Atomic coherent states as energy eigenstates of a Hamiltonian describing a two-dimensional anisotropic harmonic potential in a uniform magnetic field*

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(Received 2 April 2010; revised manuscript received 16 August 2010)

In this paper we find that a set of energy eigenstates of a two-dimensional anisotropic harmonic potential in a uniform magnetic field is classified as the atomic coherent states $|\tau\rangle$ in terms of the spin values of j in the Schwinger bosonic realization. The correctness of the above conclusions can be verified by virtue of the entangled state $\langle\eta|$ representation of the state $|\tau\rangle$.

Keywords: two-dimensional anisotropic harmonic oscillator, uniform magnetic field, atomic coherent state, entangled state representation

PACC: 4250, 0365, 0530

1. Introduction

In quantum physics atomic coherent state (ACS), which was first defined by Arecchi *et al.*[1] is regarded as an important kind of two-mode correlated state. As a description of the assembly of two-level atoms, the ACS has many attractive properties analogous to those of the fields of coherent states, so in quantum optics the properties[2−4] and applications[5,6] of the ACS were widely discussed. For instance, Agarwal[2] investigated the properties, such as orthogonal property and overcomplete relation, of the ACS and obtained the intimate relationship between the ACS representation and the idea of state multipoles. Gerry and Mark[4] discussed the spin squeezing and entanglement via-burning in ACS. In Refs. [3], [5] and [6], the authors gave the explicit Wigner distribution function of the ACS and some of its practical applications.

The purpose of this paper is to demonstrate that the ACSs $|\tau\rangle$, each with a phase factor $\exp[\mathrm{i}\,\varphi(t)]$, in the Schwinger bosonic realization are a set of energy eigenstates of the two-dimensional (2D) time-independent (time-dependent) anisotropic harmonic potential in a uniform magnetic field (UMF), where τ and $\varphi(t)$ are determined by the dynamic parameters in the Hamiltonian. The rest of the present paper is organized as follows. In Section 2 we briefly recall the properties of the ACS in Schwinger Bose operator realization. In Section 3 we show that the ACSs in the Schwinger Bose operator realization are just the stationary eigenstates of time-independent 2D anisotropic harmonic potential in a UMF. In Section 4 we consider the case of a time-dependent Hamiltonian.

2. Schwinger representation of ACS

The ACS $|\tau\rangle$ is defined as[7]

$$|\tau\rangle = \exp\left(\mu J_+ - \mu^* J_-\right)|\,j,-j\rangle\,, \tag{1}$$

where J_+ is the raising operator of $|j,m\rangle$, $|\,j,-j\rangle$ is the lowest weight state annihilated by J_-, $\mu = \frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}$, and $\tau = \mathrm{e}^{-\mathrm{i}\varphi}\tan\frac{\theta}{2}$. The set of the states $|\tau\rangle$ satisfies the complete relation in the j-subspace

$$\int\frac{\mathrm{d}\Omega}{4\pi}\,|\tau\rangle\langle\tau| = \sum_{m=-j}^{j}|\,j,m\rangle\langle\,j,m| = 1,$$
$$\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10574060), the Natural Science Foundation of Shandong Province of China (Grant No. Y2008A23) and the Shandong Provincal Higher Educational Science and Technology Program of China (Grant Nos. J09LA07 and J10LA15).

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and the non-orthogonal property, owing to the inner product

$$\langle\tau'\,|\tau\rangle = \frac{(1+\tau'\tau^*)^{2j}}{(1+|\tau|^2)^j(1+|\tau'|^2)^j}. \tag{3}$$

Using the Schwinger Bose operator realization of the angular momentum operator[8]

$$J_+ = a^\dagger b,\quad J_- = ab^\dagger,\quad J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right) \tag{4}$$

and the eigenstates of J_z and the square of angular momentum operator J^2 in two-mode Fock space

$$\begin{aligned}|\,j,m\rangle &= \frac{a^{\dagger j+m}b^{\dagger j-m}}{\sqrt{(\,j+m)!\,(\,j-m)!}}\,|00\rangle \\ &= |\,j+m\rangle\otimes|\,j-m\rangle\,,\end{aligned} \tag{5}$$

the state $|\tau\rangle$ can be converted into

$$\begin{aligned}|\tau\rangle &= \exp\left(\,\mu J_+ - \mu^* J_-\right)|0\rangle\otimes|2j\rangle \\ &= \frac{1}{(1+|\tau|^2)^j} \\ &\quad\times\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\tau^{2j-l}\,|2j-l\rangle\otimes|l\rangle\,.\end{aligned} \tag{6}$$

Noticing the normal ordering form of the two-mode vacuum state projector

$$|00\rangle\langle 00| =: \exp\left(a^\dagger a - b^\dagger b\right):, \tag{7}$$

where the symbol : : denotes normal ordering, and using the technique of integration within an ordered product (IWOP) of operator,[9] we can also prove that the completeness relation of $|\tau\rangle$ in the whole Fock space after summing over the spin values j is satisfied.[6]

3. Eigenstates of 2D time-independent anisotropic harmonic potential in UMF

We consider the Hamiltonian of a 2D time-independent anisotropic harmonic oscillator in a UMF $\boldsymbol{B}$

$$H = \frac{1}{2m}(\boldsymbol{P} - e\boldsymbol{A})^2 + \frac{1}{2}m(\omega_{0x}^2x^2 + \omega_{0y}^2y^2), \tag{8}$$

where $\boldsymbol{P}$ is the electron canonical momentum, $\boldsymbol{A}$ is the vector potential of the field. If the magnetic vector potential is taken as the symmetric gauge $\boldsymbol{A} = \frac{1}{2}(-By, Bx, 0)$, then the Hamiltonian H is changed into

$$\begin{aligned}H &= \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{1}{2}m\left(\omega_x^2x^2 + \omega_y^2y^2\right) \\ &\quad - \omega_L\left(xP_y - yP_x\right),\end{aligned} \tag{9}$$

where

$$\omega_L = \frac{eB}{2m},\quad \omega_i^2 = \omega_{0i}^2 + \omega_L^2,\quad i = x, y. \tag{10}$$

Letting

$$a_i = \left(\frac{m\omega_i}{2\hbar}\right)^{1/2} i + \mathrm{i}\left(2m\omega_i\hbar\right)^{-1/2}P_i, \tag{11}$$

we obtain the bosonic form of H in Eq. (9) as

$$\begin{aligned}H &= \hbar\omega_1\left(a_1^\dagger a_1 + \frac{1}{2}\right) + \hbar\omega_2\left(a_2^\dagger a_2 + \frac{1}{2}\right) \\ &\quad + \mathrm{i}\hbar\omega_L\left(a_1^\dagger a_2 - a_1a_2^\dagger\right),\end{aligned} \tag{12}$$

where the subscripts x and y are respectively replaced by 1 and 2 for convenience.

In order to ascertain whether the ACS is the stationary eigenstate of the time-independent 2D anisotropic harmonic potential in a UMF, now we directly act the Hamiltonian H in Eq. (12) on $|\tau\rangle$ in Eq. (6) and then use the relations $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, thus we have

$$\begin{aligned}H|\tau\rangle &= \frac{1}{(1+|\tau|^2)^j}\left\{\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\tau^{2j-l}\hbar[\omega_1\left(2j-l+\frac{1}{2}\right) + \omega_2\left(l+\frac{1}{2}\right)|2j-l\rangle\otimes|l\rangle\right. \\ &\quad + \mathrm{i}\hbar\omega_L\sum_{l=1}^{2j}\left[\frac{(2j)!}{(l-1)!\,(2j-l+1)!}\right]^{1/2}\tau^{2j-l}(2j-l+1)\,|2j-l+1\rangle\otimes|l-1\rangle \\ &\quad \left. - \mathrm{i}\hbar\omega_L\sum_{l=0}^{2j}\left[\frac{(2j)!}{(l+1)!\,(2j-l-1)!}\right]^{1/2}\tau^{2j-l}(l+1)\,|2j-l-1\rangle\otimes|l+1\rangle\right\}.\end{aligned} \tag{13}$$

Let $l' = l \mp 1$ in the latter two terms on the right-hand side of Eq. (13) respectively, then we will have

$$H|\tau\rangle = \hbar\left[\omega_1\left(2j+\frac{1}{2}\right)+\frac{\omega_2}{2}+\frac{\mathrm{i}2\omega_L j}{\tau}\right]|\tau\rangle - \frac{1}{(1+|\tau|^2)^j} \times \sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\tau^{2j-l}\hbar\left[(\omega_1-\omega_2)+\mathrm{i}\omega_L\left(\tau+\frac{1}{\tau}\right)\right]l\,|2j-l\rangle\otimes|l\rangle\,. \quad (14)$$

We see when the following condition is satisfied,

$$\mathrm{i}\omega_L\tau^2+(\omega_1-\omega_2)\tau+\mathrm{i}\omega_L=0, \quad (15)$$

where its solutions are

$$\tau_\pm=\frac{(\omega_2-\omega_1)\pm\sqrt{(\omega_1-\omega_2)^2+4\omega_L^2}}{\mathrm{i}2\omega_L}, \quad (16)$$

then the states $|\tau_\pm\rangle$ given by Eq. (6) are the eigenstates of H with the eigenvalues

$$E_\pm=\hbar\left[\omega_1\left(2j+\frac{1}{2}\right)+\frac{\omega_2}{2}+\frac{\mathrm{i}2\omega_L j}{\tau_\pm}\right]. \quad (17)$$

Therefore, we find that a set of energy eigenstates of the time-independent 2D anisotropic harmonic potential in a UMF is classified as the ACSs $|\tau\rangle$ in terms of the spin values j in the Schwinger bosonic realization, where τ is determined by the dynamic parameters in Hamiltonian H. In particular, when $\omega_1=\omega_2=\omega$, the Hamiltonian of Eq. (9) describes a 2D time-independent isotropic harmonic potential in a UMF, thus from Eqs. (16) and (17) we know $\tau_\pm=\mp\mathrm{i}$, $E_\pm=\hbar[\omega+2(\omega\mp\omega_L)j]$. Further, taking the spin value $j=1/2$, from Eq. (6) it follows that the eigenstates of H are

$$|\tau_\pm\rangle=\frac{1}{\sqrt{2}}(|0,1\rangle\pm\mathrm{i}\,|1,0\rangle). \quad (18)$$

From Ref. [6] we know that all values of τ are allowed because the states $|\tau\rangle$ will form a complete set. However, from Eqs. (16) and (17) we find that, for the definite Hamiltonian, only two very specific eigenstates corresponding to two specific values $\tau_\pm$ are selected.

4. Eigenvectors of time-dependent 2D anisotropic harmonic potential in UMF

When the magnetic field $\boldsymbol{B}$ is time-dependent in a UMF, the Hamiltonian of the 2D anisotropic harmonic potential in a UMF is also time-dependent, so we have

$$H(t)=\hbar\omega_1(t)\left(a_1^\dagger a_1+\frac{1}{2}\right)+\hbar\omega_2(t)\left(a_2^\dagger a_2+\frac{1}{2}\right)+\mathrm{i}\hbar\omega_L(t)\left(a_1^\dagger a_2-a_1 a_2^\dagger\right). \quad (19)$$

In order to obtain the energy eigenstates and the energy eigenvlaues of the Hamiltonian $H(t)$, we must solve the following Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial|\phi(t)\rangle}{\partial t}=H(t)\,|\phi(t)\rangle\,. \quad (20)$$

Supposing

$$|\phi(t)\rangle=f(t)\,|\tau\rangle\,, \quad (21)$$

and substituting Eq. (21) into the terms of Eq. (20) lead to the differential equation

$$\mathrm{i}\hbar\frac{\mathrm{d}f(t)}{\mathrm{d}t}|\tau\rangle+\mathrm{i}\hbar f(t)\frac{\mathrm{d}}{\mathrm{d}t}|\tau\rangle=H(t)f(t)\,|\tau\rangle\,. \quad (22)$$

Using the explicit expression of $|\tau\rangle$ in Eq. (6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tau\rangle=\frac{2j}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}|\tau\rangle-\frac{j}{1+|\tau|^2}\frac{\mathrm{d}(|\tau|^2)}{\mathrm{d}t}|\tau\rangle - \frac{1}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}l\tau^{2j-l-1}\,|2j-l\rangle\otimes|l\rangle\,\frac{\mathrm{d}\tau}{\mathrm{d}t}. \quad (23)$$

In addition, using the same method as that of deriving Eq. (14), we have

$$H(t)f(t)\,|\tau\rangle=\hbar\left[\omega_1(t)\left(2j+\frac{1}{2}\right)+\frac{\omega_2(t)}{2}+\frac{\mathrm{i}2\omega_L(t)j}{\tau}\right]f(t)\,|\tau\rangle-\frac{\hbar f(t)}{(1+|\tau|^2)^j} \times\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\left[(\omega_1(t)-\omega_2(t))+\mathrm{i}\omega_L(t)\left(\tau+\frac{1}{\tau}\right)\right]l\tau^{2j-l}\,|2j-l\rangle\otimes|l\rangle\,. \quad (24)$$

Substituting Eqs. (23) and (24) into Eq. (22) and using the linear independence of the Fock states, we have

$$\mathrm{i}\frac{\mathrm{d}\tau}{\mathrm{d}t} = [\omega_1(t) - \omega_2(t)]\tau + \mathrm{i}\omega_L(t)(\tau^2 + 1) \tag{25}$$

and

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}f(t)}{\mathrm{d}t} =\ & f(t)\Bigg[\omega_1(t)\left(2j + \frac{1}{2}\right) + \frac{\omega_2(t)}{2} \\ & + \frac{\mathrm{i}2\omega_L(t)j}{\tau} - \frac{\mathrm{i}j}{1 + |\tau|^2} \\ & \times \left(\tau^*\frac{\mathrm{d}\tau}{\mathrm{d}t} + \tau\frac{\mathrm{d}\tau^*}{\mathrm{d}t}\right) + \frac{\mathrm{i}2j}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\Bigg].\end{aligned} \tag{26}$$

Taking $\mathrm{d}\tau/\mathrm{d}t = 0$ in Eq. (25), the Hamiltonian $H(t)$ reduces to H. Substituting Eq. (25) into Eq. (26) leads to the simplified form of Eq. (26), i.e.,

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}f(t)}{\mathrm{d}t} =\ & f(t)\Bigg[\omega_2(t)\left(2j + \frac{1}{2}\right) + \frac{\omega_1(t)}{2} \\ & + \mathrm{i}j\omega_L(t)\left(\tau^* - \tau\right)\Bigg].\end{aligned} \tag{27}$$

If we take

$$f(t) = \exp[\mathrm{i}\varphi(t)], \tag{28}$$

then the phase $\varphi(t)$ is defined as

$$\begin{aligned}\varphi(t) =\ & -\int_0^t f(\varepsilon)\{\omega_2(\varepsilon)\left(2j + \frac{1}{2}\right) + \frac{\omega_1(\varepsilon)}{2} \\ & + \mathrm{i}j\omega_L(\varepsilon)[\tau^*(\varepsilon) - \tau(\varepsilon)]\}\mathrm{d}\varepsilon.\end{aligned} \tag{29}$$

So we conclude that the state $|\tau\rangle$ with a phase factor $\exp[\mathrm{i}\varphi(t)]$ is the solution of Schrödinger equation of the time-dependent 2D anisotropic harmonic potential in a UMF. At this point we mention that the above results can also be obtained by virtue of the wave function of the ACS in the entangled state $\langle\eta|$ representation. The state $\langle\eta|$ is defined as[9]

$$\langle\eta| = \langle 00|\exp\left[\eta^* a - \eta b + ab\right], \quad \eta = \eta_1 + \mathrm{i}\eta_2, \tag{30}$$

which obeys the eigenvetor equations

$$\langle\eta|\left(a - b^\dagger\right) = \langle\eta|\,\eta, \quad \langle\eta|\left(a^\dagger - b\right) = \langle\eta|\,\eta^* \tag{31}$$

and the differential relations

$$-\frac{\partial}{\partial\eta}\langle\eta| = \langle\eta|\,b, \quad \frac{\partial}{\partial\eta^*}\langle\eta| = \langle\eta|\,a. \tag{32}$$

So in the $\langle\eta|$ basis we have the following corresponding relations:

$$a \to \frac{\partial}{\partial\eta^*}, \quad b \to -\frac{\partial}{\partial\eta},$$

$$a^\dagger \to -\frac{\partial}{\partial\eta} + \eta^*, \quad b^\dagger \to \frac{\partial}{\partial\eta^*} - \eta. \tag{33}$$

Using the relations between the coordinate (momentum) operator and the Bose creation and annihilation operators, we find that the state $\exp(-|\eta|^2/2)|\eta\rangle$ is the common eigenstate of two-particle relative position $(X_1 - X_2)$ and the total momentum $(P_1 + P_2)$ in two-mode Fock space. On the other hand, using the IWOP technique[9−12] we can obtain the completeness property and the orthonormal relation

$$\begin{aligned}&\int\frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{-|\eta|^2}\,|\eta\rangle\,\langle\eta| = 1, \\ &\langle\eta|\,\eta'\rangle = \pi\delta\left(\eta - \eta'\right)\delta\left(\eta^* - \eta'^*\right).\end{aligned} \tag{34}$$

Using the above remarkable property of the entangled state $\langle\eta|$, we can obtain the wave function of the ACS in $\langle\eta|$ representation, which is just proportional to a single-variable ordinary Hermite polynomial of order $2j$, i.e.,

$$\langle\eta|\,\tau\rangle = \frac{(-1)^{j+1}\tau^j}{(1 + |\tau|^2)^j\sqrt{(2j)!}}H_{2j}\left(\mathrm{i}\frac{\eta - \tau\eta^*}{2\sqrt{\tau}}\right). \tag{35}$$

The orthogonality and the recursive relation of the single-variable Hermite polynomials are conducive to deriving the above results. However, the corresponding derivation is not simpler than the above process, so we do not present them here in this paper.

5. Conclusions

In this paper using the Schwinger representation[13−16] of the ACS we find that the states $|\tau\rangle$ are a set of energy eigenstates of the 2D time-independent anisotropic harmonic potential in a UMF. By solving the Schrödinger equation we also find that $|\tau\rangle$ with a phase factor $\exp[\mathrm{i}\,\varphi(t)]$ are just the eigenstates of the 2D time-dependent anisotropic harmonic potential in a UMF. The above conclusions are verified by virtue of the entangled state $\langle\eta|$ representation of $|\tau\rangle$, which is proportional to a single-variable ordinary Hermite polynomial of order $2j$. In doing so, in this paper the results can enrich the theory about the ACS in quantum optics. Also we expect the results can be widely used in quantum optics and quantum information.

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