Nonlinear interactions and quantum entanglement for collective fields in near-resonantly driven systems*

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This paper proposes a novel form of multimode nonlinear interactions by using a near-resonantly dressed atomic ensemble in an optical cavity. Due to quantum interference, a pair of collective fields come into the bilinear interactions, whose strengths are proportional to the population difference between dressed states which are coupled to the collective fields. By such an interaction, it is possible to obtain perfect multimode squeezing and collective Einstein–Podolsky–Rosen (EPR) entanglement in the cavity output.

Keywords: collective parametric interaction, quantum interference, collective EPR entanglement, multimode squeezing

PACC: 4250, 4265

1. Introduction

Optical parametric processes turn out to be the most effective ways for the generation of the squeezed and entangled light sources.[1−15] However, the existing schemes have been limited in two aspects. First, although various forms of parametric interactions have been established for multiple fields, the mechanisms are limited to the one-by-one interactions between individual fields.[16−20] So far, to our knowledge, the interactions between a pair of collective fields have not been presented yet, these fields consist of two or more field components. In some cases, one needs entanglement between two different groups but does not necessarily need entanglement between two individuals from the same groups.[21−23] This form of entanglement may have potentials for the quantum networks and communications.[24] Second, the applied driving fields are far off resonant with the involved transitions in the media.[13] In this case, the atoms are hardly excited and the susceptibilities are extremely low. In the case of near resonance, the atoms are much excited and the susceptibilities are enhanced remarkably. As a consequence, the spontaneous emission has been expected to spoil the quantum correlations. However, it has recently been shown that fluctuations from spontaneous emission can be cancelled at the cavity output due to destructive interference.[25]

Here we engineer a different form of multimode parametric interactions by using optical cavities with near-resonantly driven atomic ensembles. The amplified fields behave as two different collective modes and the parametric interaction occurs between these two collective modes but not between two individuals from the same collective modes. The mechanism is based on the quantum-beats, by which the individual fields are divided into two groups and each group corresponds to one collective field. By this kind of interactions, we obtain the multimode squeezing and collective entanglement. In this case, entanglement occurs between any two components from different collections but not between two in the same collection. In addition to the great potentials in quantum networks and communications, the collective interactions also give the first step for desirable and arbitrary control of multimode parametric interactions and correlations. With this step, then one can manipulate the individuals in the same collective modes. Recently, Kofler *et al.* presented such collective entanglement for a linear chain of harmonic oscillators by phenomenalistically introducing harmonic nearest-neighbour interactions.[21] Linear interaction between the collective fields based on multiwave mixing in a near-resonantly driven system was also presented.[23] The purpose of the present paper is to present col-

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lective nonlinear interactions based on the quantum interference.

2. Model and equation

We first consider the four-mode case. An ensemble of N atoms in V configuration (Fig. 1) is placed in a four-mode optical cavity. In our system, the atom has one ground state $|0\rangle$ and two excited states $|1\rangle$ and $|2\rangle$. The atoms are driven on the dipole-allowed transitions $|0\rangle-|1\rangle$ and $|2\rangle$ by two strong classical fields of circular frequencies ω_j $(j=1,2)$, respectively. Four quantified cavity fields with circular frequencies ω_{a_j} and ω_{b_j} are generated from sidebands of the two transitions, respectively. The master equation for the atom–field density operator is derived in the dipole approximation and in an appropriate rotating frame as[26]

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}[H_0 + H_{\mathrm{I}}, \rho] + \mathscr{L}_{\mathrm{a}}\rho + \mathscr{L}_{\mathrm{c}}\rho, \tag{1}$$

where H_0 describes the free term of atoms and the interaction between atoms and the classical fields, and H_{I} characterizes the interaction of cavity fields with the atoms,

$$H_0 = \sum_{\mu=1}^{N}\sum_{j=1,2}\hbar\left[\Delta_j\sigma_{jj}^{\mu} + \frac{1}{2}(\Omega_l\sigma_{j0}^{\mu} + \Omega_l^*\sigma_{0j}^{\mu})\right], \tag{2}$$

$$H_{\mathrm{I}} = \sum_{\mu=1}^{N}\sum_{j=1,2}\hbar(\, g_{a_j}a_j\,\mathrm{e}^{-\mathrm{i}\,\delta_{a_j}t}\sigma_{j0}^{\mu} + g_{b_j}b_j\,\mathrm{e}^{-\mathrm{i}\,\delta_{b_j}t}\sigma_{j0}^{\mu}) + \mathrm{H.c.} \tag{3}$$

Here H.c. represents the complex conjugate. The $\sigma_{jk}^{\mu} = |j^{\mu}\rangle\langle k^{\mu}|$ $(j,k=0,1,2)$ are the atomic projection operators when $j=k$ and the flip operators for $j\neq k$. The a_j, b_j and $a_j^{\dagger}, b_j^{\dagger}$ are annihilation and creation operators for the cavity fields, respectively. The g_{a_j} and g_{b_j} describe coupling constants between the cavity fields and the atoms, and Ω_j are Rabi frequencies. $\Delta_j = \omega_{j0} - \omega_j$, $\delta_{a_j} = \omega_{a_j} - \omega_j$, $\delta_{b_j} = \omega_{b_j} - \omega_j$ are respectively the detunings of atomic resonance frequencies ω_{j0} and the cavity fields frequencies ω_{a_j} and ω_{b_j} from the corresponding driving field frequencies ω_j. The damping terms $\mathscr{L}_{\mathrm{a}}\rho = \sum_{j=1,2}\mathscr{L}_{0j}^{(\mathrm{a})}\rho$ and $\mathscr{L}_{\mathrm{c}}\rho = \sum_{j=1,2}(\mathscr{L}_{a_j}^{(\mathrm{c})}\rho + \mathscr{L}_{b_j}^{(\mathrm{c})}\rho)$ describe the atomic decay and the cavity losses, respectively, where

$$\mathscr{L}_{0j}^{(\mathrm{a})}\rho = \sum_{\mu=1}^{N}\frac{\gamma_j}{2}(2\sigma_{0j}^{\mu}\rho\sigma_{j0}^{\mu} - \sigma_{j0}^{\mu}\sigma_{0j}^{\mu}\rho - \rho\sigma_{j0}^{\mu}\sigma_{0j}^{\mu}) \tag{4}$$

denotes atomic decay from $|\,j^{\mu}\rangle$ to $|0^{\mu}\rangle$ with rate γ_j, and

$$\mathscr{L}_{a_j}^{(\mathrm{c})}\rho = \frac{\kappa_{a_j}}{2}(2a_j\rho a_j^{\dagger} - a_j^{\dagger}a_j\rho - \rho a_j^{\dagger}a_j) \tag{5}$$

stands for the cavity loss of mode a_j with rate κ_{a_j}, $\mathscr{L}_{b_j}^{(\mathrm{c})}\rho$ has the same form as Eq. (5) except for substitution of b for a, which represents the cavity loss of mode b_j with rate κ_{b_j}.

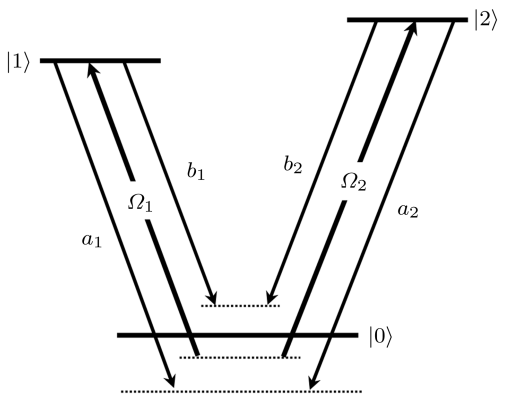


Fig. 1. Level diagram for three-level V-type atom. Two external fields are applied to the atomic transitions $|0\rangle-|1\rangle$ and $|2\rangle$ on Raman two-photon resonance with Rabi frequencies $\Omega_{1,2}$, and four cavity fields (annihilation operators $a_{1,2}$ and $b_{1,2}$) are produced from sideband resonances.

3. Collective parametric interaction

To simplify the analysis and to clarify the physics, we consider the Raman two-photon resonance case $\Delta_1 = \Delta_2 = \Delta$. Two orthogonal coherent superposition states of the two excited states are defined as $|\tilde{1}^{\mu}\rangle = \cos\phi|1^{\mu}\rangle + \sin\phi|2^{\mu}\rangle$ and $|\tilde{2}^{\mu}\rangle = -\sin\phi|1^{\mu}\rangle + \cos\phi|2^{\mu}\rangle$, where we have assumed that $\Omega_{1,2}$ are real for simplicity and have defined $\tan\phi = \Omega_2/\Omega_1$ and $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2}$. The Hamiltonian H_0 is rewritten in terms of the superposition states as

$$H_0 = \sum_{\mu=1}^{N}\hbar[\Delta(\sigma_{\tilde{1}\tilde{1}}^{\mu} + \sigma_{\tilde{2}\tilde{2}}^{\mu}) + \frac{\Omega}{2}(\sigma_{\tilde{1}0}^{\mu} + \sigma_{0\tilde{1}}^{\mu})]. \tag{6}$$

It is seen from Eq. (6) that only one superposition state $|\tilde{1}^{\mu}\rangle$ is coupled to the equivalent field with Rabi frequency Ω, while the other coherent superposition state $|\tilde{2}^{\mu}\rangle$ is decoupled. It is reasonable to assume $\gamma_1 = \gamma_2 = \gamma$ for the near-degenerate excited states. We note that the superposition state $|\tilde{2}^{\mu}\rangle$ is not populated, $\langle\sigma_{\tilde{2}\tilde{2}}^{\mu}\rangle = \sin^2\phi\langle\sigma_{11}^{\mu}\rangle + \cos^2\phi\langle\sigma_{22}^{\mu}\rangle - \sin(2\phi)\,\mathrm{Re}\langle\sigma_{12}^{\mu}\rangle = 0$. In what follows, for the sake of simplicity we assume that $\delta_{a_l} = \delta_{\mathrm{a}}$, $\delta_{b_l} = \delta_b$, $g_{a_l} = g_{\mathrm{a}}$, $g_{b_l} = g_b$, $\kappa_{a_l} = \kappa_{\mathrm{a}}$,

$\kappa_{b_l} = \kappa_b$. The interaction Hamiltonian for the cavity fields and the atoms becomes

$$\begin{aligned} H_{\rm I} = & \sum_{\mu=1}^{N} \hbar(g_{\rm a} A_1 \, {\rm e}^{-{\rm i}\,\delta_{\rm a} t} + g_b B_1 \, {\rm e}^{-{\rm i}\,\delta_b t}) \sigma_{\tilde{1}0}^{\mu} \\ & + \sum_{\mu=1}^{N} \hbar(g_{\rm a} A_2 \, {\rm e}^{-{\rm i}\,\delta_{\rm a} t} + g_b B_2 \, {\rm e}^{-{\rm i}\,\delta_b t}) \sigma_{\tilde{2}0}^{\mu} \\ & + {\rm H.c.}, \end{aligned} \tag{7}$$

where we have introduced two pairs of orthonormal collective modes

$$\begin{aligned} A_1 &= a_1 \cos\phi + a_2 \sin\phi, \\ B_1 &= b_1 \cos\phi + b_2 \sin\phi, \\ A_2 &= -a_1 \sin\phi + a_2 \cos\phi, \\ B_2 &= -b_1 \sin\phi + b_2 \cos\phi. \end{aligned} \tag{8}$$

In view of the combination modes, we call A_1, B_1 the sum modes and A_2, B_2 the difference modes. It is seen from Eq. (7) that the sum modes and difference modes are coupled to the transition $|0^{\mu}\rangle - |\tilde{1}^{\mu}\rangle$ and $|0^{\mu}\rangle - |\tilde{2}^{\mu}\rangle$, respectively. Since the superposition state $|\tilde{2}^{\mu}\rangle$ is empty, the difference modes undergo only absorption and thus there is no gain for them. This fact determines that the difference modes A_2 and B_2 always stay in the vacuum states,[27−30] $\langle A_2\rangle = \langle A_2^2\rangle = \langle A_2^{\dagger} A_2\rangle = 0$ and $\langle B_2\rangle = \langle B_2^2\rangle = \langle B_2^{\dagger} B_2\rangle = 0$. This gives rise to the following three consequences.

(i) The relative phases are locked, $\varphi_{a_1} - \varphi_{a_2} = 0$ and $\varphi_{b_1} - \varphi_{b_2} = 0$, where we have defined the phases through the expressions $\langle X\rangle = |\langle X\rangle| \exp(-{\rm i}\varphi_X)$, $X = a_{1,2}, b_{1,2}, A_1, B_1$. This fact determines the relations between the phases of the collective fields and the individual fields, $\varphi_{A_1} = \varphi_{a_1}$ and $\varphi_{B_1} = \varphi_{b_1}$.

(ii) We can drop the terms involving A_2 and B_2 and simplify the Hamiltonian (7) to

$$H_{\rm I} = \sum_{\mu=1}^{N} \hbar(g_{\rm a} A_1 \, {\rm e}^{-{\rm i}\,\delta_{\rm a} t} + g_b B_1 \, {\rm e}^{-{\rm i}\,\delta_b t}) \sigma_{\tilde{1}0}^{\mu} + {\rm H.c.}, \tag{9}$$

which indicates that the four-mode system reduces to the two-collective-field case.

(iii) The density operator can be separated into two parts, $\rho = \rho_1 \rho_2$,[29,30] where ρ_1 is for the coupling system of sum modes A_1 and B_1 and atoms, and ρ_2 is for the difference modes A_2 and B_2. This leads us to remove the part for ρ_2 and obtain the master equation for ρ_1 as

$$\dot{\rho}_1 = -\frac{\rm i}{\hbar}[H_0 + H_{\rm I}, \rho_1] + \mathscr{L}_a \rho_1 + \mathscr{L}_{\rm c} \rho_1, \tag{10}$$

where $\mathscr{L}_a \rho_1 = \mathscr{L}_{0\tilde{1}}^{(\rm a)} \rho_1$ and $\mathscr{L}_{\rm c} \rho_1 = \mathscr{L}_{A_1}^{(\rm c)} \rho_1 + \mathscr{L}_{B_1}^{(\rm c)} \rho_1$. The $\mathscr{L}_{0\tilde{1}}^{(\rm a)} \rho_1$ has the same form as $\mathscr{L}_{0j}^{(\rm a)} \rho$ in Eq. (4) except for the substitution of $\tilde{1}$ for j and describes the atomic decay from $|\tilde{1}^{\mu}\rangle$ to $|0^{\mu}\rangle$ with rate $\gamma_{\tilde{1}} = \gamma_1$, and $\mathscr{L}_{A_1}^{(\rm c)} \rho_1$ ($\mathscr{L}_{B_1}^{(\rm c)} \rho_1$) have the same form as $\mathscr{L}_{a_j}^{(\rm c)} \rho$ in Eq. (5) except for the substitution of A_1 (B_1) for a_j and represent the cavity loss of the sum mode A_1 (B_1) with $\kappa_{A_1} = \kappa_{\rm a}$ ($\kappa_{B_1} = \kappa_b$). We have removed the term $\mathscr{L}_{0\tilde{2}}^{(\rm a)} \rho_1$ since the empty state $|\tilde{2}^{\mu}\rangle$ is decoupled. This means that the present case is reduced to an equivalent two-level system ($|0^{\mu}\rangle$ and $|\tilde{1}^{\mu}\rangle$).

It is convenient to employ the dressed-atom approach.[31] By diagonalizing the Hamiltonian H_0, we write the dressed states in terms of the bare atomic state $|0^{\mu}\rangle$ and superposition state $|\tilde{1}^{\mu}\rangle$ as $|+^{\mu}\rangle = \cos\theta |\tilde{1}^{\mu}\rangle + \sin\theta |0^{\mu}\rangle$ and $|-^{\mu}\rangle = -\sin\theta |\tilde{1}^{\mu}\rangle + \cos\theta |0^{\mu}\rangle$, where $\tan(2\theta) = -\Omega/\Delta$ ($0 < \theta < \pi/2$). The dressed states $|\pm\rangle$ have their eigenvalues $\lambda_{\pm} = (1/2)(\Delta \pm \Omega_{\rm R})$, where $\Omega_{\rm R} = \sqrt{\Delta^2 + \Omega^2}$. When the driving fields are so strong that $\Omega_{\rm R} \gg \gamma$, the dressed states are well separated from each other. In terms of the dressed atomic states, the Hamiltonian H_0 is rewritten as $H_0 = \sum_{\mu=1}^{N} \hbar(\lambda_+ \sigma_{++}^{\mu} + \lambda_- \sigma_{--}^{\mu})$. We tune the cavity fields detunings $|\delta_{a,b}| \gg |\delta_{\rm a} + \delta_b|$, $(\Omega_{\rm R}, \delta_{a,b}) \gg |\Omega_{\rm R} - \delta_{a,b}| \gg (\gamma, |g_{\rm a}\langle A_1\rangle|, |g_b\langle B_1\rangle|)$, make a unitary transformation $U = \exp(-{\rm i} H_0 t/\hbar)$, and neglect rapidly oscillating terms. The effective Hamiltonian can be obtained from $H_{\rm eff} = -({\rm i}/\hbar) H_{\rm I}(t) \int H_{\rm I}(t') {\rm d}t'$.[32] Taking detunings $\delta_{\rm a} + \delta_b = (|g_b|^2 \sin^4\theta + |g_{\rm a}|^2 \cos^4\theta)/(\delta_{\rm a} - \Omega_{\rm R})$, we can remove the Stark shifts and write the effective Hamiltonian as

$$H_{\rm eff} = \hbar(\xi^* A_1 B_1 + \xi A_1^{\dagger} B_1^{\dagger})(\sigma_{--} - \sigma_{++}), \tag{11}$$

where $\xi = [N g_{\rm a}^* g_b^*/(\Omega_{\rm R} - \delta_{\rm a})] \cos^2\theta \sin^2\theta$ and $\sigma_{\pm\pm} = (1/N) \sum_{\mu=1}^{N} \sigma_{\pm\pm}^{\mu}$. It is seen that the effective Hamiltonian is proportional to the dressed population difference $(\sigma_{--} - \sigma_{++})$. If we go to the picture where the equivalent driving field Ω is treated quantum mechanically, the effective Hamiltonian is intuitively understood as follows.[33] In this picture we have a ladder of doublets $|+^{\mu}, n\rangle = \cos\theta |\tilde{1}^{\mu}, n\rangle + \sin\theta |0^{\mu}, n+1\rangle$ and $|-^{\mu}, n\rangle = -\sin\theta |\tilde{1}^{\mu}, n\rangle + \cos\theta |0^{\mu}, n+1\rangle$. The effective Hamiltonian corresponds to the two-photon transitions $|+^{\mu}, n+1\rangle \to |-^{\mu}, n\rangle \to |+^{\mu}, n-1\rangle$ and $|-^{\mu}, n+1\rangle \to |+^{\mu}, n\rangle \to |-^{\mu}, n-1\rangle$. Since the equivalent driving field Ω is strong and has a narrow distribution of photons, we can neglect the differences between n and $n \pm 1$. As a result, these two-photon transitions as one-step processes give rise to the dependence of dressed populations $\sigma_{\pm\pm}$. The difference

between dressed populations $(\sigma_{--}-\sigma_{++})$ is due to the destructive interference between the two sets of transitions.[34] With the effective Hamiltonian and the dressed states we rewrite the master equation (10) as

$$\dot{\rho}_1 = -\frac{\mathrm{i}}{\hbar}[H_{\rm eff}, \rho_1] + \mathscr{L}'_{\rm a}\rho_1 + \mathscr{L}_{\rm c}\rho_1, \qquad (12)$$

where $\mathscr{L}'_{\rm a}\rho_1 = \mathscr{L}^{(\rm a)}_{+-}\rho_1 + \mathscr{L}^{(\rm a)}_{-+}\rho_1 + \mathscr{L}^{(\rm a)}_{\rm ph}\rho_1$, $\mathscr{L}^{(\rm a)}_{\rm ph}\rho_1 = \sum_{\mu=1}^{N}(\gamma_{\rm ph}/4)(2\sigma_p^\mu\rho_1\sigma_p^\mu - \sigma_p^\mu\sigma_p^\mu\rho_1 - \rho_1\sigma_p^\mu\sigma_p^\mu)$, $\gamma_{+-} = \gamma\sin^4\theta$, $\gamma_{-+} = \gamma\cos^4\theta$, $\gamma_{\rm ph} = 2\gamma\sin^4\theta\cos^4\theta$, $\sigma_p^\mu = \sigma_{++}^\mu - \sigma_{--}^\mu$. Since we treat weak cavity fields, the expectation value for the population difference can be obtained from the master equation in the absence of cavity fields as $\langle\sigma_{--}\rangle - \langle\sigma_{++}\rangle = \cos(2\theta)/(\sin^4\theta + \cos^4\theta)$. Now we can make an estimate of the parametric interaction strength $\xi(\langle\sigma_{--}\rangle - \langle\sigma_{++}\rangle)$. Note that $|\langle\sigma_{--}\rangle - \langle\sigma_{++}\rangle|$ takes the maximal value of 1 when $|\Delta| \gg \Omega$, and the minimal value of zero when $\Delta = 0$. The parametric interaction strength is $\eta = \cos^2\theta\sin^2\theta\cos(2\theta)/(\cos^4\theta + \sin^4\theta)$ in the unit of $|Ng_{\rm a}g_b/(\Omega_{\rm R} - \delta_{\rm a})|$. So far, the parametric interaction Hamiltonian has been mainly confined to the far off-resonantly driven system.[13] In that case all atoms are in the ground state and $\sigma_{--} - \sigma_{++} \approx \pm 1$, i.e., $|\Delta| \gg \Omega$, $\cos\theta \approx 1$, $\sin\theta \approx 0.1$, or $\sin\theta \approx 1$, $\cos\theta \approx 0.1$, one has $|\eta| \approx 0.01$. Here, we focus on the near-resonantly driven system, $\Delta \sim \Omega$. We have the maximal value $|\eta_{\max}| \approx 0.15$ at $|\Delta|/\Omega = 0.55$, as shown in Fig. 2. We can also see that $|\eta| \geq 0.1$ in a very wide range of Δ/Ω. It is clear that in the near-resonant case the parametric interaction strength is larger than that in the far-resonant case by one order.

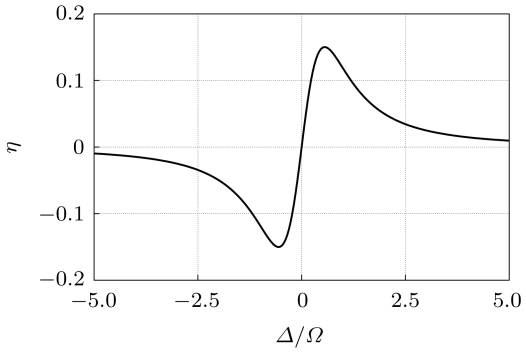


Fig. 2. The function η versus the normalized detuning Δ/Ω.

4. Quantum correlations

In this section, we will discuss the output quantum correlations by including the fluctuations caused by the population difference $(\sigma_{--} - \sigma_{++})$ in Hamiltonian (11). Choosing the normal ordering $A_1^\dagger$, $B_1^\dagger$, σ_{++}, B_1, A_1, defining the corresponding c-number vector $O(t) = (\alpha^*, \beta^*, z, \beta, \alpha)$, and using the closure relation for the atomic population $\sigma_{++} + \sigma_{--} = 1$, we derive the set of Langevin equations from Eq. (12) as follows:

$$\dot{\alpha} = -\mathrm{i}\varepsilon_\alpha^* + \xi'(1-2z)\beta^* - \frac{\kappa_{\rm a}}{2}\alpha + F_\alpha(t), \quad (13)$$

$$\dot{\beta} = -\mathrm{i}\varepsilon_\beta^* + \xi'(1-2z)\alpha^* - \frac{\kappa_b}{2}\beta + F_\beta(t), \quad (14)$$

$$\dot{z} = \gamma\sin^4\theta - \gamma(\sin^4\theta + \cos^4\theta)z + F_z(t) \quad (15)$$

with $\varepsilon_\alpha^* = \varepsilon_{a_1}^*\cos\phi + \varepsilon_{a_2}^*\sin\phi$ and $\varepsilon_\beta^* = \varepsilon_{b_1}^*\cos\phi + \varepsilon_{b_2}^*\sin\phi$, where $\varepsilon_{a_{1,2}}$ and $\varepsilon_{b_{1,2}}$ are the input signals for the fields $a_{1,2}$ and $b_{1,2}$. For simplicity we have assumed that $\xi' = \mathrm{i}\xi$ is assumed to be real. The fluctuation forces have zero averages $\langle F_x(t)\rangle = 0$ and delta corrections $\langle F_x(t)F_y(t')\rangle = D_{xy}\delta(t-t')$ with nonzero diffusion coefficients $D_{\alpha\beta} = D_{\alpha^*\beta^*} = d/2$, $D_{zz} = 2\gamma\bar{z}\cos^4\theta$, where we have used $d = 2\xi'(1-2\bar{z})$ with $\bar{z} = \langle\sigma_{++}\rangle = \sin^4\theta/(\cos^4\theta + \sin^4\theta)$ being the steady state population for the state $|+\rangle$. The steady state solutions for the cavity fields can be obtained as $\bar{\alpha} = 2\mathrm{i}(\kappa_{\rm b}\varepsilon_{\rm a}^* - d\varepsilon_{\rm b})/(d^2 - \kappa_{\rm a}\kappa_{\rm b})$, $\bar{\beta} = 2\mathrm{i}(\kappa_{\rm a}\varepsilon_{\rm b}^* - d\varepsilon_{\rm a})/(d^2 - \kappa_{\rm a}\kappa_{\rm b})$. Using $\delta O(t) = O(t) - \bar{O}$, we obtain the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta O(t) = -R\delta O(t) + F(t), \qquad (16)$$

where the drift matrix R and the noise correlation matrix D for $\langle F(t)F^{\rm T}(t')\rangle = D\delta(t+t')$ can be easily obtained from the Langevin equations. The procedure of assuming small fluctuations about a steady-state deterministic solutions is only consistent if the deterministic solution $\bar{\alpha}$, $\bar{\beta}$ and $\bar{z}$ are stable. This is the case only when all eigenvalues of the matrix R have positive real parts. The normally ordered intracavity correlation spectrum $S(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega\tau}\langle\delta O(t)\delta O^{\rm T}(t+\tau)\rangle\,\mathrm{d}\tau$ can be derived as[35]

$$S(\omega) = (R - \mathrm{i}\omega I)^{-1}D(R^{\rm T} + \mathrm{i}\omega I)^{-1}, \qquad (17)$$

where I is a unit matrix.

Now we are in a position to discuss the collective correlations. To do this, we define a pair of Einstein–Podolsky–Rosen (EPR) like operators, $u = x_A - x_B$, $v = p_A + p_B$, in which

$$x_A = \frac{1}{\sqrt{2}}(A_1^{\rm out}\,\mathrm{e}^{\mathrm{i}\,\varphi_A} + A_1^{\dagger\rm out}\,\mathrm{e}^{-\mathrm{i}\,\varphi_A}),$$

$$p_A = \frac{-\mathrm{i}}{\sqrt{2}}(A_1^{\rm out}\,\mathrm{e}^{\mathrm{i}\,\varphi_A} - A_1^{\dagger\rm out}\,\mathrm{e}^{-\mathrm{i}\,\varphi_A}),$$

$$x_B = x_A(A \to B), \quad p_B = p_A(A \to B),$$

where the superscripts "out" denote the output fields. The collective modes at the output port are entangled

if the sum of variances of EPR-like operators u and v satisfy the inequality[36]

$$V = \langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle < 2. \tag{18}$$

The equation $V = 0$ corresponds to EPR entanglement. At the same time, we should note $\langle(\Delta u)^2\rangle = \langle(\Delta v)^2\rangle$, which means that $V < 2$ corresponds to squeezing of the multimode operators u and v. Using the standard input–output theory,[35] we obtain the fluctuation correlation spectrum for the collective output fields

$$\begin{aligned} &V(\omega) \\ &= 2 + \kappa_{\rm a}[S_{15}(\omega) + S_{51}(\omega)] + \kappa_{\rm b}[S_{24}(\omega) + S_{42}(\omega)] \\ &\quad -\sqrt{\kappa_{\rm a}\kappa_{\rm b}}[S_{12}(\omega) + S_{21}(\omega) + S_{45}(\omega) + S_{54}(\omega)]. \end{aligned} \tag{19}$$

At zero-frequency we have

$$\begin{aligned} &V(0) \\ &= 2\left|\frac{d-\sqrt{\kappa_{\rm a}\kappa_{\rm b}}}{d+\sqrt{\kappa_{\rm a}\kappa_{\rm b}}}\right|^2 \\ &\quad + \frac{32(\xi')^2\bar{z}\cos^4\theta}{\gamma(\sin^4\theta+\cos^4\theta)^2}\left|\frac{\bar{\alpha}\sqrt{\kappa_{\rm a}}-\bar{\beta}^*\sqrt{\kappa_{\rm b}}}{d+\sqrt{\kappa_{\rm a}\kappa_{\rm b}}}\right|^2. \end{aligned} \tag{20}$$

In Eq. (20), the first term originates from the vacuum fluctuations and the second term is due to the spontaneous emission. For the far off-resonant case,[13] i.e., $|\Delta|/\Omega \gg 1$, we have $\sin\theta \to 0$, then $\bar{z} \to 0$. This shows that the contribution due to spontaneous emission is negligibly small. However, for the near-resonant case, $|\Delta|/\Omega \sim 1$, the fluctuations from spontaneous emission generally are non-negligible. We note that the destructive interference occurs when $d = \sqrt{\kappa_{\rm a}\kappa_{\rm b}}$, from which we have $\bar{\alpha}\sqrt{\kappa_{\rm a}} = \bar{\beta}^*\sqrt{\kappa_{\rm b}}$. This means that the two terms in Eq. (20) trend to vanish when destructive interference takes place. As a result, we have $V(0) = 0$, which indicates that one can have perfect multimode squeezing and the collective EPR entanglement. At the same time, we can also see that the larger the parameter d the larger the cavity damping rates $\kappa_{\rm a}$ and $\kappa_{\rm b}$ for the above ideal squeezing and entanglement. From the above analyses, we can summarize the quantum interference effects as follows. First, two quantum beats are created, this determines that the sum modes A_1 and B_1 mediate into interaction but the difference modes A_2 and B_2 remain in the vacuum state and decouple from the system. Second, two sum modes A_1 and B_1 are detuned and enter into parametric interactions, the fluctuations due to spontaneous emission and the vacuum couplings are cancelled due to destructive interference. These two effects combine to take effect, and so almost perfect multimode squeezing and collective EPR entanglement are obtained.

Next, we analyse the correlations for the individual fields. For the individual modes a_k and b_j $(k, j = 1, 2)$ we define the pair of EPR-like operators $u_{kj} = x_{a_k} - x_{b_j}$ and $v_{kj} = p_{a_k} + p_{b_j}$, where

$$\begin{aligned} x_{a_k} &= \frac{1}{\sqrt{2}}(a_k^{\rm out}\,{\rm e}^{{\rm i}\,\varphi_{a_k}} + a_k^{\dagger{\rm out}}\,{\rm e}^{-{\rm i}\,\varphi_{a_k}}), \\ p_{a_k} &= \frac{-{\rm i}}{\sqrt{2}}(a_k^{\rm out}\,{\rm e}^{{\rm i}\,\varphi_{a_k}} - a_k^{\dagger{\rm out}}\,{\rm e}^{-{\rm i}\,\varphi_{a_k}}), \\ x_{b_j} &= x_{a_k}(a_k \to b_j), \quad p_{b_j} = p_{a_k}(a_k \to b_j). \end{aligned}$$

We have $\langle(\Delta u_{kj})^2\rangle = \langle(\Delta v_{kj})^2\rangle$. Noise squeezing for quadrature u_{kj} and v_{kj} and entanglement between the individual modes a_k and b_j are obtained if the following inequality is satisfied:

$$V_{kj} = \langle(\Delta u_{kj})^2\rangle + \langle(\Delta v_{kj})^2\rangle < 2. \tag{21}$$

The correlation spectra $V_{kj}(\omega)$ for the pairs of individual fields at the output port are related to the collective mode correlation spectrum $V(\omega)$ by the following relations:

$$\begin{aligned} V_{11}(\omega) &= V(\omega)\cos^2\phi < 2, \\ V_{12}(\omega) &= V_{21}(\omega) = V(\omega)\cos\phi\sin\phi < 2, \\ V_{22}(\omega) &= V(\omega)\sin^2\phi < 2. \end{aligned} \tag{22}$$

It is seen from Eq. (22) that the squeezing and entanglement for the collective fields correspond to those for any two individual components from different collective modes. When $\cos\phi = \sin\phi$, i.e., $\Omega_1 = \Omega_2$, we have $V_{11}(\omega) = V_{12}(\omega) = V_{21}(\omega) = V_{22}(\omega) = (1/2)V(\omega)$. By varying the ratio Ω_2/Ω_1, we can manipulate the degree of squeezing and entanglement for different pairs of cavity fields. When $\cos\phi \to 1$, i.e., $\Omega_1 \gg \Omega_2$, the pairs of individual modes from different collections have the correlation spectra $V_{11}(\omega) \approx V(\omega)$, $V_{12}(\omega) \approx V_{21}(\omega) \approx V_{22}(\omega) \approx 0$. Similarly, for $\sin\phi \to 1$, i.e., $\Omega_2 \gg \Omega_1$, we have $V_{22}(\omega) \approx V(\omega)$, $V_{11}(\omega) \approx V_{12}(\omega) \approx V_{21}(\omega) \approx 0$. For the case of extreme disparity in the Rabi frequencies of applied fields, the perfect squeezing and the EPR entanglement for the collective modes correspond to the perfect squeezing and the EPR entanglement for individual modes a_1 and b_1 or for individual modes a_2 and b_2. In this case, the squeezing and entanglement for individual modes a_1 and b_2 or for individual modes a_2 and b_1 almost vanishes.

Finally, it is easy to generalize the present scheme to a general case. For a $(j+1)$-level system, there are

j ($j \geq 3$) excited states and one ground state. The j driving fields couple the j excited states to the common ground state. We assume that all driving fields are cross-correlated and the coherent depopulation effect is preserved. When all excited states have the same decay rates, the $(j-1)$ superposition states are decoupled. We obtain a pair of quantum beats as above, each of which has at most j cavity fields. Then we have entanglement between two collective fields, each has $m_{1,2} = j$ components. We also have squeezing and entanglement for collective fields, in each of which there are different numbers of modes, $m_1 \leq j$, $m_2 \leq j$, $m_1 \neq m_2$. In a special case, one group contains a single mode and the other consists of j modes. This gives squeezing and entanglement for a single mode ($m_1 = 1$) and a group of $m_2 \geq 2$ modes.

5. Conclusion

In conclusion, we have shown the multimode squeezing and collective continuous-variable entanglement in a near-resonantly driven three-level V system. Two quantum beats are induced via atomic coherence effects. This yields the collective parametric interactions, whose coupling strength is scaled up by the population difference of the atomic ensemble. The fluctuations due to spontaneous emission and the vacuum fluctuations are cancelled due to quantum interference at the output port. One has almost perfect squeezing and EPR entanglement for the collective modes. This also corresponds to squeezing and entanglement for two individual fields from different collections.

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