Scheme for direct measurement of Wigner function in two-mode cavity QED driven by classical fields*

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We propose a scheme for the direct measurement of Wigner function in two-mode cavity QED. The atoms are sent to resonantly interact with two orthogonally polarized cavity modes in the presence of strong classical field. The probability of measuring the atom in the ground state directly gives the useful information of the cavity field. This method can be used for quantum non-demolition measurement of the photon number.

Keywords: two-mode cavity, Wigner function, quantum non-demolition measurement

PACC: 4250, 0365

1. Introduction

Preparation, measurement, and reconstruction of nonclassical states have been studied extensively in the past two decades. Several schemes with quantum optical systems have been proposed for this purpose.[1−7] In the cavity QED system, the scheme for direct observation of Wigner characteristic function of a cavity field was first proposed by Wilkens and Meystre via the nonlinear atomic homodyne detection.[8] Kim *et al.* have made a similar proposal based on current experimental conditions.[9] In Ref. [10], Lutterbach and Davidovich presented a scheme for directly measuring Wigner function applicable to cavity QED and ion trap, which has recently been experimentally demonstrated by Haroche's group.[11] In Ref. [12], Bardroff *et al.* proposed a simple and rapid scheme to measure the Wigner characteristic function of the motional state of a trapped ion. This scheme can be applied to measure the quantum state of a cavity field via realizing a displacement operator for the cavity field correlated to the internal state of the atom. Zou *et al.* proposed a scheme for reconstruction of the cavity field by considering the resonant atom–cavity interaction in the presence of a strong classical field.[13] All the above reconstruction schemes concentrate on the single-mode case. In Ref. [14], Kim and Agarwal presented a scheme to reconstruct a two-mode entangled cavity field states via the interaction of a V-type three-level atom with displaced field. One of our authors has also proposed a scheme for measuring the Wigner function of two-mode cavity field based on the dispersive interaction, in which the atom passing through the cavity should separately interact with two cavity modes with different frequencies.[15] The experimental realization of this scheme is difficult.

In this paper, we present an alternative scheme for the direct measurement of Wigner function of two-mode field in both microwave and optical cavity QED systems. The multi-level atoms resonantly interact with the two cavity modes with orthogonal polarization and an external strong classical field. Unlike the previous scheme,[15] the atom can simultaneously interact with two cavity modes, and thus the process is simplified. Besides, using the method of atomic spin analysis, our scheme can be applied to the quantum non-demolition (QND) measurement of the two-mode cavity field.

The paper is organized as follows. In Section 2, we deduce an effective Hamiltonian of an atom interacting with two-mode cavity field in the presence of a strong classical field. In Section 3, applications of this effective Hamiltonian, including the measurement of Wigner function and QND measurement, are consid-

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ered. We summarize our results in Section 4.

2. Derivation and dynamics of the effective system Hamiltonian

The schematic setup of our scheme is depicted in Fig. 1(a). For microwave cavity setup, the high-Q superconducting cavity C has two non-degenerate modes with orthogonal polarizations, which are probed by a pulsed monokinetic stream of rubidium atoms excited in box B to the circular Rydberg state. The atoms are coupled to a classical field injected by the microwave source S when they are passing through the cavity, and are detected by ionization counter D discriminating the states $|i\rangle$ and $|r\rangle$. For optical cavity setup, cold atoms are delivered into the cavity mode from a small cloud of caesium atoms. The atoms are shined by an external laser and can finally be detected by direct detection of spontaneously emitted photons using high quality optics. The idea here will be generalized in optical domain with the ultrahigh-Q toroidal cavity setup, in which the atom–cavity coupling strength can achieve a few hundred MHz and the critical photon number and critical atomic number can be in the order of 10^{-5} and 10^{-7}, respectively.[16] The energy level structure of atom is shown in Fig. 1(b), including three ground states $|e\rangle$, $|i\rangle$ and $|g\rangle$, and an excited state $|r\rangle$.

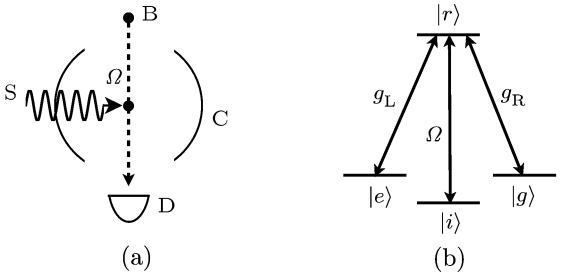


Fig. 1. (a) Schematic setup of the scheme. The atoms interact with the two-mode cavity field and are driven by an external strong classical field with Rabi frequency Ω. The atoms are finally detected. (b) Atomic level structure. The atom has three ground states $|e\rangle$, $|i\rangle$ and $|g\rangle$, and an excited state $|r\rangle$. The transitions $|r\rangle \to |e\rangle$ and $|r\rangle \to |g\rangle$ are coupled to left-circularly polarized mode and right-circularly polarized mode with the coupling strength $g_{\rm L}$ and $g_{\rm R}$, respectively, while the transition $|r\rangle \to |i\rangle$ is driven by an external classical field with Rabi frequency Ω.

The transitions $|r\rangle \to |e\rangle$ and $|r\rangle \to |g\rangle$ are coupled to left-circularly polarized mode and right-circularly polarized mode, with the coupling strengths $g_{\rm L}$ and $g_{\rm R}$, respectively. The transition $|r\rangle \to |i\rangle$ is driven by an external classical field (frequency w), with Rabi frequency Ω. The associated Hamiltonian reads

$$H = H_0 + H_I \tag{1}$$

with

$$\begin{aligned} H_0 =\ & w_{\rm r}|r\rangle\langle r| + w_{\rm e}|e\rangle\langle e| + w_{\rm g}|g\rangle\langle g| \\ & + w_{\rm ca}a^\dagger a + w_{\rm cb}b^\dagger b, \end{aligned} \tag{2}$$

and

$$\begin{aligned} H_I =\ & g_{\rm L}(a|r\rangle\langle e| + a^\dagger|e\rangle\langle r|) + g_{\rm R}(b|r\rangle\langle g| + b^\dagger|g\rangle\langle r|) \\ & + \Omega(\mathrm{e}^{-\mathrm{i}\,wt}|r\rangle\langle i| + \mathrm{e}^{\mathrm{i}\,wt}|i\rangle\langle r|), \end{aligned} \tag{3}$$

where a (b) is the annihilation operator of the left-circularly (right-circularly) polarized mode, with frequency $w_{\rm ca}$ ($w_{\rm cb}$). $w_{\rm r}$, $w_{\rm e}$ and $w_{\rm g}$ are Bohr frequencies of atomic states $|r\rangle$, $|e\rangle$ and $|g\rangle$, respectively. Then the interaction Hamiltonian, in the interaction picture, is

$$\begin{aligned} H_{\rm I}^1 =\ & \mathrm{e}^{\mathrm{i}\,H_0 t}H_{\rm I}\mathrm{e}^{-\mathrm{i}\,H_0 t} \\ =\ & g_{\rm L}(a|r\rangle\langle e| + a^\dagger|e\rangle\langle r|) + g_{\rm R}(b|r\rangle\langle g| + b^\dagger|g\rangle\langle r|) \\ & + \Omega(|r\rangle\langle i| + |i\rangle\langle r|). \end{aligned} \tag{4}$$

In order to obtain physical insight into the dynamics of such a system, we introduce the atomic dressed basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|i\rangle \pm |r\rangle)$. Then the Hamiltonian can be rewritten as

$$\begin{aligned} H_{\rm I}^1 =\ & \frac{1}{\sqrt{2}}g_{\rm L}[a(|+\rangle - |-\rangle)\langle e| + a^\dagger|e\rangle(\langle +| - \langle -|)] \\ & + \frac{1}{\sqrt{2}}g_{\rm R}[b(|+\rangle - |-\rangle)\langle g| + b^\dagger|g\rangle(\langle +| - \langle -|)] \\ & + \Omega(|+\rangle\langle +| - |-\rangle\langle -|). \end{aligned} \tag{5}$$

After that, we switch to the interaction picture with respect to $\Omega(|+\rangle\langle +| - |-\rangle\langle -|)$, and rewrite $H_{\rm I}^1$ as

$$\begin{aligned} H_{\rm I}^2 =\ & \frac{1}{\sqrt{2}}g_{\rm L}[a(\mathrm{e}^{\mathrm{i}\,\Omega t}|+\rangle - \mathrm{e}^{-\mathrm{i}\,\Omega t}|-\rangle)\langle e| \\ & + a^\dagger|e\rangle(\langle +|\mathrm{e}^{-\mathrm{i}\,\Omega t} - \langle -|\mathrm{e}^{\mathrm{i}\,\Omega t})] \\ & + \frac{1}{\sqrt{2}}g_{\rm R}[b(\mathrm{e}^{\mathrm{i}\,\Omega t}|+\rangle - \mathrm{e}^{-\mathrm{i}\,\Omega t}|-\rangle)\langle g| \\ & + b^\dagger|g\rangle(\langle +|\mathrm{e}^{-\mathrm{i}\,\Omega t} - \langle -|\mathrm{e}^{\mathrm{i}\,\Omega t})] \\ =\ & -\frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\,\Omega t}(g_{\rm L}a|-\rangle\langle e| + g_{\rm R}b|-\rangle\langle g|) + \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\,\Omega t} \\ & \times (g_{\rm L}a^\dagger|e\rangle\langle +| + g_{\rm R}b^\dagger|g\rangle\langle +|) + \mathrm{H.c.} \end{aligned} \tag{6}$$

In the strong driving regime, i.e., with the choice of $\Omega \gg g_{\rm L}, g_{\rm R}$, neglecting the effect of rapidly oscillating terms and using the time averaging methods of

Ref. [17], we can further reduce $H_{\rm I}^2$ to an effective interaction Hamiltonian

$$H_{\rm eff} = \left(\frac{g_{\rm L}^2}{2\Omega} aa^\dagger + \frac{g_{\rm R}^2}{2\Omega} bb^\dagger\right)(|+\rangle\langle +| - |-\rangle\langle -|). \quad (7)$$

We have numerically simulate the validity of this effective Hamiltonian by taking the system evolution into consideration, as shown in Fig. 2(a). In addition, the probabilities of the states $|e\rangle$ and $|g\rangle$ being populated are shown in Fig. 2(b).

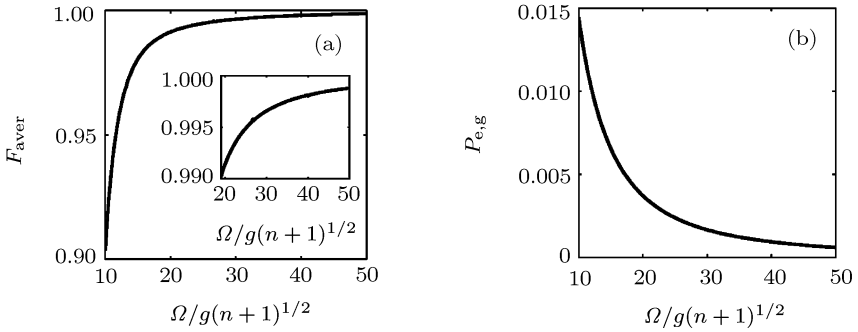


Fig. 2. (a) The time-averaged fidelity $F_{\rm aver}$ of state evolution without decay in the first time interval $t_{\rm d} = 1$ ms ($t_{\rm d} = 0.1$ μs) versus $\Omega/g\sqrt{n+1}$ in the microwave (optical) cavity setup. (b) The probabilities $P_{\rm e}$ and $P_{\rm g}$ of the atom being found in the states $|e\rangle$ and $|g\rangle$ versus $\Omega/g\sqrt{n+1}$. The atom is supposed to be initially in the state $|i\rangle$, and the two-mode cavity field is in Fock state $|n_{\rm L}\rangle|n_{\rm R}\rangle$ in both (a) and (b). Here we have set $g_{\rm L} = g_{\rm R} = g$, $n_{\rm L} = n_{\rm R} = 8$, and $g/2\pi = 50$ kHz (500 MHz) in microwave (optical) setup.

The atom is supposed to be initially in the state $|i\rangle$, and the two-mode cavity field is in Fock state $|n_a\rangle|n_b\rangle$. The system state governed by Hamiltonian (4) at time t is assumed to be $|\phi_{\rm real}(t)\rangle$, and governed by Eq. (7) is $|\phi_{\rm eff}(t)\rangle$. It is found that the time-averaged fidelity $F_{\rm aver} = \langle|\langle\phi_{\rm eff}(t)|\phi_{\rm real}(t)\rangle|^2\rangle$ in the first time interval $t_{\rm d} = 1$ ms ($t_{\rm d} = 0.1$ μs) in microwave (optical) cavity setup is over 0.99 if $\Omega > 19g_\mu\sqrt{n+1}$, μ=L, R, where we have set $g_{\rm L} = g_{\rm R} = g$ and $n_a = n_b = n$ for simplicity. In this case, the time-averaged probability $P_{\rm e}$ ($P_{\rm g}$) for the atom being found in the state $|e\rangle$ ($|g\rangle$) is less than 0.005. To evaluate the effect of decoherence on the time evolution of the system, we will make use of the quantum jump approach. Under the condition that no photon is emitted and the interaction time is much shorter than the spontaneous lifetime, the system evolution can be described by a non-Hermite Hamiltonian

$$H_{\rm nh} = H_{\rm I}^1 - {\rm i}\,\frac{\kappa}{2}(a^\dagger a + b^\dagger b) - {\rm i}\,\frac{\Gamma}{2}|r\rangle\langle r|, \quad (8)$$

where we assume that the two cavity modes have the same decay rate κ, and Γ is the atomic spontaneous emission rate. The temporal evolution of the system state $\phi_{\rm nh}(t)$ is now governed by $H_{\rm nh}$. We redefine the time-averaged fidelity $F'_{\rm aver} = \langle|\langle\phi_{\rm eff}(t)|\phi_{\rm nh}(t)\rangle|^2\rangle$. The numerical result of $F'_{\rm aver}$ as a function of $\Omega/g\sqrt{n+1}$ is given in Fig. 3 with $n = 1, 3, 8, 15$.[18] It is seen that $F'_{\rm aver}$ increases with the growth of $\Omega/g\sqrt{n+1}$ and finally achieves the saturated plateau value. Since the loss of decoherence, the saturation value will decrease with the increase of photon number n. In microwave cavity setup, the average fidelity is larger than 0.93 for $n = 1, 3$ and $\Omega/g\sqrt{n+1} > 15$. In the case that $n = 8$ and $n = 15$, the respective fidelity $F'_{\rm aver}$ can still exceed 0.85 and 0.75 by appropriately selecting the rate $\Omega/g\sqrt{n+1}$. In optical cavity setup, the optimal fidelity for the corresponding photon number is lower than that in microwave cavity setup. It is mainly limited by the atomic spontaneous decay. However, it is possible to suppress the spontaneous emission by means of Purcell effect.[19] Indeed, in the experiment of Haroche's group,[18] the two non-degenerate modes with orthogonal polarizations in the Fabre Perot cavity have exactly the same geometry. When the atomic orbit plane is perpendicular to the cavity axis, the couplings are thus exactly equal. However, this situation might be changed by orienting, with a proper static electric field, the atomic orbit plane. If polarization of mode 1 is still on the plane, but that of mode 2 off plane, the coupling to mode 2 will be somewhat reduced. In their experiment, the orbit plane is perpendicular to the static

directing electric field, naturally oriented along the cavity axis by applying a potential difference between the mirrors. However, the electrodes on the side of the cavity might be used to change the direction of this static field a little bit and to achieve the uneven coupling. On the other hand, the position-dependent coupling strength makes it possible to realize uneven interaction of the atom with two orthogonally polarized cavity modes in optical cavity setup. Using the above methods, we can realize the direct measurement of Wigner function and QND measurement of photon number of two-mode field.

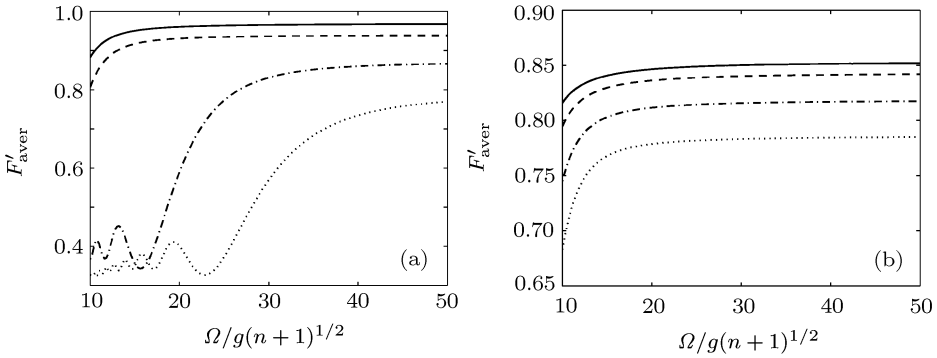


Fig. 3. The time-averaged fidelity $F'_{\rm aver}$ as a function of $\Omega/g\sqrt{n+1}$ for $n = 1$, 3, 8, 15 (from top to bottom) in (a) microwave cavity setup: $(g/2\pi,\ \kappa,\ \Gamma)$=(50, 1/130, 1/30) kHz, and (b) optical cavity setup: $(g, \kappa, \Gamma)/2\pi = (500, 0.01, 1)$ MHz.

3. Applications

3.1. Direct measurement of Wigner function

The effective Hamiltonian $H_{\rm eff}$, which describes the interaction of an atom with two cavity modes, fulfills the condition

$$[aa^\dagger, H_{\rm eff}] = [bb^\dagger, H_{\rm eff}] = 0, \tag{9}$$

thus it can be used to implement schemes for measuring Wigner function and photon number of the cavity field.

We assume the initial density operator of the cavity field is $\hat{\rho}_{\rm f}$. In microwave cavity setup, the pulsed microwave source, coupled by diffraction on the mirrors' edges, can inject into the cavity a coherent field, which induces displacements for the respective cavity modes. This leads to

$$\hat{\rho}_{\rm f}(\alpha,\beta) = \hat{D}_a(\alpha)\hat{D}_b(\beta)\hat{\rho}_{\rm f}\hat{D}_b^\dagger(\beta)\hat{D}_a^\dagger(\alpha), \tag{10}$$

where $\hat{D}_a(\alpha) = {\rm e}^{\alpha a^\dagger - \alpha^* a}$ and $\hat{D}_b(\beta) = {\rm e}^{\beta b^\dagger - \beta^* b}$ are displacement operators. Then an atom, initially prepared in the state $|i\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, is sent through the cavity. After an interaction time t, the density operator of the whole system is

$$\begin{aligned}
\hat{\rho}_{\rm s}(t) =\ & \frac{1}{2}\left[\exp\left[-{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]|+\rangle + \exp\left[{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]|-\rangle\right] \\
& \times \hat{\rho}_{\rm f}(\alpha,\beta)\left[\langle+|\exp\left[{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right] + \langle-|\exp\left[-{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]\right] \\
=\ & \frac{1}{4}\left\{\left[\exp\left[-{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right] + \exp\left[{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]\right]|i\rangle \right. \\
& \left. + \left[\exp\left[-{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right] - \exp\left[{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]\right]|r\rangle\right\}\hat{\rho}_{\rm f}(\alpha,\beta) \\
& \times \left\{\langle i|\left[\exp\left[-{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right] + \exp\left[{\rm i}\left(\frac{g_{\rm L}^2}{2\Omega}aa^\dagger + \frac{g_{\rm R}^2}{2\Omega}bb^\dagger\right)t\right]\right]\right.
\end{aligned}$$

$$+\langle r|\left[\exp\left[\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]-\exp\left[-\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]\right]\Bigg\}. \tag{11}$$

We choose the atomic velocity carefully so that

$$\frac{g_{\mathrm{L}}^2}{2\Omega}t=\left(2m+\frac{1}{2}\right)\pi,\qquad \frac{g_{\mathrm{R}}^2}{2\Omega}t=\left(2n+\frac{1}{2}\right)\pi,\qquad m,n\in N, \tag{12}$$

where $g_{\mathrm{L}}/g_{\mathrm{R}}=1$ is not necessarily required. Indeed, if only $g_{\mathrm{L}}/g_{\mathrm{R}}=\sqrt{(4m+1)/(4n+1)}$ will be good for the case in our consideration. Then we have

$$\begin{aligned}\hat{\rho}_{\mathrm{s}}=&\frac{1}{4}\Bigg\{\left[\exp-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]|i\rangle+\left[\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)t\right]\right.\\&\left.-\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]|r\rangle\Bigg\}\hat{\rho}_{\mathrm{f}}(\alpha,\beta)\Bigg\{\langle i|\left[\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]\\&+\langle r|\left[\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)t\right]-\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]\Bigg\}.\end{aligned} \tag{13}$$

We now measure the state of the atom. The probability of finding the atom in the state $|i\rangle$ is

$$P_i(\alpha,\beta)=\frac{1}{2}+\frac{1}{2}\,\mathrm{Tr}\left[\exp\left[\mathrm{i}\,\pi(a^\dagger a+b^\dagger b)\right]\hat{\rho}_{\mathrm{f}}(\alpha,\beta)\right]. \tag{14}$$

The Wigner function for a two-mode field is expressed as[20]

$$\begin{aligned}W(\alpha,\beta)=&\frac{4}{\pi^2}\,\mathrm{Tr}\left[\exp\left[\mathrm{i}\,\pi(a^\dagger a+b^\dagger b)\right]\widehat{D}_a(-\alpha)\widehat{D}_b(-\beta)\widehat{\rho}_{\mathrm{f}}\widehat{D}_b^\dagger(-\beta)\widehat{D}_a^\dagger(-\alpha)\right]\\=&\frac{4}{\pi^2}\,\mathrm{Tr}\left[\exp\left[\mathrm{i}\,\pi(a^\dagger a+b^\dagger b)\right]\widehat{\rho}_{\mathrm{f}}(-\alpha,-\beta)\right].\end{aligned} \tag{15}$$

Combining Eqs. (14) and (15) we obtain

$$W(-\alpha,-\beta)=\frac{8}{\pi^2}\left[P_i(\alpha,\beta)-\frac{1}{2}\right]. \tag{16}$$

Therefore the measurement of the probability at the point (α,β) directly yields the two-mode Wigner function at the point $(-\alpha,-\beta)$. Since $P_i(\alpha,\beta)$ lies between 0 and 1, it follows that the Wigner function is bounded:

$$-\frac{4}{\pi^2}\le W(-\alpha,-\beta)\le\frac{4}{\pi^2}. \tag{17}$$

Now suppose the q-th atom is not detected after interacting with the cavity in the process of measurement. Tracing out the atomic degree of freedom, we find the reduced density operator of the cavity field,

$$\begin{aligned}\hat{\rho}_{\mathrm{f}}'(\alpha,\beta)=&\,\mathrm{Tr}_A(\hat{\rho}_{\mathrm{s}})\\=&\frac{1}{2}\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\hat{\rho}_{\mathrm{f}}(\alpha,\beta)\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\\&+\frac{1}{2}\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\hat{\rho}_{\mathrm{f}}(\alpha,\beta)\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right].\end{aligned} \tag{18}$$

After the $(q+1)$-th atom passing through the cavity, the system becomes

$$\begin{aligned}\hat{\rho}_{\mathrm{s}}'=&\frac{1}{8}\Bigg\{\left[\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]|i\rangle+\left[\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)t\right]\right.\\&\left.-\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]|r\rangle\Bigg\}\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\hat{\rho}_{\mathrm{f}}(\alpha,\beta)\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\\&\times\Bigg\{\langle i|\left[\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]+\langle r|\left[\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)t\right]\right.\\&\left.-\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]\Bigg\}+\frac{1}{8}\Bigg\{\left[\exp\left[-\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[\mathrm{i}\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]|i\rangle\end{aligned}$$

$$+\left[\exp\left[-\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)t\right]-\exp\left[\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]|r\rangle\Big\}\exp\left[\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\hat{\rho}_{\mathrm{f}}(\alpha,\beta)$$
$$\times\exp\left[-\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\Big\{\langle i|\left[\exp\left[\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]+\exp\left[-\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]$$
$$+\langle r|\left[\exp\left[\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)t-\exp\left[-\mathrm{i}\,\frac{\pi}{2}(aa^\dagger+bb^\dagger)\right]\right]\right]\Big\}. \tag{19}$$

The probability of detecting this atom in state $|i\rangle$ is again given by equation (14). Thus the scheme is insensitive to the detection efficiency. On the other hand, the least time needed for our scheme requires $(g_\mu^2/2\Omega)t=\frac{\pi}{2}$ (μ=L, R), i.e., $g_u t=(\Omega/g_\mu)\pi$. In the F-P cavity of Ref. [18], $g_u/2\pi=50$ kHz, we obtain $t=0.5$ ms with $\Omega/g_\mu=50$, which is much less than the lifetime of a single photon $T_{\mathrm{c}}\simeq 130$ ms. Assume that the cavity field is initially in the state $|n_a\rangle|n_b\rangle$, then after the first atom passing through the cavity the fidelity $F=|\langle\phi_{\mathrm{eff}}(t)|\phi_{\mathrm{nh}}(t)\rangle|^2$ of the system state for $t=0.5$ ms with $n=0,1,\ldots,15$ is shown in Fig. 4(a). It is seen that the fidelity is larger than 0.9 for $n\leq 8$. Thus our scheme is experimentally feasible in this regime by combining effective data analysis and making full use of atomic detection rate.[18] Similar results can be obtained in optical cavity setup if the cavity field is initially displaced, which can be implemented by sending an atom driven by strong classical field through the cavity firstly.[21] Likewise, we plot the fidelity F as a function of n in Fig. 4(b) with $g_u/2\pi=500$ MHz, $\Omega/g_\mu=50$, and $t=0.05$ μs. The fidelity can exceed 0.8 for $n\leq 6$, which is still of experimental interest.

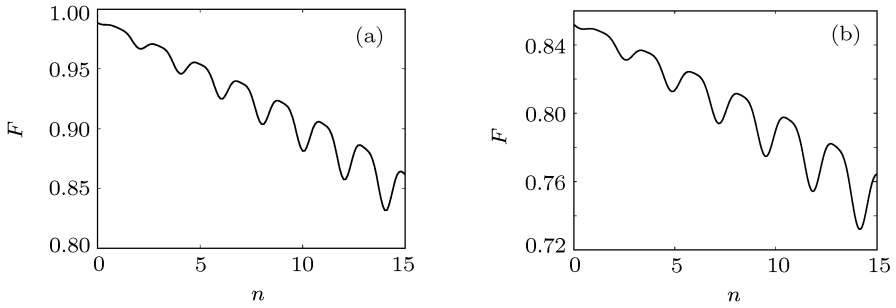


Fig. 4. The fidelity F of the system state versus photon number n at time t in (a) microwave cavity setup: $t=0.5$ ms, $(g/2\pi,\,\kappa,\,\Gamma)$=(50, 1/130, 1/30) kHz, and (b) optical cavity setup: $(g,\kappa,\Gamma)/2\pi=(500,0.01,1)$ MHz. Here we have set $n_{\mathrm{L}}=n_{\mathrm{R}}=n$ and $g_{\mathrm{L}}=g_{\mathrm{R}}=g$.

3.2. QND measurement of photon number

A QND measurement is a detection process which, instead of erasing entirely all information in the state of system it has been detected, projects it into the eigenstate corresponding to the measurement result and thus preserves useful information for subsequent processing. The process should be repeatable. QND measurements can not only be used as very sensitive probes of small perturbations acting on a system, but also used for unlimited distribution of optical signals in optics.[22] Various schemes for QND measurements with different physical systems have been experimentally realized.[23−25] Recently, Guerlin *et al.* have observed a step-by-step collapse by non-destructively measuring the photon number of a field stored in a cavity.[26] More recently, the same group completed another experiment about process tomography of field damping and measurement of Fock state lifetimes by QND photon counting in a cavity.[27] A closely related work has also been done on a superconducting quantum circuit.[28] These experiments provided us a nearly ideal system to implement the following scheme for QND measurement of two-mode field.

Suppose that the two-mode cavity field are initially in Fock state $|n_a\rangle|n_b\rangle$. An atom, prepared in the state $|i\rangle=\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)$, is sent through the cavity with interaction time t, then the system state is given by

$$|\Psi_{\mathrm{s}}(t)\rangle=\mathrm{e}^{-\mathrm{i}\,H_{\mathrm{eff}}t}\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)|n_a\rangle|n_b\rangle$$
$$=\left\{\exp\left[-\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]\frac{|+\rangle}{\sqrt{2}}+\exp\left[\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]\frac{|-\rangle}{\sqrt{2}}\right\}|n_a\rangle|n_b\rangle$$

$$= \frac{1}{2}\left\{\left[\exp\left[-\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]+\exp\left[\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]\right]|i\rangle \right.$$
$$\left.+\left[\exp\left[-\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]-\exp\left[\mathrm{i}\left(\frac{g_{\mathrm{L}}^2}{2\Omega}aa^\dagger+\frac{g_{\mathrm{R}}^2}{2\Omega}bb^\dagger\right)t\right]\right]|r\rangle\right\}|n_a\rangle|n_b\rangle$$
$$= \left\{\cos\left[\left(\frac{g_{\mathrm{L}}^2}{2\Omega}(n_a+1)+\frac{g_{\mathrm{R}}^2}{2\Omega}(n_b+1)\right)t\right]|i\rangle-\mathrm{i}\sin\left[\left(\frac{g_{\mathrm{L}}^2}{2\Omega}(n_a+1)+\frac{g_{\mathrm{R}}^2}{2\Omega}(n_b+1)\right)t\right]|r\rangle\right\}|n_a\rangle|n_b\rangle. \quad (20)$$

The probability of finding the atom in the state $|i\rangle$ is

$$P_i=\cos^2\left[\left(\frac{g_{\mathrm{L}}^2}{2\Omega}(n_a+1)+\frac{g_{\mathrm{R}}^2}{2\Omega}(n_b+1)\right)t\right]. \quad (21)$$

Applying a static electric field to the atom when it was passing through cavity or delivering the atom to different positions, we can realize different values of $g_{\mathrm{L}}/g_{\mathrm{R}}$, and thus change the detection probability. Then the photon number of two cavity modes can be calculated by

$$n_a=\frac{G_{i1}-G_{i2}}{C_1^2-C_2^2},\quad n_b=\frac{C_2^2G_{i1}-C_1^2G_{i2}}{C_2^2-C_1^2}, \quad (22)$$

where $G_{\mathrm{i}\,k}=(\Omega/g_{\mathrm{R}}^2t)\arccos(2P_{ik}-1)$, $k=1,2$. P_{i1} and P_{i2} are detection probabilities in the case of $g_{\mathrm{L}}/g_{\mathrm{R}}=C_1$ and $g_{\mathrm{L}}/g_{\mathrm{R}}=C_2$, respectively. Thus, the measurement of these probabilities gives information on the photon number.

Assume that after k atoms passing through the cavity, the density operator of the cavity field is $\hat{\rho}^{\mathrm{f}}(k)=\sum_{n_a,n_b,n_a',n_b'}\rho^{\mathrm{f}}_{n_a,n_bn_a',n_b'}(k)|n_a,n_b\rangle\langle n_a',n_b'|$. Then after the $(k+1)$-th atom going through cavity, the density operator of the whole system evolves to

$$\hat{\rho}(k+1)=\sum_{n_a,n_bn_a',n_b'}\sum\rho^{\mathrm{f}}_{n_a,n_bn_a',n_b'}(k)|n_a,n_b\rangle\langle n_a',n_b'|$$
$$\otimes\left[d_{\mathrm{i},n_a,n_b}(\tau_{k+1})\,|i\rangle+d_{\mathrm{r},n_a,n_b}(\tau_{k+1})\,|r\rangle\right]$$
$$\times\left[\langle i|d^*_{\mathrm{i},n_a',n_b'}(\tau_{k+1})+\langle r|d^*_{\mathrm{r},n_a',n_b'}(\tau_{k+1})\right], \quad (23)$$

where

$$d_{\mathrm{i},n_a,n_b}(\tau_{k+1})=\cos\left[\left(\frac{g_{\mathrm{L}}^2}{2\Omega}(n_a+1)+\frac{g_{\mathrm{R}}^2}{2\Omega}(n_b+1)\right)\tau_{k+1}\right],$$
$$d_{\mathrm{r,n_a,n_b}}(\tau_{k+1})=\mathrm{i}\sin\left[\left(\frac{g_{\mathrm{L}}^2}{2\Omega}(n_a+1)+\frac{g_{\mathrm{R}}^2}{2\Omega}(n_b+1)\right)\tau_{k+1}\right], \quad (24)$$

τ_{k+1} is the interaction time of the cavity field with the $(k+1)$-th measuring atom. If we find this atom in the state $|\ell\rangle$ $(\ell=i,r)$, the cavity field collapses to

$$\hat{\rho}^{\mathrm{f}}(k+1)=\frac{\sum_{n_a,n_b}\sum_{n_a',n_b'}\rho^{\mathrm{f}}_{n_a,n_b,n_a',n_b'}(k)d_{\ell,n_a,n_b}(\tau_{k+1})d^*_{\ell,n_a',n_b'}(\tau_{k+1})|n_a,n_b\rangle\langle n_a',n_b'|}{\sum_{n_a,n_b}|d_{\ell,n_a,n_b}(\tau_{k+1})|^2\rho^{\mathrm{f}}_{n_a,n_b,n_a,n_b}(k)}. \quad (25)$$

Repeat the procedure, which reduces the density operator of the field $\hat{\rho}^{\mathrm{f}}$ until the two-mode field collapses to a Fock state $|n_a,n_b\rangle$. Further measurements do not change the field state any more. We cannot predict the final Fock state, but only a probability of its occurrence depending on the photon distribution of the initial field state. Repeating the experiment many times we can obtain the probability of the occurrence of each Fock state and thus determine the initial photon number distribution of the two-mode field.[27]

4. Summary

We have derived an effective Hamiltonian by using time averaging methods from the precise atom–cavity coupled model. The dynamical evolution of this effective Hamiltonian can be used for direct measurement of Wigner function. By controlling the orbit plane of Rydberg atom in microwave cavity setup or changing the position-dependent coupling strength in optical cavity setup, the effective Hamiltonian can also be used for QND measurement, according to which the photon number of two orthogonally polarized modes can be calculated by

successively sending and detecting the atoms and given by atomic spin analysis. Our ideas can be generalized for reconstruction and test of nonlocality of the cavity field state.

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