Symmetry and conserved quantities of discrete generalized Birkhoffian system*

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The Noether symmetry, the Mei symmetry and the conserved quantities of discrete generalized Birkhoffian system are studied in this paper. Using the difference discrete variational approach, the difference discrete variational principle of discrete generalized Birkhoffian system is derived. The discrete equations of motion of the system are established. The criterion of Noether symmetry and Mei symmetry of the system is given. The discrete Noether and Mei conserved quantities and the conditions for their existence are obtained. Finally, an example is given to show the applications of the results.

Keywords: discrete generalized Birkhoffian system, Noether symmetry, Mei symmetry, conserved quantity

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1. Introduction

The symmetries of mechanical systems have close connection with the associated conserved quantities. The modern methods of finding conserved quantities use the symmetries which are mainly Noether symmetry,[1] Lie symmetry[2] and Mei symmetry.[3,4] The Noether symmetry is the invariance of Hamiltonian action functional under infinitesimal transformation groups. The Lie symmetry is the invariance of differential equations of motion under infinitesimal transformation groups. Mei presented a new theory of symmetries, the form invariance of mechanical systems. Mei's form invariance is also called Mei symmetry, which means that the dynamical function in the equation of motion still satisfies the equation's primary form after some infinitesimal transformations. Recently, the symmetries and conserved quantities have been thoroughly studied in continuous mechanical systems.[5−15]

However, these symmetries and conserved quantities have been extended to discrete systems in the last few years. A number of authors have since proposed variational treatments of discrete mechanical systems. Zhang *et al.*[16,17] and Fu *et al.*[18] studied the discrete varitional principles of several constrained mechanical systems. Shi *et al.*[19,20] studied Lie symmetry, Mei symmetry and conserved quantities of general discrete holonomic systems. Lu *et al.*[21] studied Noether symmetry, Mei symmetry and the conserved quantities of discrete holonomic system. However, the symmetries and conserved quantities of the discrete generalized Birkhoffian system have never been studied so far.

In 2002, Guo *et al.*[22−26] presented a discrete difference variational principle and introduced the concept of Euler–Lagrange cohomology. Using the difference discrete variational approach, we study the Noether symmetry and the Mei symmetry of discrete generalized Birkhoffian system, give the criterion of Noether symmetry and Mei symmetry of the system, and obtain the discrete Noether and Mei conserved quantities.

2. Discrete equations of motion of the system

The generalized Birkhoffian equation can be written as

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=-\Lambda_\mu,\quad(\mu,\nu=1,\ldots,2n),\tag{1}$$

where $B=B(t,\boldsymbol{a})$ is called a Birkhoffian, $R_\mu=R_\mu(t,\boldsymbol{a})$ are called Birkhoffian functions, a^μ are variables, Λ_μ are additional items.

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The Birkhoffian action of the system can be expressed as

$$A = \int_{t_1}^{t_2} (R_\mu \dot{a}^\mu - B)\mathrm{d}t. \tag{2}$$

Choosing the infinitesimal transformations of time and coordinates as

$$\begin{aligned} &t^* = t + \delta_t t = t + \varepsilon\xi_0(t, \boldsymbol{a}), \\ &a^{\mu *}(t^*) = a^\mu(t) + \delta_t a^\mu(t) = a^\mu(t) + \varepsilon\xi_\mu(t, \boldsymbol{a}), \end{aligned} \tag{3}$$

where ε is an infinitesimal parameter, and δ_t represents the total variation, while ξ_0, ξ_μ are infinitesimal generators.

If the infinitesimal transformations (3) are the generalized quasi-symmetry transform of system (1), we have

$$\delta_t A = -\int_{t_1}^{t_2} \Lambda_\mu \delta a^\mu \mathrm{d}t. \tag{4}$$

In discrete mechanics, time interval (t_1, t_2) is discretized into a point sequence (t_k), $k = 0, 1, \ldots, N$, the discrete versions of a^μ, $\Lambda_\mu(t, \boldsymbol{a})$, $B(t, \boldsymbol{a})$, $R_\mu(t, \boldsymbol{a})$ are $a^{\mu,k} = a^\mu(t_k)$, $\Lambda_\mu^k = \Lambda_\mu(t_k, \boldsymbol{a}^k)$, $B_D^k = B_D(t_k, \boldsymbol{a}^k)$ and $R_\mu^k = R_\mu(t_k, \boldsymbol{a}^k)$, respectively.

Choosing the infinitesimal transformations of discrete time and discrete coordinates as

$$\begin{aligned} t_k^* &= t_k + \delta_t t_k = t_k + \varepsilon\xi_0^k(t_k, \boldsymbol{a}^k), \\ a^{\mu,k*}(t_k^*) &= a^{\mu,k}(t_k) + \delta_t a^{\mu,k}(t_k) \\ &= a^{\mu,k}(t_k) + \varepsilon\xi_\mu^k(t_k, \boldsymbol{a}^k), \end{aligned} \tag{5}$$

where ε is an infinitesimal parameter, while ξ_0^k, ξ_μ^k are discrete infinitesimal generators.

Under the infinitesimal transformations (5), the corresponding version of Eqs. (2) and (4) becomes

$$A_D = \sum_k (t_{k+1} - t_k)(R_\mu^k \Delta a^{\mu,k} - B_D^k), \tag{6}$$

$$\delta_t A_D = -\sum_k (t_{k+1} - t_k)(\Lambda_\mu^k \delta a^{\mu,k}) \tag{7}$$

with the difference $\Delta a^{\mu,k}$ of $a^{\mu,k}$ at t_k defined by

$$\Delta a^{\mu,k} = \frac{a^{\mu,k+1} - a^{\mu,k}}{t_{k+1} - t_k}, \tag{8}$$

where $\delta a^{\mu,k}$ satisfy the following relational expressions[25]

$$\delta a^{\mu,k} = \delta_t a^{\mu,k} - \delta_t t_k \Delta a^{\mu,k}. \tag{9}$$

From Eqs. (6) and (7), we obtain

$$\begin{aligned} &\delta_t \sum_k (t_{k+1} - t_k)(R_\mu^k \Delta a^{\mu,k} - B_D^k) + \sum_k (t_{k+1} - t_k)(\Lambda_\mu^k \delta a^{\mu,k}) \\ &= \sum_k \{(R_\mu^k \Delta a^{\mu,k} - B_D^k)\delta_t(t_{k+1} - t_k) + (t_{k+1} - t_k)\delta_t(R_\mu^k \Delta a^{\mu,k} - B_D^k) + (t_{k+1} - t_k)\Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k \Delta a^{\mu,k})\} \\ &= \sum_k (t_{k+1} - t_k)\{R_\mu^k \Delta a^{\mu,k} \Delta\delta_t t_k - B_D^k \Delta\delta_t t_k + \Delta a^{\mu,k}\delta_t R_\mu^k + R_\mu^k \delta_t \Delta a^{\mu,k} \\ &\quad - \delta_t B_D^k + \Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k \Delta a^{\mu,k})\} = 0. \end{aligned} \tag{10}$$

By using the Leibniz law

$$\Delta(f_k g_k) = \Delta f_k \cdot g_k + f_{k+1} \cdot \Delta g_k, \tag{11}$$

and the following relational expressions[25]

$$\delta_t \Delta a^{\mu,k} = \Delta\delta_t a^{\mu,k} - \Delta a^{\mu,k} \Delta\delta_t t_k. \tag{12}$$

Then equation (10) turns into

$$\begin{aligned} &\sum_k (t_{k+1} - t_k)\Big\{ R_\mu^k \Delta a^{\mu,k} \Delta\delta_t t_k - \Delta(B_D^{k-1}\delta_t t_k) + \Delta B_D^{k-1}\delta_t t_k \\ &+ \frac{\partial R_\mu^k}{\partial t_k}\delta_t t_k \Delta a^{\mu,k} + \frac{\partial R_\nu^k}{\partial a^{\mu,k}}\delta_t a^{\mu,k}\Delta a^{\nu,k} + R_\mu^k(\Delta\delta_t a^{\mu,k} - \Delta a^{\mu,k}\Delta\delta_t t_k) \\ &- \frac{\partial B_D^k}{\partial a^{\mu,k}}\delta_t a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\delta_t t_k + \Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k \Delta a^{\mu,k})\Big\} \end{aligned}$$

$$= \sum_k (t_{k+1} - t_k)\Big\{\Delta(R_\mu^{k-1}\delta_t a^{\mu,k} - B_D^{k-1}\delta_t t_k) + \Big(\Delta B_D^{k-1} - \frac{\partial B_D^k}{\partial t_k} + \frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \Lambda_\mu^k \Delta a^{\mu,k}\Big)\delta_t t_k$$
$$+ \Big(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \Delta R_\mu^{k-1} - \frac{\partial B_D^k}{\partial a^{\mu,k}} + \Lambda_\mu^k\Big)\delta_t a^{\mu,k}\Big\} = 0. \tag{13}$$

Adopting the fixed boundary $\delta_t t_0 = \delta_t t_N = 0$ and $\delta_t a^{\mu,0} = \delta_t a^{\mu,N} = 0$, from Eq. (13), we obtain the discrete equations of motion of the system

$$\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \Delta R_\mu^{k-1} - \frac{\partial B_D^k}{\partial a^{\mu,k}} + \Lambda_\mu^k = 0, \tag{14}$$
$$\Delta B_D^{k-1} - \frac{\partial B_D^k}{\partial t_k} + \frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \Lambda_\mu^k \Delta a^{\mu,k} = 0. \tag{15}$$

3. Noether symmetry and Noether conserved quantities of discrete generalized Birkhoffian system

In order to obtain the discrete Noether conserved quantities and the conditions for their existence, we give a Noether equation first.

According to Eqs. (10) and (12), we obtain

$$R_\mu^k\Delta a^{\mu,k}\Delta\delta_t t_k - B_D^k\Delta\delta_t t_k + \Delta a^{\mu,k}\delta_t R_\mu^k + R_\mu^k\delta_t\Delta a^{\mu,k} - \delta_t B_D^k + \Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k\Delta a^{\mu,k})$$
$$= R_\mu^k\Delta a^{\mu,k}\Delta\delta_t t_k - B_D^k\Delta\delta_t t_k + \frac{\partial R_\mu^k}{\partial t_k}\delta_t t_k\Delta a^{\mu,k} + \frac{\partial R_\nu^k}{\partial a^{\mu,k}}\delta_t a^{\mu,k}\Delta a^{\nu,k}$$
$$+ R_\mu^k(\Delta\delta_t a^{\mu,k} - \Delta a^{\mu,k}\Delta\delta_t t_k) - \frac{\partial B_D^k}{\partial a^{\mu,k}}\delta_t a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\delta_t t_k + \Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k\Delta a^{\mu,k})$$
$$= \Big(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\Big)\delta_t t_k + \Big(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}}\Big)\delta_t a^{\mu,k} - B_D^k\Delta\delta_t t_k$$
$$+ R_\mu^k\Delta\delta_t a^{\mu,k} + \Lambda_\mu^k(\delta_t a^{\mu,k} - \delta_t t_k\Delta a^{\mu,k}) = 0. \tag{16}$$

Take note of

$$\delta_t t_k = \varepsilon\xi_0^k, \quad \delta_t a^{\mu,k} = \varepsilon\xi_\mu^k, \tag{17}$$

due to the randomicity of ε, we obtain

$$\Big(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\Big)\xi_0^k + \Big(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}}\Big)\xi_\mu^k - B_D^k\Delta\xi_0^k + R_\mu^k\Delta\xi_\mu^k + \Lambda_\mu^k(\xi_\mu^k - \xi_0^k\Delta a^{\mu,k}) = 0. \tag{18}$$

Proposition 1 For discrete generalized Birkhoffian systems (14) and (15), if there exists a discrete gauge function $G_N^k = G_N(t_k, \boldsymbol{a}^k)$ satisfying equation

$$\Big(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\Big)\xi_0^k + \Big(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}}\Big)\xi_\mu^k - B_D^k\Delta\xi_0^k$$
$$+ R_\mu^k\Delta\xi_\mu^k + \Lambda_\mu^k(\xi_\mu^k - \xi_0^k\Delta a^{\mu,k}) + \Delta G_N^k = 0, \tag{19}$$

then the Noether symmetry of the system leads to the discrete Noether conserved quantities

$$I_D = R_\mu^{k-1}\xi_\mu^k - B_D^{k-1}\xi_0^k + G_N^k = \text{const.}, \tag{20}$$

and equation (19) is called the discrete Noether identity.

Proof By using the Leibniz law (11), then equation (19) turns into

$$\Big(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\Big)\xi_0^k + \Big(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}}\Big)\xi_\mu^k - B_D^k\Delta\xi_0^k + R_\mu^k\Delta\xi_\mu^k + \Lambda_\mu^k(\xi_\mu^k - \xi_0^k\Delta a^{\mu,k}) + \Delta G_N^k$$

$$
\begin{aligned}
&= \left(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k}\right)\xi_0^k + \left(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}}\right)\xi_\mu^k - \Delta(B_D^{k-1}\xi_0^k) \\
&\quad + \Delta B_D^{k-1}\xi_0^k + \Delta(R_\mu^{k-1}\xi_\mu^k) - \Delta R_\mu^{k-1}\xi_\mu^k + \Lambda_\mu^k(\xi_\mu^k - \xi_0^k\Delta a^{\mu,k}) + \Delta G_N^k \\
&= \left(\frac{\partial R_\mu^k}{\partial t_k}\Delta a^{\mu,k} - \frac{\partial B_D^k}{\partial t_k} + \Delta B_D^{k-1} - \Lambda_\mu^k\Delta a^{\mu,k}\right)\xi_0^k \\
&\quad + \left(\frac{\partial R_\nu^k}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \frac{\partial B_D^k}{\partial a^{\mu,k}} - \Delta R_\mu^{k-1} + \Lambda_\mu^k\right)\xi_\mu^k + \Delta(R_\mu^{k-1}\xi_\mu^k - B_D^{k-1}\xi_0^k + G_N^k) = 0, \qquad (21)
\end{aligned}
$$

substituting Eqs. (14) and (15) into Eq. (21), we obtain

$$\Delta(R_\mu^{k-1}\xi_\mu^k - B_D^{k-1}\xi_0^k + G_N^k) = 0. \qquad (22)$$

4. Mei symmetry and Mei conserved quantities of discrete generalized Birkhoffian system

We introduce the recursive operators of discrete system for any discrete variable or function

$$R_\pm f(z_k) = f(z_{k\pm1}), \qquad (23)$$

the vector field of generators is

$$X_D^{(0)} = \xi_0^k\frac{\partial}{\partial t_k} + \xi_\mu^k\frac{\partial}{\partial a^{\mu,k}}. \qquad (24)$$

The dynamic functions B_D^k, B_D^{k-1}, R_μ^k, R_μ^{k-1} and Λ_μ^k become $B_D^{k^*}$, $B_D^{k-1^*}$, $R_\mu^{k^*}$, $R_\mu^{k-1^*}$ and $\Lambda_\mu^{k^*}$ respectively under the infinitesimal transformations (5), and expanding $B_D^{k^*}$, $B_D^{k-1^*}$, $R_\mu^{k^*}$, $R_\mu^{k-1^*}$ and $\Lambda_\mu^{k^*}$, we have

$$B_D^{k^*} = B_D(t_k^*, \boldsymbol{a}^{k^*}) = B_D^k + \varepsilon X_D^{(0)}(B_D^k) + O(\varepsilon^2) + \cdots, \qquad (25)$$

$$B_D^{k-1^*} = B_D(t_{k-1}^*, \boldsymbol{a}^{k-1^*}) = B_D^{k-1} + \varepsilon R_- X_D^{(0)}(B_D^k) + O(\varepsilon^2) + \cdots, \qquad (26)$$

$$R_\mu^{k^*} = R_\mu(t_k^*, \boldsymbol{a}^{k^*}) = R_\mu^k + \varepsilon X_D^{(0)}(R_\mu^k) + O(\varepsilon^2) + \cdots, \qquad (27)$$

$$R_\mu^{k-1^*} = R_\mu(t_{k-1}^*, \boldsymbol{a}^{k-1^*}) = R_\mu^{k-1} + \varepsilon R_- X_D^{(0)}(R_\mu^k) + O(\varepsilon^2) + \cdots, \qquad (28)$$

$$\Lambda_\mu^{k^*} = \Lambda_\mu(t_k^*, \boldsymbol{a}^{k^*}) = \Lambda_\mu^k + \varepsilon X_D^{(0)}(\Lambda_\mu^k) + O(\varepsilon^2) + \cdots. \qquad (29)$$

Definition If the forms of Eqs. (14) and (15) keep invariant, when replacing dynamic functions B_D^k, B_D^{k-1}, R_μ^k, R_μ^{k-1} and Λ_μ^k before transforming by dynamic functions $B_D^{k^*}$, $B_D^{k-1^*}$, $R_\mu^{k^*}$, $R_\mu^{k-1^*}$ and $\Lambda_\mu^{k^*}$ after transforming, then the form invariance is called Mei symmetry of discrete generalized Birkhoffian system.

Substituting Eqs. (25)–(29) into Eqs. (14) and (15), ignoring ε^2 and the higher-order infinitesimal terms, we obtain the following criterion of Mei symmetry.

Criterion For the discrete generalized Birkhoffian systems (14) and (15), if the infinitesimal generators ξ_0^k, ξ_μ^k satisfy

$$\frac{\partial X_D^{(0)}(R_\nu^k)}{\partial a^{\mu,k}}\Delta a^{\nu,k} - \Delta R_- X_D^{(0)}(R_\mu^k) - \frac{\partial X_D^{(0)}(B_D^k)}{\partial a^{\mu,k}} + X_D^{(0)}(\Lambda_\mu^k) = 0, \qquad (30)$$

$$\Delta R_- X_D^{(0)}(B_D^k) - \frac{\partial X_D^{(0)}(B_D^k)}{\partial t_k} + \frac{\partial X_D^{(0)}(R_\mu^k)}{\partial t_k}\Delta a^{\mu,k} - X_D^{(0)}(\Lambda_\mu^k)\Delta a^{\mu,k} = 0, \qquad (31)$$

the invariance is called Mei symmetry of discrete generalized Birkhoffian system.

Proposition 2 For the discrete Birkhoffian systems (14) and (15), if the infinitesimal generators ξ_0^k, ξ_μ^k of Mei symmetry and a discrete gauge function $G_M^k = G_M(t_k, \boldsymbol{a}^k)$ satisfy the following structure equation

$$X_D^{(0)}\{X_D^{(0)}(R_\mu^k)\}\Delta a^{\mu,k} - X_D^{(0)}\{X_D^{(0)}(B_D^k)\} - X_D^{(0)}(B_D^k)\Delta\xi_0^k$$

$$+X_D^{(0)}(R_\mu^k)\Delta\xi_\mu^k+X_D^{(0)}(\Lambda_\mu^k)(\xi_\mu^k-\xi_0^k\Delta a^{\mu,k})+\Delta G_M^k=0, \tag{32}$$

then the Mei symmetry of the system leads to the discrete Mei conserved quantities

$$I_D=R_-X_D^{(0)}(R_\mu^k)\xi_\mu^k-R_-X_D^{(0)}(B_D^k)\xi_0^k+G_M^k=\text{const.} \tag{33}$$

Equation (32) is called the structure equations of Mei symmetry of discrete generalized Birkhoffian system.

Proof By using the Leibniz law (11), then equation (32) turns into

$$\begin{aligned}
&X_D^{(0)}\{X_D^{(0)}(R_\mu^k)\}\Delta a^{\mu,k}-X_D^{(0)}\{X_D^{(0)}(B_D^k)\}-X_D^{(0)}(B_D^k)\Delta\xi_0^k\\
&+X_D^{(0)}(R_\mu^k)\Delta\xi_\mu^k+X_D^{(0)}(\Lambda_\mu^k)(\xi_\mu^k-\xi_0^k\Delta a^{\mu,k})+\Delta G_M^k\\
=&\ \frac{\partial X_D^{(0)}(R_\nu^k)}{\partial a^{\mu,k}}\Delta a^{\nu,k}\xi_\mu^k+\frac{\partial X_D^{(0)}(R_\mu^k)}{\partial t_k}\Delta a^{\mu,k}\xi_0^k-\frac{\partial X_D^{(0)}(B_D^k)}{\partial a^{\mu,k}}\xi_\mu^k-\frac{\partial X_D^{(0)}(B_D^k)}{\partial t_k}\xi_0^k\\
&-\Delta[R_-X_D^{(0)}(B_D^k)\xi_0^k]+\Delta[R_-X_D^{(0)}(B_D^k)]\xi_0^k+\Delta[R_-X_D^{(0)}(R_\mu^k)\xi_\mu^k]\\
&-\Delta[R_-X_D^{(0)}(R_\mu^k)]\xi_\mu^k+X_D^{(0)}(\Lambda_\mu^k)(\xi_\mu^k-\xi_0^k\Delta a^{\mu,k})+\Delta G_M^k\\
=&\left\{\frac{\partial X_D^{(0)}(R_\nu^k)}{\partial a^{\mu,k}}\Delta a^{\nu,k}-\Delta R_-X_D^{(0)}(R_\mu^k)-\frac{\partial X_D^{(0)}(B_D^k)}{\partial a^{\mu,k}}+X_D^{(0)}(\Lambda_\mu^k)\right\}\xi_\mu^k\\
&+\left\{\Delta R_-X_D^{(0)}(B_D^k)-\frac{\partial X_D^{(0)}(B_D^k)}{\partial t_k}+\frac{\partial X_D^{(0)}(R_\mu^k)}{\partial t_k}\Delta a^{\mu,k}-X_D^{(0)}(\Lambda_\mu^k)\Delta a^{\mu,k}\right\}\xi_0^k\\
&+\Delta\{R_-X_D^{(0)}(R_\mu^k)\xi_\mu^k-R_-X_D^{(0)}(B_D^k)\xi_0^k+G_M^k\},
\end{aligned} \tag{34}$$

substituting Eqs. (30) and (31) into Eq. (34), we have

$$\Delta\{R_-X_D^{(0)}(R_\mu^k)\xi_\mu^k-R_-X_D^{(0)}(B_D^k)\xi_0^k+G_M^k\}=0. \tag{35}$$

5. An example

The discrete Birkhoffian system is

$$\begin{aligned}
&R_1^k=a^{3,k},\quad R_2^k=a^{4,k},\quad R_3^k=R_4^k=0,\\
&B_D^k=a^{2,k}+\frac{1}{2}\{(a^{3,k})^2+(a^{4,k})^2\},
\end{aligned} \tag{36}$$

the additional items are

$$\Lambda_1^k=a^{1,k}+t_k,\quad \Lambda_2^k=0,\quad \Lambda_3^k=a^{3,k},\quad \Lambda_4^k=0. \tag{37}$$

Let us study the Noether symmetry, the Mei symmetry and the resulting conserved quantities of discrete generalized Birkhoffian system.

According to Eqs. (14) and (15), we obtain the discrete equations of motion of the system

$$\begin{aligned}
&-\Delta a^{3,k-1}+a^{1,k}+t_k=0,\quad -\Delta a^{4,k-1}-1=0,\\
&\Delta a^{1,k}=0,\quad \Delta a^{2,k}-a^{4,k}=0,\\
&\Delta B_D^{k-1}-a^{3,k}\Delta a^{3,k}=0.
\end{aligned} \tag{38}$$

Firstly, we study the Noether symmetry. Choose infinitesimal generators as

$$\xi_0^k=0,\quad \xi_1^k=1,\quad \xi_2^k=a^{4,k-1},\quad \xi_3^k=\xi_4^k=0. \tag{39}$$

Substituting Eq. (39) into the discrete Noether identity (19), we obtain the gauge function

$$G_N^k=a^{2,k-1}+a^{2,k}-a^{3,k-1}. \tag{40}$$

According to Proposition 1, Noether symmetry of the system (36) can lead to a Noether conserved quantity

$$I_D=a^{2,k-1}+a^{2,k}+(a^{4,k-1})^2=\text{const.} \tag{41}$$

Then we study Mei symmetry. Choose infinitesimal generators as

$$\begin{aligned}
&\xi_0^k=-a^{4,k},\quad \xi_1^k=a^{4,k}-1,\\
&\xi_2^k=a^{4,k},\quad \xi_3^k=0,\quad \xi_4^k=-1.
\end{aligned} \tag{42}$$

It is easy to verify that they satisfy Eqs. (30) and (31), so the infinitesimal generators (42) are Mei symmetrical for the system (36). Also, they satisfy Eq. (32).

From Eq. (32), we obtain the gauge function

$$G_M^k=a^{4,k}+a^{2,k}-t_k. \tag{43}$$

According to Proposition 2, Mei symmetry of the system (36) can lead to a Mei conserved quantity

$$I_D=a^{2,k}-t_k=\text{const.} \tag{44}$$

6. Conclusion

We studied Noether symmetry and Mei symmetry of the discrete generalized Birkhoffian system by using the difference discrete variational approach, gave discrete Noether equation and the structure equation of Mei symmetry of the system and obtained the discrete Noether and Mei conserved quantities. Using the difference discrete variational approach, we dispersed the original continuous system and retained the structure and nature of the original system as much as possible. Many geometric properties of continuous mechanical systems are also the corresponding existence in discrete systems. When $t_{k+1} - t_k \to 0$, the results presented in this paper will turn into the corresponding continuous situation naturally.

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