, R_0] \times [-R_0, R_0], \boldsymbol{R})$ and

$$F_2(t, R_0, R_0) = f(t, \beta(t), \beta(t-\delta)) + R_0 - \beta(t) > 0, \tag{13}$$

$$F_2(t, -R_0, -R_0) = f(t, \alpha(t), \alpha(t-\delta)) - R_0 - \alpha(t) < 0. \tag{14}$$

From Eqs. (13) and (14), we know that the function $F_2(t, x, y)$ satisfies all the conditions of Theorem 1. Thus

$$\deg\{I - M_{F_2}, \Omega, \theta\} \neq 0, \tag{15}$$

where $\Omega = \{T : T \in C_\omega, |T|_0 < R_0\}$. Further, we let $\Delta = \{T : T \in C_\omega, \alpha(t) < T(t) < \beta(t)\}$. Obviously, $\Delta \subset \Omega$. In what follows, we shall prove that

$$LT \neq N_{F_2} T, \quad \forall T \in \Omega \setminus \overline{\Delta}.$$

If the statement is not true, there must be a $t_0 \in [0, \omega]$ such that

$$\max_{t\in[0,\omega]} \{T(t) - \beta(t)\} = T(t_0) - \beta(t_0) \geq 0, \tag{16}$$

or

$$\min_{t\in[0,\omega]} \{T(t) - \alpha(t)\} = T(t_0) - \alpha(t_0) \leq 0. \tag{17}$$

If expression (16) holds, then

$$T'(t_0) = \beta'(t_0). \tag{18}$$

But, from expression (16) and the definition of $\beta(t)$, we have

$$\begin{aligned} &T'(t_0) - \beta'(t_0) \\ &= F_2(t_0, T(t_0), T(t_0 - \delta)) - \beta'(t_0) \\ &= f(t_0, \beta(t_0), \beta(t_0 - \delta)) + T(t_0) - \beta(t_0) - \beta'(t_0) \\ &> 0, \end{aligned}$$

which contradicts to Eq. (18). This contradiction implies that $T(t) < \beta(t), \forall t \in [0, \omega]$. Similarly, we have $T(t) > \alpha(t), \forall t \in [0, \omega]$. So

$$LT \neq N_{F_2} T, \quad \forall T \in \Omega \setminus \overline{\Delta}.$$

Thus,

$$\begin{aligned} &\deg\{I - M, \Delta, \theta\} \\ &= \deg\{I - M_{F_2}, \Delta, \theta\} \\ &= \deg\{I - M_{F_2}, \Omega, \theta\} - \deg\{I - M_{F_2}, \Omega \setminus \overline{\Delta}, \theta\} \\ &\neq 0. \end{aligned}$$

Thus equation (2) exists at least one $\omega$-periodic solution $T(t) \in \Delta$.[24] That is

$$\alpha(t) < T(t) < \beta(t), \quad \forall t \in [0, \omega].$$

The proof is completed.

## 3. The periodic solutions for model (1)

By applying Theorem 2, we study the existence and asymptotic estimates of the periodic solutions of Eq. (1).

i) We first consider the reduced problem of Eq. (1), i.e.

$$T'(t) = AT - BT^3(t-\delta). \tag{19}$$

By Theorem 2, we let $f(t,x,y) = Ax - By^3, \alpha(t) \equiv -R_1 < 0, \beta(t) \equiv R_1 > 0$, where $R_1 < 1$ is a constant. Obviously, $f(t,x,y)$ is non-increasing with respect to $y$, and $\alpha(t), \beta(t)$ are the strict lower and strict upper $\omega$-periodic solutions of Eq. (19) with $\alpha(t) < \beta(t), \forall t \in [0,\omega]$, if $A > B$.

By applying Theorem 2, we know that the reduced problem of Eq. (1) exists at least one $\omega$-periodic solution $T_0(t)$ satisfying

$$-R_1 < T_0(t) < R_1, \quad \forall t \in [0,\omega].$$

ii) We now consider Eq. (1)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = AT - \epsilon T^3 - BT^3(t-\delta).$$

From conclusion i), we have $|T_0(t)| \leq R_1$. By Theorem 2, we let $f(t,x,y) = Ax - \epsilon x^3 - By^3, \alpha(t) = T_0(t) - \gamma\epsilon, \beta(t) = T_0(t) + \gamma\epsilon$, where $\gamma$ is a constant which satisfies $A\gamma - R_1^3 - 3BR_1^2\gamma > 0$. In the following, we assume sufficiently small $\epsilon > 0$. Then $f(t,x,y,\epsilon)$ is not increasing with respect to $y$, and

$$\begin{aligned}&\alpha'(t) - f(t,\alpha(t),\alpha(t-\delta))\\ &> (A\gamma - R_1^3 - 3BR_1^2\gamma)\epsilon + o(\epsilon) > 0, \quad 0 < \epsilon \ll 1,\end{aligned}$$

i.e.

$$\alpha'(t) > f(t,\alpha(t),\alpha(t-\delta)), \forall t \in [0,\omega].$$

Similarly,

$$\beta'(t) < f(t,\beta(t),\beta(t-\delta)), \forall t \in [0,\omega].$$

By applying Theorem 2, for sufficiently small $\epsilon$ ($0 < \epsilon \ll 1$), equation (1) has at least one $\omega$-periodic solution $T(t,\epsilon)$ such that

$$|T(t,\epsilon) - T_0(t)| \leq o(\epsilon).$$

## 4. Conclusions

Atmospheric physics studies a very complicated natural phenomenon. Hence we need to reduce basic models for the sea–air oscillator and study the periodic behaviours of the temperature anomaly in the equatorial eastern Pacific. The interesting thing is that the asymptotic estimates of periodic solutions can be used to obtain the approximation of the period. Thus, we can forecast when the ENSO appears.

## References


[1] McPhaden M J and Zhang D 2002 *Nature* **415** 603
[2] Gu D F and Philander S G H 1997 *Science* **275** 805
[3] Feng G L, Dong W J, Jia X J and Cao H X 2002 *Acta Phys. Sin.* **51** 1181 (in Chinese)
[4] Liu S K, Fu Z T, Liu S D and Zhao Q 2002 *Acta Phys. Sin.* **51** 10 (in Chinese)
[5] Wang C Z 2001 *Advances in Atmospheric Sciences* **18** 674
[6] An S I and Wang B 2000 *J. Climate* **13** 2044
[7] Mo J Q, Lin W T and Zhu J 2004 *Acta Phys. Sin.* **53** 3245 (in Chinese)
[8] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 993 (in Chinese)
[9] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 1081 (in Chinese)
[10] Mo J Q and Lin W T 2005 *Chin. Phys.* **14** 875
[11] Mo J Q, Wang H and Lin W T 2006 *Acta Phys. Sin.* **55** 3229 (in Chinese)
[12] Mo J Q, Lin W T and Wang H 2007 *Chin. Phys.* **16** 578
[13] Mo J Q, Lin W T and Lin Y H 2007 *Chin. Phys.* **16** 1908
[14] Mo J Q and Lin W T 2007 *Acta Phys. Sin.* **56** 5565 (in Chinese)
[15] Mo J Q, Lin Y H and Lin W T 2005 *Acta Phys. Sin.* **54** 3971 (in Chinese)
[16] Lin W T, Ji Z Z, Wang B and Zhang X 2002 *Progress in Natural Sci.* **12** 102
[17] Lin W T and Mo J Q 2004 *Chin. Sci. Bull.* **48 II** 5
[18] Lin Y H, Ji Z Z and Zeng Q C 1999 *Progress in Natural Sci.* **9** 532
[19] Lin Y H, Xue F and Lian S M 2001 *Chinese J. Atmospheric Sci.* **25** 111
[20] Lin Y H and Fu Y F 2001 *Acta Meteorologica Sin.* **59** 1
[21] Li X J 2007 *Chin. Phys.* **16** 2837
[22] Li X J 2008 *Chin. Phys.* B **17** 1946
[23] Li X J 2008 *Acta Phys. Sin.* **57** 5366 (in Chinese)
[24] Gaines R E and Mawhin J L 1977 *Coincidence Degree and Nonlinear Differential Equations* (Berlin: Springer)