Wave growth rate in a cylindrical metal waveguide with ion-channel guiding of a relativistic electron beam*

Li Hai-Rong(李海容)[a)b)†], Tang Chang-Jian(唐昌建)[a)], and Wang Shun-Jin(王顺金)[a)]

[a)] School of Physics Science and Technology, Sichuan University, Chengdu 610064, China
[b)] College of Science, University of Science and Technology Liaoning, Anshan 114051, China

(Received 24 March 2010; revised manuscript received 27 April 2010)

This paper addresses the formulae and numerical issues related to the possibility that fast wave may be grown when a relativistic electron beam through an ion channel in a cylindrical metal waveguide. To derive the dispersion equations of the beam–wave interaction, it solves relativistic Lorentz equation and Maxwell's equations for appropriate boundary conditions. It has been found in this waveguide structure that the TM_{0m} modes are the rational operating modes of coupling between the electromagnetic modes and the betatron modes. The interaction of the dispersion curves of the electromagnetic TM_{0m} modes and the upper betatron modes is studied. The growth rates of the wave are obtained, and the effects of the beam radius, the beam energy, the plasma frequency, and the beam plasma frequency on the wave growth rate are numerically calculated and discussed.

Keywords: relativistic electron beam, plasma, ion channel, dispersion relations, wave growth rate

PACC: 4170, 4775, 5235H

1. Introduction

In recent times a great deal of interest has developed in ion-channel guiding (ICG) of a relativistic electron beam (REB) since the focusing effect of the ion channel (IC) was proposed as a novel focusing mechanism for the transport of the REB.[1] As the REB is injected into the preformed plasma, in a blow-out region, it expels all the background plasma electrons out from a region around the axis and forms the ion channel.[2−4] An ion channel can be used to guide and focus the REB, which allows higher beam current than the vacuum limit, helps radiation guiding, and also eliminates the need for conventional focusing magnets. There are important applications in this subject to accelerators[5,6] and radiation sources, such as the plasma wake-field accelerator, the free electron laser with ICG,[7−10] the ion-channel laser (ICL),[11−14] and those devices including an intense electron beam as their active gain medium as the dielectric Cherenkov maser, traveling wave tube (TWT)[15] and backward-wave oscillator (BWO).[16]

To create an ion channel, the electron beam density n_b is higher than the preformed plasma density n_p. The channel radius $r_i \simeq r_0(n_b/n_p)^{1/2}$ is larger than the electron beam radius r_0. It is assumed that the transverse electric field due to the ion charge is much larger than the transverse force on the beam due to self-fields, which requires $n_p \gg n_b/\gamma^2$, where γ is the Lorentz factor. The beam electrons undergo transverse oscillations at a betatron frequency $\omega_b = \omega_p/\sqrt{2\gamma}$, where $\omega_p = (n_p e^2/\varepsilon_0 m)^{1/2}$ is the plasma frequency, $-e$ is the electron charge, m is the electron rest mass, and ε_0 denotes the permittivity in vacuum. The plasma wiggler strength is given by $K = \gamma\omega_b r_0/c$, where c is the speed of light. As an electron undergoes betatron oscillation in an ion channel, it will emit synchrotron radiation. The ICL was conceived as a means to generate tunable coherent radiation. In this device, ion focused force provides a much higher wiggler strength than that of a conventional magnetic wiggler, leads to a more effective generation of radiation.[17] If $K \gg 1$, high harmonic radiation dominates the spectrum and a broadband spectrum is generated.[18] When $K \ll 1$, radiation is emitted primarily at the fundamental frequency $\omega = 2\gamma_{z0}^2\omega_b$, where γ_{z0} is the initial Lorentz factor for the electron entering the channel. As K approaches unity, scattered radiation will appear at harmonics of the resonant frequency.

*Project supported in part by the National Natural Science Foundation of China (Grant Nos. 10775100 and 90503008), the Science Foundation of China Academy of Engineering Physics (Grant No. 10576019), and the Fund of Theoretical Nuclear Physics Center, National Laboratory of Heavy Ion Accelerator Facility of Lanzhou.

†Corresponding author. E-mail: haeroy@sohu.com

© 2010 Chinese Physical Society and IOP Publishing Ltd

This paper analyses the axial thin REB in an ICG-metal waveguide, which is an empty metal cylindrical waveguide partly filled by an ion channel, operated in the fast-wave regime. We consider that the preformed plasma radius $r_{\rm p}$ is less than the self-consistent ion channel radius $r_0(n_{\rm b}/n_{\rm p})^{1/2}$ and the neutral gas can be dealt as vacuum just like the ion channel, which means an axial thin REB with ion channel inside a loss-free metal wall. Hence, when a and R are the beam radius and inner radius of metal wall respectively, as illustrated in Fig. 1, the a/R is arbitrary. In addition, we deal the tiny REB with $n_{\rm b}/\gamma^2 < n_{\rm p} < n_{\rm b}$, and all transverse and longitudinal oscillations to be unrelativistic and small are assumed. In this work, the electrons motion is approximately linear. We shall present a self-consistent relativistic field theory of the REB betatron resonance radiation and take self-fields of the electron beam into account. It will be shown that, under certain conditions, a fast-wave electron betatron resonance radiation can be operated with direct axial injection of an electron beam. The presented analysis is based on Maxwell's equations, the cold-fluid continuity equation and relativistic momentum-transfer equation. It will be carried out in the laboratory frame by using a cylindrical coordinate system (r, θ, z) with the beam and waveguide axes taken as the z axis.

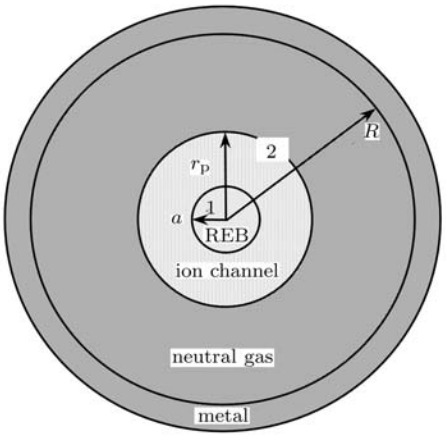


Fig. 1. The cross section of the ICG waveguide. The region 1 denotes the REB, the region 2 including the IC and the neutral gas.

The paper is organized as follows. Section 2 deals with the basic formulas such as the expressions of transverse components of the field in terms of longitudinal components and the longitudinal wave equations from relativistic Lorentz equation, Maxwell's equations and the equation of continuity. In Section 3 the dispersion relation is derived. In Section 4, we present the wave growth rate. Section 5 includes results of calculation and analysis of wave growth rate. Section 6 contains the conclusions.

2. Basic analysis of the REB with ICG

In the present calculations, we consider that the system consists of a net ion channel and equilibrium REB, where the equilibrium REB employs an ion channel density $n_{\rm p0} = n_{\rm b}/\gamma^2$ and the net ion channel density is $n_{\rm p,net} = n_{\rm p} - n_{\rm p0}$. The analysis will be limited to near-axis betatron motion of beam electron with an equilibrium position $r_0 = 0$. It assumes that the electrons beam radius is unaltered under the little betatron motion and the perturbation due to the net ion channel and the propagating electromagnetic (EM) wave respectively. In this model, the eigenmodes of cylindrical metal waveguide without beam are $\rm TM_{nm}$ and $\rm TE_{nm}$ modes, in which the asymmetric TM and TE modes will be replaced by HE and EH modes when an REB presents. Therefore, the interaction of the electron beam with transverse magnetic $\rm TM_{0m}$ and $\rm TE_{0m}$ mode should be discussed. The remainder of this section will treat the perturbations and find the basic formulas. The perturbed term is assumed to have azimuthal symmetry and depend on $\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}$, where ω and k_z are the angular frequency and the axial wave number.

2.1. Relativistic single electron dynamics

To study electron beam dynamics, we integrate the fully relativistic electron equations of motion in the ion channel field, the REB self-field, and the disturbed EM field determined by Maxwell equations. Consider a relativistic and cold electron beam moving with velocity $\boldsymbol{u}_0$ ($\boldsymbol{u}_0 = \boldsymbol{\beta} c$ and $\gamma = \left(1-\beta^2\right)^{-1/2}$) along the z axis in the presence of an ion channel having its axis coincident with the z axis. The total velocity $\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}$ is the sum of steady state and perturbed components of particle velocity, and the total charge density $n = n_0 + \tilde{n}$ is the sum of the steady state and oscillatory components of the electron density.

The electron motion equation will be obtained by solving relativistic Lorentz equation. Here the transverse focusing force and longitudinal space charge force are both considered. The component equations of relativistic Lorentz equation can be written as

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\cdot\boldsymbol{\nabla}\right]\tilde{\boldsymbol{u}}_{\perp}$$
$$=-\frac{e}{m\gamma}\left\{\tilde{\boldsymbol{E}}_{\perp}+\frac{e\left(n_{\mathrm{p}}-n_{\mathrm{e}}\right)}{2\varepsilon_0}\tilde{\boldsymbol{r}}+\left[(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\times\tilde{\boldsymbol{B}}\right]_{\perp}-\frac{n_{\mathrm{e}}e}{2c\varepsilon_0}\left[(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\times(\boldsymbol{\beta}\times\tilde{\boldsymbol{r}})\right]_{\perp}\right\}, \quad (1)$$

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\cdot\boldsymbol{\nabla}\right]\tilde{u}_z$$
$$=-\frac{e}{m\gamma^3}\left\{\tilde{E}_z+\left[(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\times\tilde{\boldsymbol{B}}\right]\cdot\hat{\boldsymbol{z}}-\frac{n_{\mathrm{e}}e}{2c\varepsilon_0}\left[(\boldsymbol{u}_{0z}+\tilde{\boldsymbol{u}})\times(\boldsymbol{\beta}\times\tilde{\boldsymbol{r}})\right]\cdot\hat{\boldsymbol{z}}\right\}, \quad (2)$$

where $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ are the fluctuating electric and magnetic fields, the subscript $\perp$ refers to the transverse component, the subscript z refers to the longitudinal (axial) physical quantity, $\hat{\boldsymbol{z}}$ denotes the longitudinal unit vector, and the γ^3 is originated from the variation of the total velocity and the qualification of $\tilde{u}\ll u_0$.

Retaining terms up to first order in oscillatory components and using the expression of radial velocity

$$\begin{aligned}\tilde{\boldsymbol{u}}_r&=\hat{\boldsymbol{r}}\,\mathrm{d}\tilde{r}/\mathrm{d}t\\&=\hat{\boldsymbol{r}}\left(\partial/\partial t+u_{0z}\nabla_z\right)\tilde{r}\\&=-\mathrm{i}\left(\omega-\beta ck_z\right)\tilde{\boldsymbol{r}},\end{aligned} \quad (3)$$

where $\nabla_z=\partial/\partial z$ is the longitudinal gradient functor, we reduce the component equations of motion to

$$\tilde{\boldsymbol{u}}_r=-\mathrm{i}\frac{e}{m\gamma}\frac{1}{(\omega-\beta ck_z)}\left(\tilde{\boldsymbol{E}}_r+\boldsymbol{\beta}c\times\tilde{\boldsymbol{B}}_\theta\right)\eta_r, \quad (4)$$

$$\tilde{\boldsymbol{u}}_\theta=-\mathrm{i}\frac{e}{m\gamma}\frac{1}{(\omega-\beta ck_z)}\left(\tilde{\boldsymbol{E}}_\theta+\boldsymbol{\beta}c\times\tilde{\boldsymbol{B}}_r\right), \quad (5)$$

$$\tilde{\boldsymbol{u}}_z=-\mathrm{i}\frac{e}{m\gamma^3}\frac{1}{(\omega-\beta ck_z)}\tilde{\boldsymbol{E}}_z, \quad (6)$$

where

$$\eta_r=(\omega-\beta ck_z)^2/[(\omega-\beta ck_z)^2-\omega_{\mathrm{b}}^2] \quad (7)$$

is the parameter of the REB with ICG, and the electron betatron frequency becomes

$$\omega_{\mathrm{b}}^2=(\omega_{\mathrm{i}}^2-\omega_{\mathrm{e}}^2/\gamma^2)/(2\gamma), \quad (8)$$

since the self-fields of the electron beam are taken into account, $\omega_{\mathrm{e}}=(n_{\mathrm{e}}e^2/\varepsilon_0 m)^{1/2}$ is the plasma frequency of the REB.

2.2. Perturbed current and charge densities

From now on, we shall solve the linearized Maxwell's equations

$$\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}=\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad (9)$$

$$\boldsymbol{\nabla}\times\tilde{\boldsymbol{B}}=\mu_0\tilde{\boldsymbol{J}}-\mathrm{i}\omega\varepsilon_0\mu_0\tilde{\boldsymbol{E}}, \quad (10)$$

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}}=-e\tilde{n}/\varepsilon_0, \quad (11)$$

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}}=0, \quad (12)$$

for oscillated EM fields presented in the waveguide along with the continuity equation of beam

$$\mathrm{i}\omega e\tilde{n}+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{J}}=0, \quad (13)$$

where $\tilde{\boldsymbol{J}}$ is the perturbed current density of the beam.

Total current density is $\boldsymbol{J}=-e\left(n_0+\tilde{n}\right)(\boldsymbol{u}_{z0}+\tilde{\boldsymbol{u}})$. Since the perturbed components are assumed to be small, we can neglect second order term and take the approximation of perturbed current density. Using Eqs. (4), (6), (9), and (13), we can obtain the perturbed current densities of the electron beam affected by perturbed EM field and the betatron focused force as

$$\tilde{J}_r=\mathrm{i}\omega\varepsilon_0\frac{\omega_{\mathrm{e}}^2}{\gamma}\left(\frac{1}{\omega^2}\tilde{E}_r+\frac{\beta c}{\mathrm{i}\omega^2(\omega-\beta ck_z)}\nabla_r\tilde{E}_z\right)\eta_r, \quad (14)$$

$$\tilde{J}_\theta=\mathrm{i}\omega\varepsilon_0\frac{\omega_{\mathrm{e}}^2}{\gamma}\left(\frac{1}{\omega^2}\tilde{E}_\theta+\frac{\beta c}{\mathrm{i}\omega^2(\omega-\beta ck_z)}\nabla_\theta\tilde{E}_z\right), \quad (15)$$

$$\tilde{J}_z=\mathrm{i}\omega\varepsilon_0\frac{\omega_{\mathrm{e}}^2}{\gamma}\times\left\{\frac{\beta c}{\mathrm{i}\omega^2(\omega-\beta ck)}(\eta_r\nabla_r\tilde{E}_r+\nabla_\theta\tilde{E}_\theta)+\left[\frac{1}{\gamma^2(\omega-\beta ck)^2}-\left(\frac{\beta c}{\omega(\omega-\beta ck)}\right)^2(\eta_r\nabla_r^2+\nabla_\theta^2)\right]\tilde{E}_z\right\}, \quad (16)$$

where $\nabla_r=\partial/\partial r$ and $\nabla_\theta=(1/r)\partial/\partial\theta$ are the transverse gradient functor, $\nabla_r^2=\partial/\partial r(r\partial/\partial r)/r$, and $\nabla_\theta^2=(1/r^2)\partial^2/\partial^2\theta$.

2.3. Expression of EM field components and wave equations

From Maxwell's equations, we can obtain the expressions of transverse components of the fields in terms of longitudinal components and the longitudinal wave equations. Using Eqs. (9) and (10), transverse components of the fields can be expressed in terms of longitudinal components to obtain

$$\tilde{E}_r = \frac{\mathrm{i}}{T_r^2}\left\{\left[k_z - \frac{\beta}{c}\frac{\omega_\mathrm{e}^2}{\gamma}\frac{1}{(\omega-\beta c k_z)}\eta_r\right]\nabla_r\tilde{E}_z + \omega\nabla_\theta\tilde{B}_z\right\}, \tag{17}$$

$$\tilde{E}_\theta = \frac{\mathrm{i}}{T_\theta^2}\left\{\left[k_z - \frac{\beta}{c}\frac{\omega_\mathrm{e}^2}{\gamma}\frac{1}{(\omega-\beta c k_z)}\right]\nabla_\theta\tilde{E}_z - \omega\nabla_r\tilde{B}_z\right\}, \tag{18}$$

$$\tilde{B}_r = \frac{\mathrm{i}}{T_\theta^2}\left\{k_z\nabla_r\tilde{B}_z - \left[\frac{\omega}{c^2} - \frac{\omega_\mathrm{e}^2}{\gamma c^2}\frac{1}{(\omega-\beta c k_z)}\right]\nabla_r\tilde{E}_z\right\}, \tag{19}$$

$$\tilde{B}_\theta = \frac{\mathrm{i}}{T_r^2}\left\{k_z\nabla_\theta\tilde{B}_z + \left[\frac{\omega}{c^2} - \frac{\omega_\mathrm{e}^2}{\gamma c^2}\frac{1}{(\omega-\beta c k_z)}\eta_r\right]\nabla_r\tilde{E}_z\right\}, \tag{20}$$

where

$$T_r^2 = \left(\frac{\omega^2}{c^2} - k_z^2\right) - \frac{\omega_\mathrm{e}^2}{\gamma c^2}\eta_r, \tag{21}$$

$$T_\theta^2 = \left(\frac{\omega^2}{c^2} - k_z^2\right) - \frac{\omega_\mathrm{e}^2}{\gamma c^2}. \tag{22}$$

Using Eqs. (9)–(13) and (17)–(22), the longitudinal EM function can be written as

$$\nabla_r^2\tilde{E}_z + \kappa_\mathrm{TM}^2\tilde{E}_z = 0,$$
$$\tilde{B}_z = 0 \tag{23}$$

for $\mathrm{TM}_{0\mathrm{m}}$ modes, where

$$\kappa_\mathrm{TM}^2 = \frac{(\omega-\beta c k_z)^2 - \omega_\mathrm{e}^2/\gamma^3}{(\omega-\beta c k_z)^2 - \eta_r\omega_\mathrm{e}^2/\gamma^3}T_r^2, \tag{24}$$

and

$$\nabla_r^2\tilde{B}_z + \kappa_\mathrm{TE}^2\tilde{B}_z = 0,$$
$$\tilde{E}_z = 0 \tag{25}$$

for $\mathrm{TE}_{0\mathrm{m}}$ modes, where

$$\kappa_\mathrm{TE}^2 = T_\theta^2. \tag{26}$$

Compare the equations of $\mathrm{TM}_{0\mathrm{m}}$ modes and the equations of $\mathrm{TE}_{0\mathrm{m}}$ modes, one can find that the term of betatron motion (ω_b) is absent in the $\mathrm{TE}_{0\mathrm{m}}$ modes. This means that the interaction between the EM modes and betatron modes arises only in $\mathrm{TM}_{0\mathrm{m}}$ modes.

3. Dispersion equation

From the above analysis, for $\mathrm{TM}_{0\mathrm{m}}$ modes, the wave equation inside the beam is defined by Eqs. (23) and (24). Outside the beam we have

$$\nabla_r^2\tilde{E}_{2z} + \kappa_2^2\tilde{E}_{2z} = 0, \tag{27}$$

$$\kappa_2^2 = \omega^2/c^2 - k_z^2 = T_2^2. \tag{28}$$

The appropriate solutions are

$$\tilde{E}_{1z} = A_1\mathrm{J}_0(\kappa_1 r)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{29}$$

$$\tilde{E}_{2z} = [A_2\mathrm{J}_0(\kappa_2 r) + A_3\mathrm{N}_0(\kappa_2 r)]\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{30}$$

where $\mathrm{J}_0(x)$ and $\mathrm{N}_0(x)$ are Bessel and Neumann functions of the zero order, respectively, and a top dot will denote the derivative.

The relativistic boundary conditions are

$$\tilde{E}_{1z} = \tilde{E}_{2z} \quad \text{at} \quad r = a, \tag{31}$$

$$\boldsymbol{B}_{1\theta} - \boldsymbol{\beta}/c \times \boldsymbol{E}_{1r} = \boldsymbol{B}_{2\theta} - \boldsymbol{\beta}/c \times \boldsymbol{E}_{2r} \quad \text{at} \quad r = a, \tag{32}$$

$$\tilde{E}_{2z} = 0 \quad \text{at} \quad r = R, \tag{33}$$

which will yield three equations for three unknown coefficients A_1, A_2, and A_3. The necessary and sufficient condition for nontrivial solution is obtained by equating the determinant of coefficients to zero. This will yield

$$\left(1 - \frac{1}{\gamma^3}\frac{\omega_\mathrm{e}^2}{(\omega-\beta c k_z)^2 - \omega_\mathrm{b}^2}\right)\kappa_1\frac{\dot{\mathrm{J}}_0(\kappa_1 a)}{\mathrm{J}_0(\kappa_1 a)}$$
$$= \frac{T_r^2}{\kappa_2}\frac{\dot{\mathrm{J}}_0(\kappa_2 a)\mathrm{N}_0(\kappa_2 R) - \dot{\mathrm{N}}_0(\kappa_2 a)\mathrm{J}_0(\kappa_2 R)}{\mathrm{J}_0(\kappa_2 a)\mathrm{N}_0(\kappa_2 R) - \mathrm{N}_0(\kappa_2 a)\mathrm{J}_0(\kappa_2 R)}, \tag{34}$$

where $\dot{\mathrm{J}}_0(x)$ and $\dot{\mathrm{N}}_0(x)$ are the derivative of $\mathrm{J}_0(x)$ and $\mathrm{N}_0(x)$ respectively. This is the dispersion relation for

the space-charge, betatron, and TM waves, which describes the dispersion properties of pure EM modes in the combined beam self-field and ion channel field. When $\beta \to 0$ and $\omega_{\rm e} = \omega_{\rm i}$, equation (34) reduces to the dispersion relation of a metallic tube which is partially filled with a cylindrical plasma column. Using the Lorentz transformation, we can transform equation (34) into the dispersion relation of the beam frame, which takes the same form as demonstrated in Ref. [19] (Eq. (50) in Ref. [19]). The numerical results and analysis of the dispersion relation in the beam frame had been detailed in Ref. [19] by Rouhani *et al.*

4. Growth rate of the wave

In order to display the feature of betatron resonance radiation, we rewrite the dispersion relation Eq. (34) as

$$\begin{aligned}
&\left[(\omega-\beta ck_z)^2-\omega_{\rm b}^2\right]\left\{1-\left[\frac{{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}{\dot{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-\dot{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}\frac{\dot{\rm J}_0(\kappa_1 a)}{{\rm J}_0(\kappa_1 a)}\right]^2\right\}\\
=&\ \frac{\omega_{\rm e}^2}{\gamma c^2}(\omega-\beta ck_z)^2\left(\frac{\omega^2}{c^2}-k_z^2\right)^{-1}\\
&-\omega_{\rm e}^2/\gamma^3(\omega-\beta ck_z)^{-2}\left[(\omega-\beta ck_z)^2-\omega_{\rm b}^2\right]\ \Gamma\\
&-\omega_{\rm e}^2/\gamma^3(\omega-\beta ck_z)^{-2}\left[(\omega-\beta ck_z)^2-\omega_{\rm e}^2/\gamma^3\right]\ \Gamma, \qquad (35)
\end{aligned}$$

where $\Gamma = \left[\dfrac{{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}{\dot{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-\dot{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}\dfrac{\dot{\rm J}_0(\kappa_1 a)}{{\rm J}_0(\kappa_1 a)}\right]^2$.

In Eq. (35), if we let $\omega_{\rm e} = 0$, we can obtain

$$F_{\rm B} = 1-\left[\frac{{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}{\dot{\rm J}_0(\kappa_2 a){\rm N}_0(\kappa_2 R)-\dot{\rm N}_0(\kappa_2 a){\rm J}_0(\kappa_2 R)}\frac{\dot{\rm J}_0(\kappa_1 a)}{{\rm J}_0(\kappa_1 a)}\right]^2 = 0, \qquad (36)$$

and

$$(\omega-\beta ck_z)^2-\omega_{\rm b}^2 = 0. \qquad (37)$$

Equation (36) is the no-beam dispersion equation of a $\rm TM_{0m}$ mode with the metal waveguide, and equation (37) represents a beam mode. These two modes are coupled via the term on the right-hand side of Eq. (35) and the coupling is proportional to the density of the electron beam. The betatron radiation occurs when equations (36) and (37) are satisfied simultaneously, and the electrons give net work to the radiation wave to enable the wave to grow. Consequently equations (36) and (37) are the simultaneous conditions to produce the Betatron radiation.

Equation (35) can be rewritten as

$$F_{\rm B}[(\omega-\beta ck_z)^2-\omega_{\rm b}^2] = \alpha_{\rm B}, \quad 0 < r \le a, \qquad (38)$$

where $\alpha_{\rm B}$ is the right-hand side of Eq. (35).

The no-beam dispersion equation (36) of a $\rm TM_{0m}$ mode with the metal waveguide can be cut off to

$${\rm J}_n(\kappa_2 R) = 0. \qquad (39)$$

Since wave in the empty metal waveguide is a fast wave, only the upper solution of Eq. (37) could couple with the guided EM wave. Let the solution of Eqs. (37) and (39) be ω_0, it takes

$$\omega_0 = \beta ck_z + \omega_{\rm b}. \qquad (40)$$

Expressing ω_0 as $\omega = \omega_0 + \delta\omega$ with $\delta\omega \ll \omega_0$, one can obtain

$$(\delta\omega)^3 + 2\omega_{\rm b}(\delta\omega)^2 = \left(\frac{\alpha_{\rm B}}{\partial F_{\rm B}/\partial\omega}\right)\Big|_{\omega=\omega_0}. \qquad (41)$$

Three solutions of this cubic equation (41) can be rewritten as

$$(\delta\omega)_\nu = -\frac{1}{3}p + \frac{1}{6}t_\nu + \frac{2}{3}\frac{p^2}{t_\nu}, \quad \nu = 1,2,3, \qquad (42)$$

where $p = 2\omega_{\rm b}$, $q = -(\alpha_{\rm B}/(\partial F_{\rm B}/\partial\omega))|_{\omega=\omega_0}$, $t_\nu = -|t|\ {\rm e}^{{\rm i}(\nu-1)(2\pi/3)}$, $\nu = 1,2,3$, and they meet the equation

$$t^3 = -108q - 8p^3 + 12(81q^2 + 12qp^3)^{1/2}.$$

When $q > 0$, the roots of Eq. (41) are a real $(\delta\omega)_1$ and two conjugated complex number $(\delta\omega)_{2,3}$, the imaginary part of the complex root is

$$\mathrm{Im}(\delta\omega) = \pm\frac{\sqrt{3}}{2}\left(\frac{1}{6}|t| - \frac{2}{3}p^2\frac{1}{|t|}\right). \tag{43}$$

When the beam density is neglected ($\omega_\mathrm{e} = 0$), the radiation frequency is real for real values of the axial wave number k_z and no betatron radiation is expected. To obtain the growth rate of the wave, a nonzero value of ω_e must be used and the dispersion equation must be solved for complex values of ω for real values of k_z. The wave growth rate G_B can approximately be solved as

$$G_\mathrm{B} = \mathrm{Im}(\delta\omega). \tag{44}$$

5. Numerical results and analysis

Here, a numerical computation is conducted to investigate the effect of the REB radius (a), the beam energy (γ), the plasma frequency (ω_p), and the REB plasma frequency (ω_e) on the wave growth rate (G_B). The calculation results, which are obtained by the solution of the complete dispersion equation (35), for TM_{01} are shown in Figs. 2–7. In these calculations, the parameters of the frequency, such as ω_p, ω_b, G_B, are normalized by c/R, and the beam radius a is normalized by R, the axial wave number k_z is normalized to $1/R$.

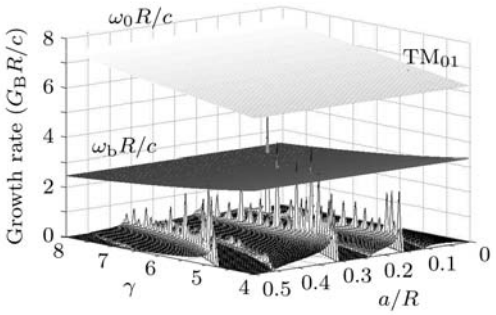


Fig. 2. Plot of the growth rate (the lower figure), the betatron frequency (the middle figure) and the operating frequency (the upper figure) of the wave as functions of the Lorentz factor and the beam radius. The normalized preformed plasma and the beam plasma frequencies are $\omega_\mathrm{p}R/c = 10$ and $\omega_\mathrm{e}R/c = 12$ respectively.

We show in Fig. 2 the normalized growth rate, the betatron frequency and the normalized operating frequency versus the beam energy and the normalized REB radius, corresponding to the normalized plasma frequency $\omega_\mathrm{p}R/c = 10$ and the normalized beam plasma frequency $\omega_\mathrm{e}R/c = 12$. The plots of the normalized growth rate as a function of the beam energy and the normalized REB radius are shown in Figs. 3 and 4, respectively. As seen from Fig. 2, if we increase the values of a/R and γ, several regions of betatron resonance radiation will present one after another. For the first region of radiation growth with smaller values of the REB radius and energy, the growth rate is small and the maximum growth rate decreases with increasing beam energy as shown in Fig. 4. The next regions will begin at some strong resonance point and attenuate quickly with increasing beam energy and REB radius as shown in Figs. 2, 3, and 4.

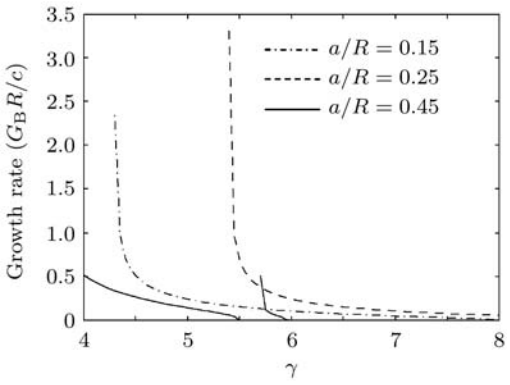


Fig. 3. Dependence of the wave growth rate on the beam energy for different radii of the REB. The normalized preformed plasma and the beam plasma frequencies are $\omega_\mathrm{p}R/c = 10$ and $\omega_\mathrm{e}R/c = 12$ respectively.

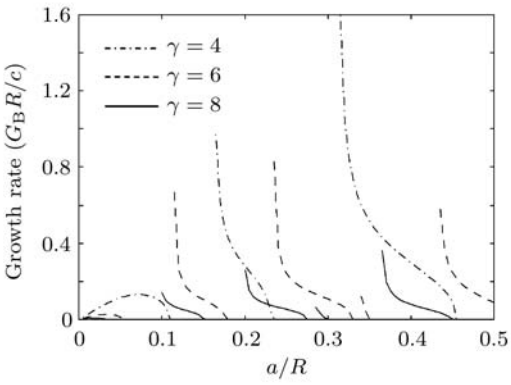


Fig. 4. Dependence of the wave growth rate on the beam radius for different Lorentz factors. The normalized preformed plasma and the beam plasma frequencies are $\omega_\mathrm{p}R/c = 10$ and $\omega_\mathrm{e}R/c = 12$ respectively.

In Fig. 5 the normalized growth rate, the betatron frequency and the normalized operating frequency versus the plasma frequency and the beam plasma frequency is plotted for the beam energy $\gamma = 6$ and the normalized REB radius $a/R = 0.3$. The dependence of the normalized growth rate on the normalized plasma frequency and the normalized beam plasma frequency are shown in Figs. 6 and 7, respectively. As seen from Fig. 5, if we increase the values of $\omega_{\rm p}R/c$ and $\omega_{\rm e}R/c$, several regions of betatron resonance radiation will present one after another. For $\omega_{\rm b}/\gamma < \omega_{\rm p} < \omega_{\rm b}$, in the first region of radiation growth with smaller values of $\omega_{\rm p}R/c$ and $\omega_{\rm e}R/c$, the maximum growth rate increases with increasing $\omega_{\rm e}R/c$ as shown in Fig. 7. The next regions will begin at some strong resonance point and attenuate quickly with increasing $\omega_{\rm p}R/c$ and $\omega_{\rm e}R/c$ as shown in Figs. 5, 6, and 7.

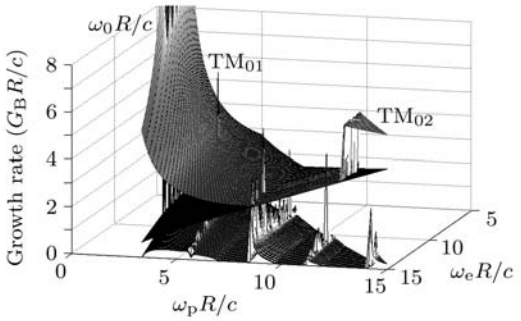


Fig. 5. Plot of the growth rate (the lower figure), the betatron frequency (the middle figure) and the operating frequency (the upper figure) of the wave as functions of the normalized preformed plasma frequency and the beam plasma frequency. The Lorentz factor and the beam radius are $\gamma = 6$ and $a/R = 0.3$ respectively.

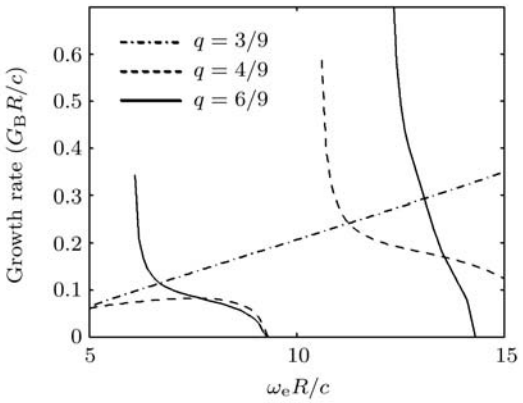


Fig. 6. Dependence of the wave growth rate on the beam plasma frequency for different preformed plasma frequencies ($\omega_{\rm p}R/c = q\omega_{\rm e}R/c$, $1/\gamma < q < 1$). The Lorentz factor and the beam radius are $\gamma = 6$ and $a/R = 0.3$ respectively.

Finally, it is instructive to compare our results with the dielectric Cherenkov maser in Refs. [20] and [21]. Quantitatively, our growth rates could be approximate or larger when some appropriate parameters are chosen. It is well known that a Cherenkov radiation needs employing a slow wave medium to slow down the phase velocity of guided EM mode to less than the velocity of light. In our design this slow wave medium is unnecessary so that we can use a much smaller inner radius of metal wall (R) and gain the much larger growth rate.

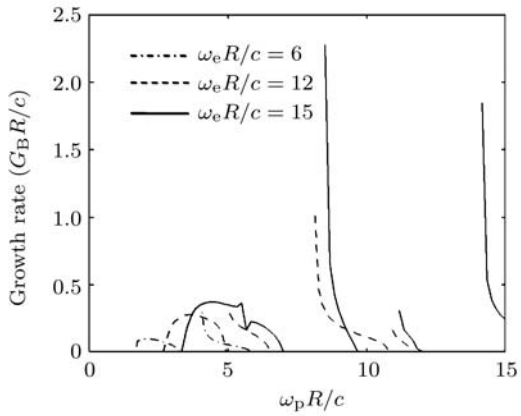


Fig. 7. Dependence of the wave growth rate on the preformed plasma frequency for different beam plasma frequencies. The Lorentz factor and the beam radius are $\gamma = 6$ and $a/R = 0.3$ respectively.

6. Conclusion

In this paper, the radiation growth characteristics of a betatron resonance radiation, which consists of a cylindrical metal waveguide and an REB with an ion channel guiding, were investigated. The performed plasma radius is considered less than the self-consistent ion channel radius, so that this waveguide have a simple structure, and the beam radius and inner radius of metal wall are arbitrary. This ICG-metal waveguide should have a significant wave growth by selecting the appropriate parameters. It is shown that only the $\mathrm{TM}_{0\rm m}$ mode can be effectively grown, and the growth rates of the wave depend on the REB radius, the beam energy, the plasma frequency, and the REB plasma frequency. Furthermore, the results obtained were compared with those for the dielectric Cherenkov maser.

References

[1] Whittum D H, Sessler A M and Dawson J M 1990 *Phys. Rev. Lett.* **64** 2511

[2] Kupershmidt H and Ron A 1994 *IEEE Trans. Plasma Sci.* **22** 674

[3] Whittum D H 1992 *Phys. Fluids* B **4** 730

[4] Whittum D H, Sharp W M, Yu S S, Lampe M and Joyce G 1991 *Phys. Rev. Lett.* **67** 991

[5] Esarey E, Sprangle P, Krall J and Ting A 1996 *IEEE Trans. Plasma Sci.* **24** 252

[6] Krall J and Joyce G 1995 *Phys. Plasmas* **2** 1326

[7] Mehdian H, Esmelzadeh M and Willett J E 2001 *Phys. Plasmas* **8** 3776

[8] Esmaeilzadeh M, Mehdian H and Willett J E 2002 *Phys. Rev.* E **65** 016501

[9] Mehdian H and Raghavi A 2007 *Plasma Phys. Control Fusion* **49** 69

[10] Mehdian H, Hasanbeigi A and Jafari S 2008 *Phys. Plasmas* **15** 073103

[11] Chen K R, Dawson J M, Lin A T and Katsouleas T 1991 *Phys. Fluids* B **3** 1270

[12] Wang S Q, Clayton C E, Blue B E, Dodd E S, Marsh K A, Mori W B, Joshi C, Lee S, Muggli P, Katsouleas T, Decker F J, Hogan M J, Iverson R H, Raimondi P, Walz D, Siemann R and Assmann R 2002 *Phys. Rev. Lett.* **88** 135004

[13] Esarey E, Shadwick B A, Catravas P and Leemans W P 2002 *Phys. Rev.* E **65** 056505

[14] Liu C S, Tripathi V K and Kumar N 2007 *Plasma Phys. Control Fusion* **49** 325

[15] Xie H Q and Liu P K 2007 *Chin. Phys.* **16** 766

[16] Li W, Gao H, Gong M L and Liu S G 2004 *Chin. Phys.* **13** 1296

[17] Kostyukov I, Kiselev S and Pukhov A 2003 *Phys. Plasmas* **10** 4818

[18] Rousse A, Phuoc K T, Shah R, Pukhov A, Lefebvre E, Malka V, Kiselev S, Burgy F, Rousseau J, Umstadter D and Hulin D 2004 *Phys. Rev. Lett.* **93** 135005

[19] Rouhani M H and Mareghechi B 2006 *Phys. Plasmas* **13** 083101

[20] Wu J Q 1997 *Phys. Plasmas* **4** 3064

[21] Shokri B, Ghomi H and Latifi H 2000 *Phys. Plasmas* **7** 2671