Three types of generalized Kadomtsev–Petviashvili equations arising from baroclinic potential vorticity equation*

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By means of the reductive perturbation method, three types of generalized (2+1)-dimensional Kadomtsev–Petviashvili (KP) equations are derived from the baroclinic potential vorticity (BPV) equation, including the modified KP (mKP) equation, standard KP equation and cylindrical KP (cKP) equation. Then some solutions of generalized cKP and KP equations with certain conditions are given directly and a relationship between the generalized mKP equation and the mKP equation is established by the symmetry group direct method proposed by Lou *et al.* From the relationship and the solutions of the mKP equation, some solutions of the generalized mKP equation can be obtained. Furthermore, some approximate solutions of the baroclinic potential vorticity equation are derived from three types of generalized KP equations.

Keywords: baroclinic potential vorticity equation, generalized Kadomtsev–Petviashvili equation, symmetry groups, approximate solution

PACC: 0220, 1130, 3310c, 0340k

1. Introduction

In recent years, the study of nonlinear partial differential equations (PDEs) have become one of the most exciting and extremely active areas of research. Many methods can be used to obtain solutions of PDEs, such as symmetry reductions, the general direct method and the extended Jacobi elliptic function rational expansion method, and so on.[1−10] However, there are many difficulties in obtaining solutions of baroclinic potential vorticity (BPV) equation[11] by these methods. Luckily, the reductive perturbation method can make an intricate equation become another new equation, then the intricate equation can be researched more succesfully with the help of a new equation. This method has been used in many fields. Lou *et al.*[12] derived coupled KdV equations from two-layer fluids. Tang *et al.*[13] obtained the variable coefficient KdV equation from the Euler equation with an earth rotation term. According to the KdV equation, they obtained the approximate solution of the Euler equation with an earth rotation term. Gao *et al.*[14] presented a coupled variable coefficient modified KdV equation from a two-layer fluid system. The main purpose of this paper is to obtain the approximate solutions of the BPV equation. The paper is organized as follows. In Section 2, we obtain generalized modified Kadomtsev–Petviashvili (mKP), KP and cylindrical KP (cKP) equations from the BPV equation by means of the reductive perturbation method. Some KP equations have been studied by some authors. Li *et al.*[15] obtained soliton-like solution and periodic form solution of a KP equation. Liu[16] gained a solution of the cKP equation by auto-Backlünd transformation. Wazwaz[17] obtained multiple-soliton solutions of the mKP equqtion by the Hirota's bilinear method. In Section 3, under certain conditions we obtain the solutions of the generalized cKP and KP equation directly, and then get an approximate solution of the BPV equation. In Section 4, by making use of the

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symmetry group direct method proposed by Lou *et al.*,[18−21] we establish a relationship between the generalized mKP equation and mKP equation, and then based on this relationship we obtain an approximate solution of the BPV equation by complex and tedious calculations. In the last section, we give the conclusion of the paper.

2. Generalized mKP, KP and cKP equations from (3+1)-dimensional BPV equation

The (3 + 1)-dimensional BPV equation may be written as

$$q_t + \psi_x q_y - \psi_y q_x + \beta\psi_x = 0, \tag{1}$$

$$q = \psi_{xx} + \psi_{yy} + \psi_{zz}, \tag{2}$$

where β is constant, $\psi \equiv \psi(x, y, z, t)$, $q \equiv q(x, y, z, t)$, and subscripts x, y, z, t represent partial derivatives.

First, by substituting Eq. (2) into Eq. (1), BVP equation becomes:

$$\begin{aligned}&\psi_{txx} + \psi_{tyy} + \psi_{tzz} + \psi_x\psi_{xxy} + \psi_x\psi_{yyy} + \psi_x\psi_{yzz}\\ &- \psi_y\psi_{xxx} - \psi_y\psi_{xyy} - \psi_y\psi_{xzz} + \beta\psi_x = 0.\end{aligned} \tag{3}$$

Now we use the long wave approximation in the x-direction and z-direction and assume the stream functions ψ to be

$$\psi = \psi_0(y, t) + \psi_1(x, y, z, t). \tag{4}$$

We introduce the stretched variables

$$X = \epsilon(x - c_0 t), \quad Z = \epsilon^2(z - c_1 t), \quad T = \epsilon^3 t, \tag{5}$$

where $X \equiv X(x, t)$, $Z \equiv Z(z, t)$, $T \equiv T(t)$, c_0 and c_1 are constants, ϵ is a small parameter. In general, the base field $\psi_0(y, t)$ is often taken only as a linear function of y, $\psi_1(x, y, z, t)$ is expanded as

$$\psi_1(x, y, z, t) = \sum_{i=1}^{\infty} \epsilon^i \phi_i(X, y, Z, T) \equiv \sum_{i=1}^{\infty} \epsilon^i \phi_i. \tag{6}$$

We also have the expansion

$$\psi_0(y, t) = U_0(y) + \sum_{i=1}^{\infty} \epsilon^i U_i(y, T). \tag{7}$$

Then substituting Eqs. (5), (6) and (7) into Eq. (3), we obtain

$$M_1\epsilon^2 + M_2\epsilon^3 + M_3\epsilon^4 + O(\epsilon^5) = 0, \tag{8}$$

where

$$\begin{aligned}M_1 =\ & -U_{0y}(y)\phi_{1Xyy} + U_{0yyy}(y)\phi_{1X}\\ & - c_0\phi_{1Xyy} + \beta\phi_{1X},\\ M_2 =\ & \phi_{1X}\phi_{1yyy} - c_0\phi_{2Xyy} - U_{1y}(y, T)\phi_{1Xyy}\\ & - \phi_{1Xyy}\phi_{1y} - c_1\phi_{1yyZ} + U_{1yyy}(y, T)\phi_{1X}\\ & - U_{0y}(y)\phi_{2Xyy} + U_{0yyy}(y)\phi_{2X} + \beta\phi_{2X},\\ M_3 =\ & \phi_{2X}\phi_{1yyy} - c_0\phi_{3Xyy} + U_{1yyy}(y, T)\phi_{2X}\\ & + \beta\phi_{3X} + \phi_{1Tyy} - U_{0y}(y)\phi_{1XXX}\\ & + U_{0yyy}(y)\phi_{3X} - U_{2y}(y, T)\phi_{1Xyy} - c_0\phi_{1XXX}\\ & - U_{0y}(y)\phi_{3Xyy} - U_{1y}(y, T)\phi_{2Xyy} - c_1\phi_{2yyZ}\\ & + U_{2yyy}\phi_{1X} + \phi_{1X}\phi_{2yyy} + U_{1yyT}(y, T)\\ & - \phi_{2Xyy}\phi_{1y} - \phi_{1Xyy}\phi_{2y}.\end{aligned}$$

Vanishing the ϵ^2 of Eq. (8) by Maple, we get a special solution

$$\phi_1 = A(X, Z, T)G(y, T) \equiv AG, \tag{9}$$

where G is determined by

$$U_{0yyy}(y)G - U_{0y}(y)G_{yy} - c_0 G_{yy} + \beta G = 0. \tag{10}$$

Then vanishing the ϵ^3 of Eq. (8), we have

$$\begin{aligned}\phi_2 =\ & G_1(y, T)A(X, Z, T)^2 + G_2(y, T)A(X, Z, T)\\ & + G_3(y, T)\int A_Z(X, Z, T)\,\mathrm{d}X\\ \equiv\ & G_1A^2 + G_2A + G_3\int A_Z\,\mathrm{d}X,\end{aligned} \tag{11}$$

where G_1, G_2 and G_3 should satisfy

$$\begin{aligned}&-G_{yy}G_y + GG_{yyy} + 2U_{0yyy}(y)G_1 - 2U_{0y}(y)G_{1yy}\\ &\quad - 2c_0G_{1yy} + 2\beta G_1 = 0,\\ &U_{1yyy}(y, T)G - U_{1y}(y, T)G_{yy} + \beta G_2 + U_{0yyy}(y)G_2\\ &\quad - U_{0y}(y)G_{2yy} - c_0G_{2yy} = 0,\\ &U_{0yyy}(y)G_3 - U_{0y}(y)G_{3yy} - c_0G_{3yy} - c_1G_{yy}\\ &\quad + \beta G_3 = 0.\end{aligned} \tag{12}$$

Substituting the solutions of Eqs. (10), (12) and $\phi_3 = 0$ into M_3, integrating the result with respect to y from 0 to y_0, then we obtain

$$\begin{aligned}&a_1A_T + a_2A_{XXX} + a_3AA_X + a_4AA_Z + a_5A^2A_X\\ &\quad + a_6A_X\int A_Z\,\mathrm{d}X + a_7\int A_{ZZ}\,\mathrm{d}X\\ &\quad + a_8A_X + a_9A_Z + a_{10}A + a_{11} = 0,\end{aligned} \tag{13}$$

where $a_i \equiv a_i(T)$, $i = 1, 2, \ldots, 11$

$$a_1 = \int_0^{y_0}\int_0^{y_1} G_{yy}\,\mathrm{d}y\,\mathrm{d}y_1,$$

$$
\begin{aligned}
a_2 &= \int_0^{y_0}\int_0^{y_1}(-U_{0y}(y)G - c_0G)\,\mathrm{d}y\mathrm{d}y_1,\\
a_3 &= \int_0^{y_0}\int_0^{y_1}[2U_{1yyy}(y,T)G_1 - 2U_{1y}(y,T)G_{1yy}\\
&\quad - G_yG_{2yy} + G_{yyy}G_2 + GG_{2yyy}\\
&\quad - G_{yy}G_{2y}]\,\mathrm{d}y\mathrm{d}y_1,\\
a_4 &= \int_0^{y_0}\int_0^{y_1}(G_{yyy}G_3 - 2c_1G_{1yy} - G_yG_{3yy})\,\mathrm{d}y\mathrm{d}y_1,\\
a_5 &= \int_0^{y_0}\int_0^{y_1}(-2G_yG_{1yy} + 2G_{yyy}G_1 + GG_{1yyy}\\
&\quad - G_{yy}G_{1y})\,\mathrm{d}y\mathrm{d}y_1,\\
a_6 &= \int_0^{y_0}\int_0^{y_1}(GG_{3yyy} - G_{yy}G_{3y})\,\mathrm{d}y\mathrm{d}y_1,\\
a_7 &= \int_0^{y_0}\int_0^{y_1}(-c_1G_{3yy})\,\mathrm{d}y\mathrm{d}y_1,\\
a_8 &= \int_0^{y_0}\int_0^{y_1}[-U_{2y}(y,T)G_{yy} + U_{2yyy}(y,T)G\\
&\quad - U_{1y}(y,T)G_{2yy} + U_{1yyy}(y,T)G_2]\,\mathrm{d}y\mathrm{d}y_1,\\
a_9 &= \int_0^{y_0}\int_0^{y_1}[U_{1yyy}(y,T)G_3 - c_1G_{2yy}\\
&\quad - U_{1y}(y,T)G_{3yy}]\,\mathrm{d}y\mathrm{d}y_1,\\
a_{10} &= \int_0^{y_0}\int_0^{y_1}G_{yyT}\,\mathrm{d}y\mathrm{d}y_1,\\
a_{11} &= \int_0^{y_0}\int_0^{y_1}U_{1yyT}(y,T)\,\mathrm{d}y\mathrm{d}y_1.
\end{aligned}
$$

Equation (13) contains the following important cases:

(i) If $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, equation (13) becomes a generalized KP equation

$$
a_1A_T + a_2A_{XXX} + a_3AA_X + a_7\int A_{ZZ}\,\mathrm{d}X = 0. \tag{14}
$$

(ii) If $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, equation (13) becomes a generalized cKP equation

$$
a_1A_T + a_2A_{XXX} + a_3AA_X + a_7\int A_{ZZ}\,\mathrm{d}X + a_{11}A = 0. \tag{15}
$$

(iii) If $a_3 = 0$, $a_4 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, equation (13) becomes a generalized mKP equation

$$
a_1A_T + a_2A_{XXX} + a_5A^2A_X + a_6A_X\int A_Z\,\mathrm{d}X + a_7\int A_{ZZ}\,\mathrm{d}X = 0. \tag{16}
$$

In the next section, we discuss the solutions of Eq. (3) with the help of Eqs. (14), (15) and (16) by a direct method and symmetry group direct method.

3. The solutions of BPV equation from generalized KP and cKP equations

In this section, we divide into two parts to obtain approximate solutions of the BPV equation with the help of the generalized KP and cKP equations.

First, we set about the generalized KP equation to obtain the approximate solution of Eq. (3). We know that equation (14) should satisfy

$$
\begin{aligned}
&a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = a_{11} = 0,\\
&U_{0yyy}(y)A_XG - U_{0y}(y)A_XG_{yy} - c_0A_XG_{yy} = 0,\\
&-G_{yy}G_y + GG_{yyy} + 2U_{0yyy}(y)G_1 - 2U_{0y}(y)G_{1yy}\\
&\qquad - 2c_0G_{1yy} = 0,\\
&U_{0yyy}(y)G_3 - U_{0y}(y)G_{3yy} - c_0G_{3yy} - c_1G_{yy} = 0,\\
&U_{1yyy}(y,T)G - U_{1y}(y,T)G_{yy} + \beta_1G\\
&\quad + U_{0yyy}(y)G_2 - U_{0y}(y)G_{2yy} - c_0G_{2yy} = 0.
\end{aligned} \tag{17}
$$

According to Eq. (17), we obtain $G(y,k)$, $G_1(y,k)$, $G_2(y,k)$, $G_3(y,k)$, $U_0(y)$, $U_1(y,k)$ and $U_2(y,k)$ after complicated computing process

$$
\begin{aligned}
&G = C_8,\ G_1 = C_7,\ G_2 = F_3(T)(C_1y + C_2),\\
&G_3 = c_1F_3(T) + F_4(T)(C_1y + C_2),\\
&U_0(y) = -\frac{1}{6}\beta y^3 + \frac{1}{2}C_4y^2 + C_5y + C_6,\\
&U_i(y,k) = 0\ \ (i \geq 2),\\
&U_1(y,T) = \frac{1}{2}C_1y^2 + C_2y + C_3 + F_1(T),
\end{aligned} \tag{18}
$$

where C_i $(i = 1,2,\ldots,8)$ are arbitrary constants, $F_1(T)$, $F_3(T)$ and $F_4(T)$ are arbitrary functions with respect to time. At the same time, we can obtain a_1, a_2, a_3 and a_7:

$$
\begin{aligned}
&a_2 = \frac{1}{2}C_8\left(\frac{1}{12}\beta y_0^4 - \frac{1}{3}C_4y_0^3 - C_5y_0^2 - c_0y_0^2\right),\\
&a_1 = C_8,\ a_3 = 2C_1C_7y_0,\\
&a_7 = -c_1(c_1F_3(T) + F_4(T)(C_1y_0 + C_2)).
\end{aligned} \tag{19}
$$

Then we discuss the solution of Eq. (14), the balancing procedure yields $n = 2$ for Eq. (14), therefore, we may choose

$$
\begin{aligned}
A = {} & f(Z,T) + f_1(T)\mathrm{sech}(W_1(T)X + W_2(T)Z\\
&+ W_3(T)) + f_2(T)\mathrm{sech}(W_1(T)X\\
&+ W_2(T)Z + W_3(T))^2,
\end{aligned} \tag{20}
$$

Substituting Eq. (20) into Eq. (14), we can get $f(Z,T)$, $f_1(T)$, $f_2(T)$, $W_1(T)$, $W_2(T)$, and $W_3(T)$ by vanishing sech,

$$f(Z,T) = g_1(T)Z + g_2(T), \quad f_1(T) = 0, \quad f_2(T) = \frac{12C_{10}^2 a_2^{1/3}}{a_3^{1/3}},$$

$$W_1(T) = \frac{C_{10}a_3^{1/3}}{a_2^{1/3}}, \quad W_2(T) = \int \frac{-a_3 g_1(T)W_1(T)}{a_1}\,\mathrm{d}T + C_9,$$

$$W_3(T) = \int \left(\frac{c_1^2 W_2(T)^2 F_3(T)}{C_8 W_1(T)} + \frac{c_1 W_2(T)^2 F_4(T)}{C_8 W_1(T)} C_1 y_0 + \frac{c_1 W_2(T)^2 F_4(T) C_2}{C_8 W_1(T)} + \frac{2}{3} W_1(T)^3 C_4 y_0^3 \right.$$
$$\left. - \frac{2W_1(T)C_1C_7 y_0 g_2(T)}{C_8} + 2W_1(T)^3 c_0 y_0^2 - \frac{1}{6} W_1(T)^3 \beta y_0^4 + 2W_1(T)^3 C_5 y_0^2 \right) \mathrm{d}T + C_{11}, \tag{21}$$

where $g_1(T)$, $g_2(T)$ are arbitrary functions of T, C_9, C_{10} and C_{11} are arbitrary constants. Then, we have

$$A = g_1(T)Z + g_2(T) + \frac{12C_{10}^2 a_2^{1/3}}{a_3^{1/3}} \mathrm{sech}(W_1(T)X + W_2(T)Z + W_3(T))^2. \tag{22}$$

According to Eqs. (20) and (21), we can obtain one possible approximate solution of Eq. (3) in the form

$$\begin{aligned} \psi &\approx U_0(y) + \epsilon U_1(y,T) + \epsilon\phi_1 + \epsilon^2\phi_2 \\ &= U_0(y) + \epsilon U_1(y,T) + \epsilon AG + \epsilon^2 \left(G_1 A^2 + G_2 A + G_3 \int A_Z \,\mathrm{d}X \right), \end{aligned} \tag{23}$$

where A, G, G_1, G_2, G_3, $U_0(y)$ and $U_1(y,k)$ satisfy Eqs.(18)–(22).

When we define uncertain parameters, some new soliton solutions can be obtained as shown in Figs. 1(a) and 1(b).

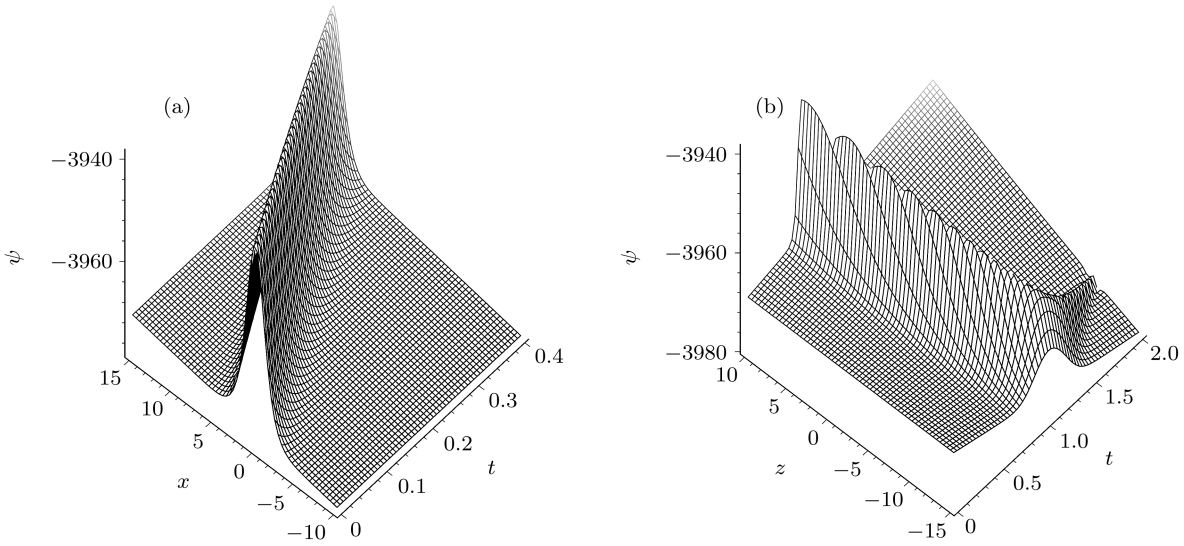


Fig. 1. Evolution of solutions (23) with the parameters: $C_1 = 1$, $\beta = 1$, $y_0 = 8$, $\epsilon = 0.1$, $c_1 = 1$, $c_0 = 1$, $g_1(T) = T$, $C_6 = 0$, $C_9 = 0$, $g_2(T) = 0$, $F_1(T) = T$, $C_8 = 1$, $C_4 = 1$, $C_5 = 1$, $C_7 = 1/12$, $C_{11} = 0$, $C_{10} = 36/106^{2/3}$, $F_3(T) = 1$, $C_2 = 0$, $F_4(T) = 1/T$, $C_3 = 0$..

Next, we discuss the solution of Eq. (3) with the generalized cKP equation. In a similar way, equation (15) should satisfy

$$\begin{aligned} &a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = 0, \\ &U_{0yyy}(y)A_X G - U_{0y}(y)A_X G_{yy} - c_0 A_X G_{yy} = 0, \\ &-G_{yy}G_y + GG_{yyy} + 2U_{0yyy}(y)G_1 - 2U_{0y}(y)G_{1yy} - 2c_0 G_{1yy} = 0, \\ &U_{0yyy}(y)G_3 - U_{0y}(y)G_{3yy} - c_0 G_{3yy} - c_1 G_{yy} = 0, \\ &U_{1yyy}(y,T)G - U_{1y}(y,T)G_{yy} + \beta_1 G + U_{0yyy}(y)G_2 - U_{0y}(y)G_{2yy} - c_0 G_{2yy} = 0, \end{aligned} \tag{24}$$

We obtain the solution of Eq. (24) by tedious calculations,

$$\begin{aligned}
&G_1 = P_5(T), \quad G_3 = c_1 P_3(T) + P_4(T)(D_1 y + D_2),\\
&G = P_5(T)P_4(T)D_1, \quad G_2 = P_3(T)(D_1 y + D_2),\\
&U_0(y) = -\frac{1}{6}\beta y^3 + \frac{1}{2}D_4 y^2 + D_5 y + D_6,\\
&U_1(y,T) = \frac{1}{2}D_1 y^2 + D_2 y + D_3 + P_1(T),\\
&U_i(y,T) = 0 \quad (i \geq 2),
\end{aligned} \tag{25}$$

where $D_i, i = 1, 2, \ldots, 6$ are arbitrary constants, $P_i(T), i = 1, 3, \ldots, 5$ are arbitrary functions of T. Then we obtain

$$\begin{aligned}
a_1 &= P_5(T)P_4(T)D_1,\\
a_2 &= \frac{1}{24}P_5(T)P_4(T)D_1 y_0^2(\beta y_0^2 - 4D_4 y_0\\
&\quad - 12c_0 - 12D_5),\\
a_3 &= P_5(T)(P_4(T)P_3(T)D_1^2 + 2D_1 y_0 - 2D_2),\\
a_7 &= -c_1^2 P_3(T) - c_1 P_4(T)(D_1 y_0 + D_2),\\
a_{11} &= P_{5T}(T)P_4(T)D_1 + P_5(T)P_{4T}(T)D_1.
\end{aligned} \tag{26}$$

Now we discuss the solution of Eq. (15). If we let

$$\begin{aligned}
P_4(T) &= \frac{-2(D_1 y_0 - D_2)}{D_1(-1 + P_3(T)D_1)},\\
D_4 &= \frac{1}{4}\frac{(-24 - 12c_0 y_0^2 + \beta y_0^4 - 12D_5 y_0^2)}{y_0^3},
\end{aligned} \tag{27}$$

and $P_3(T)$, $P_5(T)$ satisfy

$$\begin{aligned}
&\frac{1}{2}\frac{c_1(-c_1 P_3(T)D_1 + c_1 P_3(T)^2 D_1^2 - 2D_1^2 y_0^2 + 2D_2^2)}{D_1 P_5(T)(D_1 y_0 - D_2)} = \frac{3\sigma_1^2}{T^2},\\
&\frac{(-P_{5T}(T) + P_{5T}(T)P_3(T)D_1 - P_5(T)P_{3T}(T)D_1)}{(-1 + P_3(T)D_1)P_5(T)} = \frac{1}{2T},
\end{aligned} \tag{28}$$

where $\sigma_1^2 = \pm 1$, which also arises in water waves. The cKP equation is known to be completely integrable.[22] According to Eqs. (27) and (28), equation (15) becomes

$$\begin{aligned}
&A_T + A_{XXX} + AA_X\\
&\quad + \frac{3\sigma_1^2}{T^2}\int A_{ZZ}\,\mathrm{d}X + \frac{1}{2T}A = 0,
\end{aligned} \tag{29}$$

we know the solution[16] of Eq. (49):

$$\begin{aligned}
A &= 3\lambda^2 \mathrm{sech}^2(\xi),\\
\xi &= \frac{1}{2}\Bigg[\lambda X - \frac{\lambda\sigma_2 T Z^2}{12}\\
&\quad - \frac{T(\lambda^4 - \lambda\delta Z + 3\sigma_2\delta^2)}{\lambda} + \gamma\Bigg],
\end{aligned} \tag{30}$$

where $\sigma_2 = \pm 1$, λ, δ, γ are integral constants. Then we obtain an approximate solution of Eq. (3):

$$\begin{aligned}
\psi &\approx U_0(y) + \epsilon U_1(y,T) + \epsilon\phi_1 + \epsilon^2\phi_2\\
&= U_0(y) + \epsilon U_1(y,T) + \epsilon AG\\
&\quad + \epsilon^2\left(G_1 A^2 + G_2 A + G_3\int A_Z\,\mathrm{d}X\right),
\end{aligned} \tag{31}$$

where $U_0(y)$, $U_1(y,T)$, G, G_1, G_2, G_3 and A satisfy Eqs. (25)–(28) and (30).

When we define uncertain parameters, some new soliton solutions can be obtained. The evolution of solution (31) with some different parameters are shown in Figs. 2(a) and 2(b).

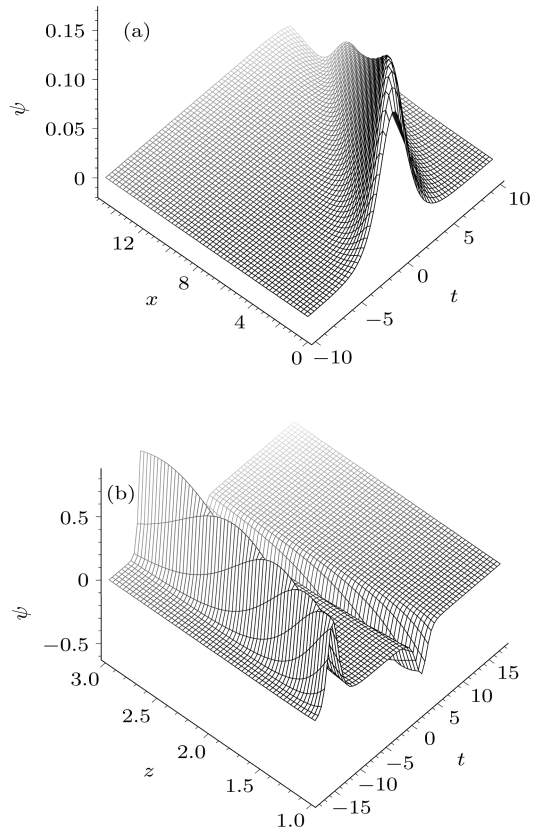


Fig. 2. Evolution of solution (31) with the parameters: $P_1(T) = 0$, $\lambda = 1$, $\epsilon 1 = -1$, $\epsilon = 0.1$, $\delta = 1$, $\gamma = 1$, $D_3 = 0$, $c_0 = 1$, $D_2 = 0$, $y_0 = 1$, $\beta = 0$, $D_1 = 1$, $D_6 = 0$, $D_5 = 0$, $D_4 = -0.1$.

4. The solution of BPV equation from generalized mKP equation by symmetry group method

As is well known, the generalized mKP equation is a universal model for the propagation of weakly nonlinear dispersive long waves which are essentially one directional with weak transverse effects. There are many difficulties in obtaining the solution of the generalized mKP equation, but it is easier to find the solution of the mKP equation than that of the generalized mKP equation. So we give a theorem to change the generalized mKP equation into the mKP equation by a generalized symmetry group method first, then we can find a more preferable research BPV equation with the help of the theorem.

First, let

$$A(X,Z,T)=v_X, \tag{32}$$

where $v\equiv v(X,Z,T)$. Substituting Eq. (32) into the generalized mKP equation, we have

$$a_1v_{XT}+v_{XXXX}+a_5v_{XX}v_X^2 + a_6v_{XX}v_Z+a_7v_{ZZ}=0. \tag{33}$$

Secondly, let

$$v=\alpha+\gamma V(\xi,\eta,\tau)=\alpha+\gamma V, \tag{34}$$

where α, γ, ξ, η, and τ are functions of $\{X,Z,T\}$, V satisfies the following equation

$$V_{\xi\xi\xi\xi}+d_1V_{\xi\tau}+d_3V_{\xi\xi}V_\xi^2 + d_4V_{\xi\xi}V_\eta+d_5V_{\eta\eta}=0, \tag{35}$$

where d_1, d_3, d_4 and d_5 are non-zero arbitrary constants. Substituting Eq. (34) into Eq. (33), then eliminating $V_{\xi\xi\xi\xi}$ by using Eq. (35), from that, the remained determining equations of the functions ξ, η, τ, α, γ can be obtained by vanishing the coefficients of V and its derivatives. Then the general solution of the determining equations are found out by tedious calculations. The result reads

$$\begin{aligned}
\xi &= c_2\left(\frac{a_1\tau_{0T}}{d_1}\right)^{1/3}X+\xi_0,\quad \tau=\tau_0,\quad \gamma=\delta\sqrt{\frac{d_3}{a_5}},\quad \eta=\sqrt{\frac{a_5}{d_3}}c_2^2\left(\frac{a_1\tau_{0T}}{d_1}\right)^{2/3}\frac{d_4Z}{a_6\delta}+\eta_0,\\
\xi_0 &= \frac{(-2a_1a_5a_{1T}\tau_{0T}-2a_1^2a_5\tau_{0TT}+3a_1^2a_{5T}\tau_{0T})}{d_5d_3a_6^2a_1^{2/3}\tau_{0T}^{2/3}d_1^{1/3}}\frac{1}{12}c_2d_4^2Z^2+\xi_1Z+\xi_2,\\
\alpha &= -\frac{1}{4}\int a_1^{1/3}\tau_{0T}^{1/3}\left(\frac{a_6^2\xi_{0Z}^2d_4^2d_1^{1/3}+4c_2\xi_{0T}a_1^{4/3}\tau_{0T}^{1/3}a_5d_4^2}{\tau_{0T}a_5d_4^2a_1a_6}+\frac{4a_6^2\xi_{0Z}^2d_3d_5d_1^{1/3}}{\tau_{0T}a_5d_4^2a_1a_6}\right)d_1^{1/3}c_2\,\mathrm{d}Z\\
&\quad -\frac{1}{2}\frac{a_6\xi_{0Z}d_1^{1/3}X}{c_2a_1^{1/3}\tau_{0T}^{1/3}a_5}+\alpha_0,
\end{aligned} \tag{36}$$

where $\tau_0\equiv\tau_0(T)$, $\xi_0\equiv\xi_0(Z,T)$, $\xi_1\equiv\xi_1(T)$, $\xi_2\equiv\xi_2(T)$, $\eta_0\equiv\eta_0(T)$, $\alpha_0\equiv\alpha_0(T)$, $\delta=\pm1$, and $c_2=1$, $-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$, $-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$. At the same time, it should satisfy four conditions:

$$\begin{aligned}
&a_7a_5d_4^2-d_5d_3a_6^2=0,\\
&6a_1\eta_{0T}\sqrt{d_3a_5}a_6^2\delta(a_1\tau_{0T})^{1/3}d_1^{2/3}d_4+(-6a_1^2a_5\tau_{0T}d_4^2a_{6T}+4a_1a_5d_4^2a_6a_{1T}\tau_{0T}+4a_1^2a_5d_4^2a_6\tau_{0TT}\\
&\quad+3a_1^2\tau_{0T}d_4^2a_{5T}a_6)c_2^2Z+12d_5d_3a_6^3\xi_{0Z}c_2a_1^{2/3}\tau_{0T}^{2/3}d_1^{1/3}=0,\\
&4\tau_{0T}a_{1T}a_5d_5d_3-3a_1d_4^2\tau_{0T}a_{5T}+2d_4^2\tau_{0T}a_5a_{1T}+2a_1d_4^2a_5\tau_{0TT}+4a_1\tau_{0TT}a_5d_5d_3=0,\\
&3a_6^3\xi_{0Z}d_4^2d_5d_3a_1^{2/3}\tau_{0T}^{2/3}d_1^{1/3}\xi_{0ZZ}+12d_5^2d_3^2a_6^3\xi_{0Z}a_1^{2/3}\tau_{0T}^{2/3}d_1^{1/3}\xi_{0ZZ}+3\tau_{0T}d_4^4a_1^2c_2a_{6T}\xi_{0Z}a_5\\
&\quad+3\tau_{0T}d_4^4a_1^2c_2a_6\xi_{0ZT}a_5-\tau_{0T}d_4^4a_1c_2a_6\xi_{0Z}a_5a_{1T}-d_4^4a_1^2c_2a_6\xi_{0Z}a_5\tau_{0TT}\\
&\quad-3\tau_{0T}d_4^4a_1^2c_2a_6\xi_{0Z}a_{5T}+6c_2\tau_{0T}\xi_{0ZT}a_1^2a_5d_4^2d_5d_3a_6=0.
\end{aligned} \tag{37}$$

In summery, we obtain the following theorem.

Theorem If $U\equiv U(X,Z,T)$ is a solution of the mKP equation (35), then

$$u=\alpha+\gamma U(\xi,\eta,\tau)$$

is the solution of the generalized mKP equation (33) on the conditions of Eq. (37), where α, γ, ξ, η, τ, δ and c_2 are given by Eq. (36).

According to the Theorem, we will research the BPV equation easier with the help of the mKP and generalized mKP equation.

Next, we will write down some types of solutions of Eq. (3) with the help of the known solutions for the (2+1)-dimensional mKP equation.

As known to all, mKP equation is

$$-u_T + u_{XXX} - 6u_X u^2 - 6u_X \int u_Z \,\mathrm{d}X + 3\int u_{ZZ}\,\mathrm{d}X = 0, \tag{38}$$

where $u \equiv u(X, Z, T)$.

Let

$$d_1 = -1, \quad d_3 = -6, \quad d_4 = -6, \quad d_5 = 3, \tag{39}$$

and replace $V(X, Z, T)$ with $V(\xi, \eta, \tau)$, equation (35) becomes

$$V_{XXXX} - V_{XT} - 6V_{XX}V_X^2 - 6V_{XX}V_Z + 3V_{ZZ} = 0, \tag{40}$$

equation (40) is equivalent to Eq. (38) under the transformation $u = V_X$. So we discuss Eq. (40) firstly in the following.

We know that the mKP equation possesses the following solution:

$$u = b(1 - \tanh(b(X - 2bZ - 16b^2T))), \tag{41}$$

where b is a non-zero arbitrary constant. According to the transformation $u = V_X$ and Eq. (41), we obtain the solution of Eq. (40):

$$V = bX + \ln(\mathrm{sech}(b(-X + 2bZ + 16b^2T)), \tag{42}$$

according to the Theorem and Eq. (32), we obtain the solution of the generalized mKP equation,

$$\begin{aligned} A = {} & \frac{1}{6a_6a_1\tau_0c_2a_5}\{6\sqrt{6}\delta\sqrt{a}bc_2^2a_1\tau_0(-a_1\tau_0)^{1/3}a_6[\tanh(b(-X + 2bZ + 16b^2T)) + 1] \\ & - 2c_2a_1a_5Z(a_{1T}\tau_0 + a_1\tau_{0TT}) - 3\xi_1a_6^2(-a_1\tau_0)^{2/3}\}, \end{aligned} \tag{43}$$

where a_5 is a non-zero arbitrary constant. In order to obtain the solution of Eq. (3), we need to know $U_0(y)$, $U_1(y, T)$, G, G_1, G_2 and G_3, which should satisfy $a_3 = 0$, $a_4 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, Eqs. (10), (12) and (37). After tedious calculations, we obtain:

$$\begin{aligned} & G = K_1y + K_2 + Q_5(T),\ G_1 = Q_6(T), \\ & G_2 = Q_2(T)K_6,\ G_3 = c_1Q_2(T) + Q_4(T)K_6, \\ & U_0(y) = -\frac{1}{6}\beta y^3 + \frac{1}{2}K_3y^2 + K_4y + K_5, \\ & U_1(y, T) = K_6y + K_7 + Q_1(T), \\ & U_i(y, T) = 0 \quad (i \geq 2), \end{aligned} \tag{44}$$

where $Q_i(T), i = 1, 2, 4, 5, 6$ are arbitrary function of T, $K_i, i = 1, 2, \ldots, 7$ are arbitrary constants. Then we get

$$\begin{aligned} a_2 = {} & \frac{1}{40}\beta K_1y_0^5 + \frac{1}{24}y_0^4(\beta K_2 - 2K_3K_1) \\ & - \frac{1}{6}y_0^3(c_0K_1 + K_4K_1 + K_3K_2) \\ & - \frac{1}{2}K_2y_0^2(c_0 + K_4), \\ a_1 = {} & K_1y_0 + K_2, \quad a_5 = 4K_1^2, \\ a_6 = {} & -2K_1(c_1Q_2(T) + Q_4(T)K_6), \\ a_7 = {} & -c_1(c_1Q_2(T) + Q_4(T)K_6). \end{aligned} \tag{45}$$

From that we obtain an approximate solution of Eq. (3)

$$\begin{aligned} \psi \approx {} & U_0(y) + \epsilon U_1(y, T) + \epsilon\phi_1 + \epsilon^2\phi_2 \\ = {} & U_0(y) + \epsilon U_1(y, T) + \epsilon AG \\ & + \epsilon^2\left(G_1A^2 + G_2A + G_3\int A_Z\,\mathrm{d}X\right), \end{aligned} \tag{46}$$

where A, G, G_1, G_2, G_3, $U_0(y)$ and $U_1(y, k)$ satisfy Eqs. (37), (43)–(44).

5. Summary and discussion

In summary, by the reductive perturbation method the generalized (2+1)-dimensional modified Kadomtsev–Petviashvili (mKP), Kadomtsev–Petviashvili (KP) and cylindrical Kadomtsev–Petviashvili (cKP) equations are derived from the baroclinic potential vorticity (BPV) equation. Some solutions of the KP, mKP and cKP equations are derived by a direct method and symmetry group direct method. Thus the solution of the generalized mKP

equation can be obtained from the mKP equation. Furthermore, some approximate solutions of the BPV equation are obtained from three types of generalized KP equations.

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