Topological horseshoe in nonlinear Bloch system*

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This paper demonstrates rigorous chaotic dynamics in nonlinear Bloch system by virtue of topological horseshoe and numerical method. It considers a properly chosen cross section and the corresponding Poincaré map, and shows the existence of horseshoe in the Poincaré map. In this way, a rigorous verification of chaos in the nonlinear Bloch system is presented.

Keywords: Bloch equation, chaos, topological horseshoe, Poincaré map

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1. Introduction

Chaos is an interesting complex dynamical phenomenon which has been extensively studied by scientists, engineers and physicists.[1−3] An understanding of dynamics of an ensemble of spins in an external magnetic field is of paramount significance to the study of various phenomena related to the extremely important field for nuclear magnetic resonance (NMR). The actual physical process is well described by the simple nonlinear Bloch equation (NLBE), which is recently reported by Abergel,[4] who discussed the dynamics of the NLBE numerically in general case. Reference [5] extended the study of Ref. [4], and obtained rich bifurcation structure of the Bloch equation by utilizing MATCONT package. Though references [4] and [5] demonstrated the chaotic behaviour of the Bloch equation by means of Lyapunov exponent and bifurcation theory, respectively, a rigorous verification of chaos is still lacking.

Topological horseshoe with symbolic dynamics is a powerful tool in rigorous studies of chaos in dynamical systems.[6−13] In this paper, we will present a rigorous verification of chaos in the nonlinear Bloch system by virtue of topological horseshoe. For this purpose, we consider a properly chosen cross section and the corresponding Poincaré map, and show the existence of horseshoe in the Poincaré map in the manner of computer-assisted proof arguments.

The organization of this paper is as follows. In Section 2, a brief revisit of the NLBE is given, and we summarize the numerical method of finding topological horseshoe in three-dimensional autonomous continuous time system. In Section 3, a rigorous computer-assisted verification of chaos is presented. Section 4 gives the conclusion of the paper.

2. Model and numerical method

2.1. Model

The dynamics of an ensemble of spins is usually described by the NLBE, which is more important to understand the underlying physical process of nuclear magnetic resonance. The basic process can be viewed as the combination of a precession about a magnetic field and a relaxation process, which gives rise to the damping of the transverse component of the magnetization with a different time constant.

The basic model is derived from a magnetization $\boldsymbol{M}$ precessing in the magnetic induction field $\boldsymbol{B}_0$ in the presence of a constant radio frequency field $\boldsymbol{B}_1$ with intensity $\boldsymbol{B}_1 = \omega_1/\lambda$ and frequency $\omega_{\rm rf}$. In dimensionless units, the dynamical model of nonlinear Bloch equation with feedback field is as follows:[4]

$$\begin{cases} \dot{x} = \delta y + \lambda z(x\sin(\psi) - y\cos(\psi)) - x/\tau_2, \\ \dot{y} = -\delta x - z + \lambda z(x\cos(\psi) + y\sin(\psi)) - y/\tau_2, \\ \dot{z} = y - \lambda\sin(\psi)(x^2 + y^2) - (z-1)/\tau_1, \end{cases} \tag{1}$$

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where δ, λ, and ψ are system parameters, τ_1 and τ_2 are the longitudinal time and transverse relation times, respectively.

For parameters

$$\delta = 0.4\pi, \quad \lambda = 10, \quad \psi = 0.7764,$$
$$\tau_1 = 0.5, \quad \tau_2 = 0.25, \tag{2}$$

equation (1) has an attractor as shown in Fig. 1. The calculated positive Lyapunov exponent for the attractor suggests that it should be chaotic. For more details about this system, see Ref. [4].

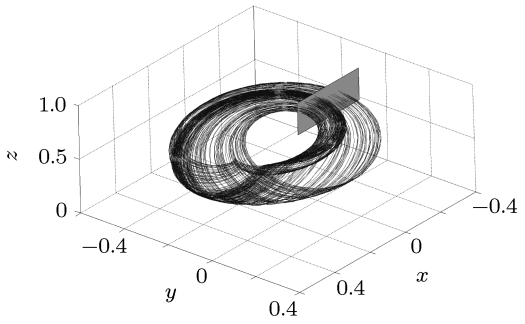


Fig. 1. The attractor of Eq. (1) and its cross section.

2.2. Numerical method

In this section, by virtue of topological horseshoe theory,[6−9] we summarize the numerical method of finding topological horseshoe in three-dimensional autonomous continuous time system. For more details about topological horseshoe theory, the reader can refer to Refs. [6]–[9], [14] and [15].

The numerical method consists of the following four steps.

Step 1 We select a cross section Π which must intersect with the orbit of the system in one direction.

Step 2 We take a quadrangle $|ABCD|$ in Π, and define the corresponding Poincaré map $P : |ABCD| \to \Pi$, such that $P(|ABCD|)$ intersects with the quadrangle $|ABCD|$. The determination of the quadrangle $|ABCD|$ and the iteration number of Poincaré map P need multiple attempts. Because of the continuity of Poincaré map P, we only discuss the images of the four sides of the quadrangle $|ABCD|$. When the image of the quadrangle $|ABCD|$ does not intersect with the quadrangle $|ABCD|$, we change the iteration number of Poincaré map P. If it still does not satisfy the above condition, we should change the vetexes of the quadrangle and iteration number repeatedly untill it satisfies the condition. Then turns to the next step.

Step 3 We discuss the images of the four sides of the quadrangle $|ABCD|$ respectively, the images of the two sides which intersect with the image $P(|ABCD|)$ should be compressed and be one side of the quadrangle $|ABCD|$, and the images of the other two sides should be extended and lie across the quadrangle $|ABCD|$. If it does not satisfy the above condition, then turns back to Step 2, else turns to the next step.

Step 4 We seek two mutually disjoint sets of $|ABCD|$, such that the map P has a P-connected family with respect to these two subsets.

Remark The numerical method is feasible for three-dimensional autonomous continuous time system which has single attractor, and it can be extended to the case of two-scroll attractors. The numerical simulations are carried out in Matlab, and we utilized the fourth-order Runge–Kutta method.

3. Numerical analysis and result

In this section, by virtue of the numerical method of Subsection 2.2, we present a rigorous verification of chaos in nonlinear Bloch system.

For the parameters (2), computer simulation shows that system (1) has single chaotic attractor as plotted in Fig. 1.

By careful selection, we consider the cross-section plane $\Pi : x_2 = 0$ as shown in Fig. 1.

In the plane $x_2 = 0$, we take a quadrangle $|ABCD|$ with its four vertexes being $A = (-0.0500, 0, 0.8000)$, $B = (-0.0461, 0, 0.8412)$, $C = (0.0166, 0, 0.8565)$, $D = (0.0132, 0, 0.8140)$, and the corresponding Poincaré map:

$$P(x) : |ABCD| \to \Pi.$$

For every $x \in |ABCD|$, $P(x)$ is defined to be the second point in P under the flow with initial condition x. Here we do not consider the situation of the first intersection, because it is not easy to find the topological horseshoe.

Under this map P, the image of the quadrangle $|ABCD|$ lies wholly across $|ABCD|$ as shown in Fig. 2.

Now we attempt to find two mutually disjoint subsets of $|ABCD|$, such that the map P has a P-family with respect to these two subsets.

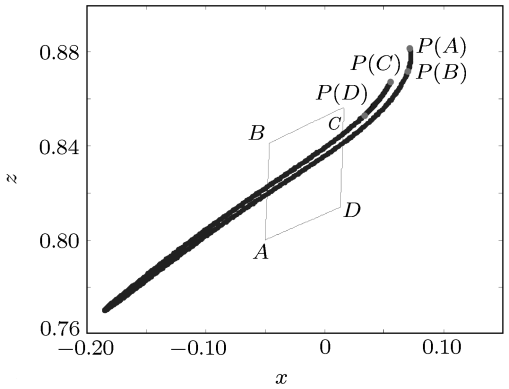


Fig. 2. The quadrangle $|ABCD|$ and its image.

After a great deal of computer simulations, we pick two disjoint subsets α and β of $|ABCD|$ as shown in Fig. 3, the vertexes of α and β are as given below. α : (–0.0500, 0, 0.8000), (–0.0461, 0, 0.8412), (–0.0164, 0, 0.8484) and (–0.0120, 0, 0.8084); β : (–0.0056, 0, 0.8098), (–0.0133, 0, 0.8492), (0.0166, 0, 0.8565) and (0.0132, 0, 0.8140). Precisely, let $\alpha_{\rm l}$ and $\alpha_{\rm r}$ denote the left and right sides of α, respectively, and $\beta_{\rm l}$ and $\beta_{\rm r}$ denote the left and right sides of β, respectively.

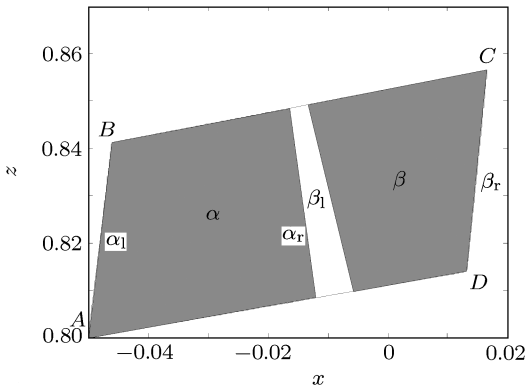


Fig. 3. The two disjoint subsets α and β.

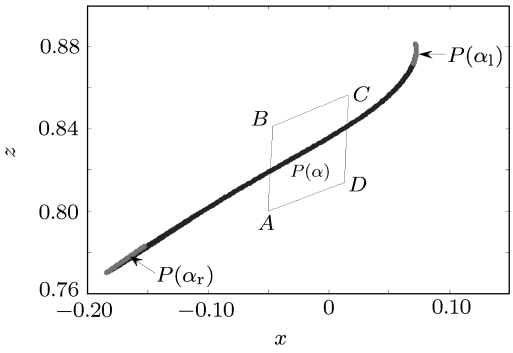


Fig. 4. The image of the subset α.

Numerical simulations show that the images $P(\alpha_)$ and $P(\beta)$ lie wholly across the quadrangle $|ABCD|$, and $P(\alpha_{\rm r})$ and $P(\beta_{\rm l})$ lie on the left side of AB, and $P(\alpha_{\rm l})$ and $P(\beta_{\rm r})$ lie on the right side of CD. For the details, see Figs. 4 and 5.

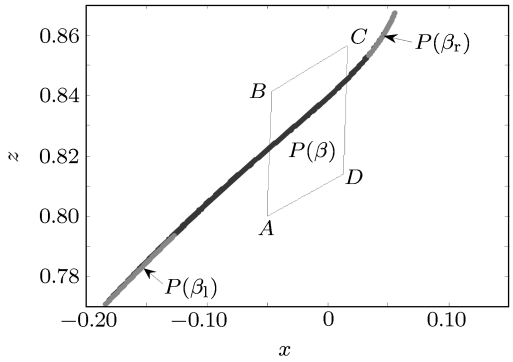


Fig. 5. The image of the subset β.

It is easy to see that for every line $l \subset |ABCD|$ that is a connection of the sides AB and CD, $P(l)$ has nonempty connections with α and β. Furthermore, both $P(l \cap \alpha)$ and $P(l \cap \beta)$ connect α and β. From these facts, it can be proved that α and β have the proper crossing property with respect to the left- and right-hand sides of α and β. According to the planar topological horseshoe theory,[8] there exists a P-family with respect to these two subsets α and β for the map, and this means that there exists a closed invariant set $\Lambda \subset (ABCD)$ for which $P|\Lambda$ is semiconjugate to a 2-shift map, thereby it indicates that the attractor illustrated in Fig. 1 is chaotic.

4. Conclusion

In this paper we have presented a rigorous computer-assisted verification for the existence of chaos in the nonlinear Bloch system. The arguments are given by using topological horseshoe with symbolic dynamics through a carefully chosen Poincaré map, which is much more convincible than the usual method of only calculating Lyapunov exponents.

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