Dynamic analysis of a new chaotic system with fractional order and its generalized projective synchronization*

Niu Yu-Jun(牛玉军), Wang Xing-Yuan(王兴元)†,
Nian Fu-Zhong(年福忠), and Wang Ming-Jun(王明军)

School of Electronic & Information Engineering, Dalian University of Technology, Dalian 116024, *China*

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Based on the stability theory of the fractional order system, the dynamic behaviours of a new fractional order system are investigated theoretically. The lowest order we found to have chaos in the new three-dimensional system is 2.46, and the period routes to chaos in the new fractional order system are also found. The effectiveness of our analysis results is further verified by numerical simulations and positive largest Lyapunov exponent. Furthermore, a nonlinear feedback controller is designed to achieve the generalized projective synchronization of the fractional order chaotic system, and its validity is proved by Laplace transformation theory.

Keywords: dynamic analysis, fractional order, generalized projective synchronization, Laplace transformation

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1. Introduction

Although the fractional differential calculus dates from the 17th century, the applications of fractional calculus to physics and engineering are just a recent focus of interest.[1,2] It was found that many systems in interdisciplinary fields can be described by the fractional differential equations, such as viscoelastic systems, dielectric polarization, electrode–electrolyte polarization and electromagnetic waves.[3−6] There are essential differences between ordinary differential equation systems and fractional-order differential systems. Most properties and conclusions of ordinary differential equation systems cannot be simply extended to that of the fractional order differential systems. Therefore, fractional order systems have attracted more and more attention. Recently, many investigations were devoted to the chaotic dynamics and chaotic control of fractional order systems.[7−21] It has been shown that the fractional order generalizations of many well-known systems can also behave chaotically.[13−21] For example, it has been shown that the fractional order Chua's circuit with an appropriate cubic nonlinearity and with an order as low as 2.7 can produce a chaotic attractor.[13] The chaotic behaviour of the fractional order Lorenz system was studied in Ref. [14], and the fractional order Chen chaotic system was also investigated in Refs. [15] and [16]. Ge and Zhang[17] investigated the chaotic behaviours of fractional order modified Duffing systems and modified van der Pol system, respectively. Li and Chen[18] found that the hyperchaos in fractional order Rössler equations has an order as low as 3.8, and the dynamic behaviours in three-dimensional Rössler system with fractional order were studied by Zhang *et al.*[19] The chaotic behaviour in the fractional order Liu system was investigated in Refs. [20] and [21], and so on.

A review for Refs. [13]–[21] indicates that phase portrait and Lyapunov exponent are always the efficient ways to research the fractional order system. However, based on the stability theory of fractional order systems, the dynamic behaviours of a new chaotic system[22] in different fractional order are investigated in our work. First, we theoretically analyse and compute the range of order in which the new fractional order system is chaotic. Then, the effectiveness of our analysis results is further verified by numerical simulations and computation of the largest Lyapunov exponent. It has been found that chaos exists in the new three-dimensional chaotic system with an order as low as 2.46. Period routes to chaos in the new

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†Corresponding author. E-mail: wangxy@dlut.edu.cn

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fractional order system are also found. Furthermore, using Laplace transformation theory, a nonlinear feedback controller is designed to achieve the generalized projective synchronization of the new fractional order chaotic system.

2. Fractional derivative and its approximation

There are several definitions of fractional derivatives.[1] The Caputo derivative[23] is a time domain computation method. In practical applications, the Caputo derivative is more popular since the in homogenous initial conditions are permitted if such conditions are necessary. Furthermore, these initial values are prone to measurement since they all have idiographic meanings.[24] The Caputo derivative definition is given by

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = J^{n-\alpha}\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n} \quad (\alpha > 0), \tag{1}$$

where n is the least integer that is not less than α, and J^β ($\beta > 0$) is the β-order Riemann–Liouville integral operator which satisfies

$$J^\beta \phi(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\phi(\tau)\mathrm{d}\tau, \tag{2}$$

where Γ stands for Gamma function, $0 < \beta \leq 1$.

Liu and Chen introduced a new three-dimensional autonomous chaotic system in Ref. [22], which is described by the following nonlinear differential equations.

$$\begin{cases} \dot{x} = ax - yz, \\ \dot{y} = by + xz, \\ \dot{z} = cz + xy, \end{cases} \tag{3}$$

where x, y, and z are state variables; and a, b, c are system parameters. When $a = 5$, $b = -10$, $c = -3.4$, system (3) exhibits a chaotic behaviour. Its attractor is similar to Chen's attractor,[25,26] as depicted in Fig. 1.

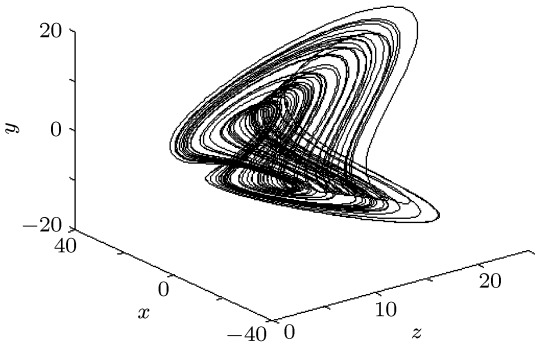


Fig. 1. The Chen-like attractor of system (3).

The fractional order system (3) is described as

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}} = ax - yz, \\ \dfrac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}} = by + xz, \\ \dfrac{\mathrm{d}^{q_3}z}{\mathrm{d}t^{q_3}} = cz + xy, \end{cases} \tag{4}$$

where q_i is the fractional order, $0 < q_i \leq 1$ $(i = 1, 2, 3)$.

According to Refs. [27]–[29], the predictor–corrector scheme for system (4) is derived. This scheme is the generalization of Adams–Bashforth–Moulton one.[29]

The following differential equation

$$\begin{aligned} &\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = f(t, x), \quad 0 \leq t \leq T, \\ &x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, 2, \ldots, n-1, \end{aligned}$$

is equivalent to the Volterra integral equation[28]

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)}\frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau, x)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau. \tag{5}$$

Let $h = T/N$, $t_n = nh$ $(n = 0, 1, 2, \ldots, N)$. Then equation (5) can be discretized as follows:

$$x_h(t_{n+1}) = \sum_{k=0}^{n-1} x_0^{(k)}\frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)}f(t_{n+1}, x_h^\rho(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum a_{j,n+1}f(t_j, x_h(t_j)),$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n, \\ 1, & j = n+1, \end{cases}$$

$$x_h^\rho(t_{n+1}) = \sum_{k=0}^{n-1} x_0^{(k)}\frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}f(t_j, x_h(t_j)), \quad b_{j,n+1} = \frac{h^\alpha}{\alpha}((n-j+1)^\alpha - (n-j)^\alpha).$$

The error estimate is $e = \mathrm{Max}\,|x(t_j) - x_h(t_j)| = O(h^\rho)$ $(j = 0, 1, \ldots, N)$, where $\rho = \mathrm{Min}(2, 1+\alpha)$.

Applying the above method, equation (4) can be discretized as follows:

$$\begin{cases} x_{n+1} = x_0 + \dfrac{h^{q_1}}{\Gamma(q_1+2)}(ax_{n+1}^\rho - y_{n+1}^\rho z_{n+1}^\rho) + \dfrac{h^{q_1}}{\Gamma(q_1+2)}\displaystyle\sum_{j=0}^{n}\beta_{1,j,n+1}(ax_j - y_j z_j), \\ y_{n+1} = y_0 + \dfrac{h^{q_2}}{\Gamma(q_2+2)}(by_{n+1}^\rho + x_{n+1}^\rho z_{n+1}^\rho) + \dfrac{h^{q_2}}{\Gamma(q_2+2)}\displaystyle\sum_{j=0}^{n}\beta_{2,j,n+1}(by_j + x_j z_j), \\ z_{n+1} = z_0 + \dfrac{h^{q_3}}{\Gamma(q_3+2)}(cz_{n+1}^\rho + x_{n+1}^\rho y_{n+1}^\rho) + \dfrac{h^{q_3}}{\Gamma(q_3+2)}\displaystyle\sum_{j=0}^{n}\beta_{3,j,n+1}(cz_j + x_j y_j), \end{cases} \tag{6}$$

where

$$\begin{cases} x_{n+1}^\rho = x_0 + \dfrac{1}{\Gamma(q_1)}\displaystyle\sum_{j=0}^{n}\gamma_{1,j,n+1}(ax_j - y_j z_j), \\ y_{n+1}^\rho = y_0 + \dfrac{1}{\Gamma(q_2)}\displaystyle\sum_{j=0}^{n}\gamma_{2,j,n+1}(by_j + x_j z_j), \\ z_{n+1}^\rho = z_0 + \dfrac{1}{\Gamma(q_3)}\displaystyle\sum_{j=0}^{n}\gamma_{3,j,n+1}(cz_j + x_j y_j), \end{cases}$$

$$\beta_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-\theta_i)(n+1)^{q_i}, & j = 0; \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1 \le j \le n; \\ 1, & j = n+1; \end{cases}$$

$$\gamma_{i,j,n+1} = \frac{h^{q_i}}{\theta_i}((n-j+1)^{q_i} - (n-j)^{q_i}), \quad 0 \le j \le n, \quad i = 1, 2, 3.$$

3. Dynamic analysis of the new fractional order chaotic system

Theorem 1 For n-dimensional fractional system, if all the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the Jacobian matrix of some equilibrium point satisfy

$$|\arg(\lambda_i)| > \alpha\pi/2, \quad \alpha = \max(q_1, q_2, \ldots, q_n) \quad (i = 1, 2, \ldots, n), \tag{7}$$

then the fractional order system is asymptotically steady at the equilibrium.

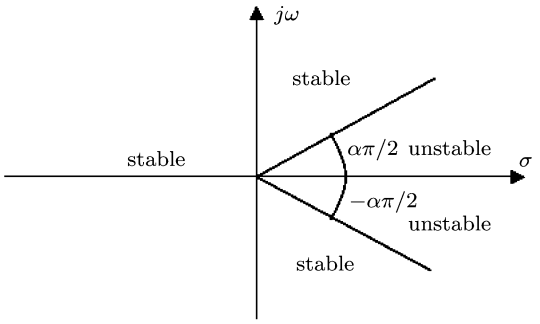


Fig. 2. Stability region of the fractional order system.

According to the stability theory of fractional order systems,[30] it is easy to prove Theorem 1. Figure 2 illustrates Theorem 1. Obviously, if the equilibrium is stable, the fractional order system will be steady at one point; only when none equilibrium is stable, is the fractional order system in chaos.

The system (4) has five equilibriums, i.e., $S_0(0,0,0)$, $S_1(x_0, y_0, z_0)$, $S_2(x_0, -y_0, -z_0)$, $S_3(-x_0, y_0, -z_0)$ and $S_4(-x_0, -y_0, z_0)$, where $x_0 = \sqrt{bc}$, $y_0 = \sqrt{-ac}$, $z_0 = \sqrt{-ab}$. When $a = 5$, $b = -10$ and $c = -3.4$, we obtain $S_0(0,0,0)$, $S_1(\sqrt{34}, \sqrt{17}, 5\sqrt{2})$, $S_2(\sqrt{34}, -\sqrt{17}, -5\sqrt{2})$, $S_3(-\sqrt{34}, \sqrt{17}, -5\sqrt{2})$ and $S_4(-\sqrt{34}, -\sqrt{17}, 5\sqrt{2})$.

First, we choose $S_0(0,0,0)$ to study: the eigenvalues of the Jacobian matrix are $\lambda_1 = 5$, $\lambda_2 = -10$ and $\lambda_3 = -3.4$. We can obtain $\arg(\lambda_1) = 0$, $\arg(\lambda_2) = \pi$ and $\arg(\lambda_3) = \pi$. According to Theorem 1, we can easily conclude that the equilibrium S_0 of Eq. (4) is unstable when $0 < \alpha < 1$, i.e., S_0 will never be stable.

When we choose $S_1(\sqrt{34}, \sqrt{17}, 5\sqrt{2})$ to study, the eigenvalues of the Jacobian matrix are $\lambda_1 = -12.6496$, $\lambda_2 = 2.1248 + 7.0172\,\mathrm{i}$ and $\lambda_3 = 2.1248 - 7.0172\,\mathrm{i}$.

We can obtain $\arg(\lambda_1) = \pi$, $\arg(\lambda_2) = 1.2768$ and $\arg(\lambda_3) = -1.2768$. According to Theorem 1, we can conclude that when q_1, q_2, and q_3 are all less than 0.8128 ($1.2768 \times 2/\pi$), the equilibrium S_1 of Eq. (4) is stable. On the contrary, when q_1, q_2, and q_3 are all greater than 0.8128, the equilibrium S_1 of Eq. (4) is unstable.

In the same way, when q_1, q_2, and q_3 are all less than 0.8128, the equilibrium S_2, S_3 and S_4 of Eq. (4) are stable; and when q_1, q_2, and q_3 are all greater than 0.8128, the equilibriums S_2, S_3, and S_4 of Eq. (4) are unstable.

To sum up, when q_1, q_2, and q_3 are all less than 0.8128, there exists at least one stable equilibrium, i.e., the system (4) will be stabilized at one point (S_1, S_2, S_3 or S_4) finally; when q_1, q_2, and q_3 are all greater than 0.8128, all the equilibriums of Eq. (4) are unstable, the system (4) will exhibit a chaotic behaviour; when $q_1 = q_2 = q_3 = 0.8128$, the system (4) will exhibit a limit cycle around some equilibrium point; when $q_i < 0.8128 < q_j$ $(i \neq j)$, the problem will be complicated, the system (4) may be convergent, periodic or chaotic, it seems hard to analyse its behaviour theoretically now, so we do not consider such a case here.

Using the discretization scheme in Section 2, we obtain the simulation results of the system (4) with different fractional orders as shown in Figs. 3–9. In these simulations, we select three kinds of values as $0.8128 < q_1, q_2, q_3 < 1$: (i) the larger values $q_1 = q_2 = q_3 = 0.95$ (Fig. 3); (ii) unequal values $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$ (Fig. 4); (iii) the minimum values $q_1 = q_2 = q_3 = 0.82$ (Fig. 5) in $0.8128 < q_1, q_2, q_3$ and two decimals are retained. Correspondingly, for $0 < q_1, q_2, q_3 < 0.8128$, we also collect three kinds of values: (I) the maximum values $q_1 = q_2 = q_3 = 0.81$ (Fig. 7) in $q_1, q_2, q_3 < 0.8128$ and two decimal is retained; (II) unequal values $q_1 = 0.6$, $q_2 = 0.7$, $q_3 = 0.8$ (Fig. 8); (III) the smaller values $q_1 = q_2 = q_3 = 0.5$ (Fig. 9).

From these simulation results (Figs. 3–9), we can see that when q_1, q_2, and q_3 are all greater than 0.8128, the system (4) will exhibit a chaotic behaviour (Figs. 3–5). When $q_1 = q_2 = q_3 = 0.8128$, the system (4) will exhibit a limit cycle around some equilibrium (Fig. 6). When q_1, q_2, and q_3 are all less than 0.8128, the system (4) will stabilize at one of fixed points S_1, S_2, S_3, and S_4, and the specific point will be determined by the initial states of the system (4) (Figs. 7–9). In this case, because S_0 is always unstable, it is not included.

We calculate the largest Lyapunov exponents of the fractional order system (4) using the well-known Wolf algorithm.[31] The value of the largest Lyapunov exponent is 0.3516 with $q_1 = q_2 = q_3 = 0.82$. Therefore, it indicates that when $q_1 = q_2 = q_3$ and two decimals are retained, the lowest order we found for the system (4) to yield chaos is 2.46.

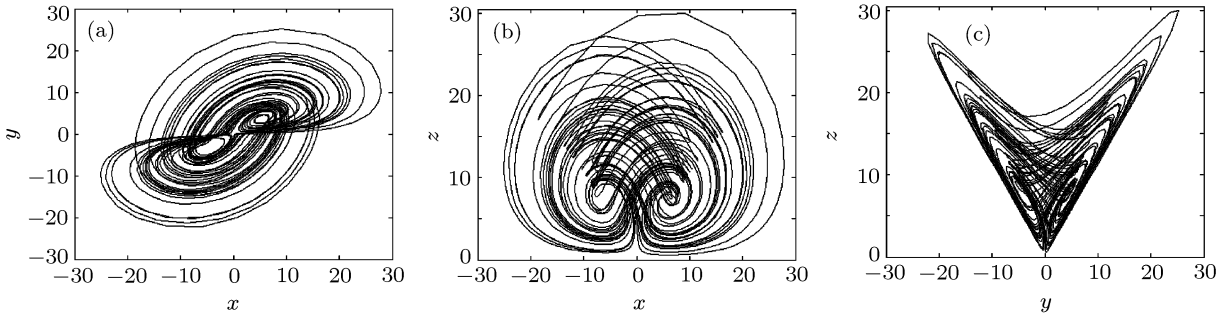


Fig. 3. The projections of the attractor of system (4), when $q_1 = 0.95$, $q_2 = 0.95$, and $q_3 = 0.95$.

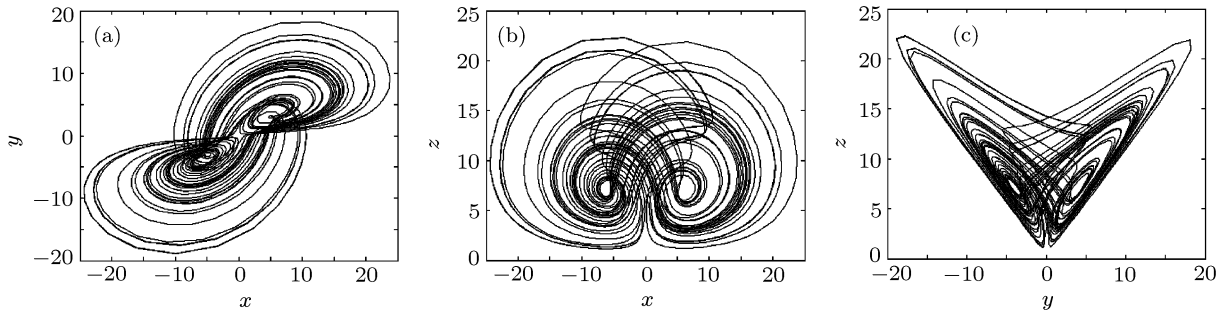


Fig. 4. The projections of the attractor of system (4), when $q_1 = 0.85$, $q_2 = 0.9$, and $q_3 = 0.95$.

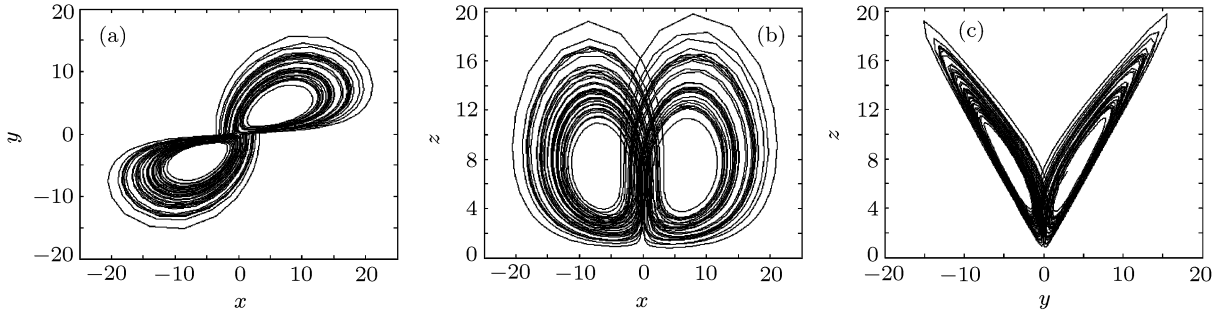


Fig. 5. The projections of the attractor for system (4), when $q_1 = 0.82$, $q_2 = 0.82$, and $q_3 = 0.82$.

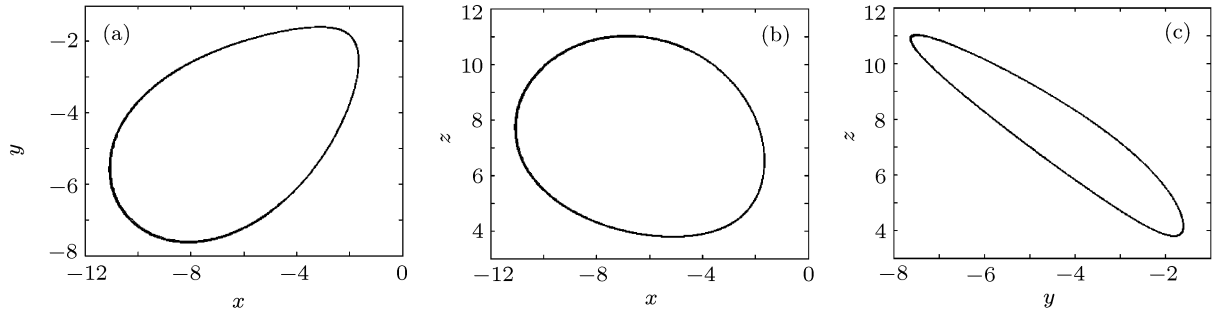


Fig. 6. The projections of the attractor for system (4), when $q_1 = 0.8128$, $q_2 = 0.8128$, and $q_3 = 0.8128$.

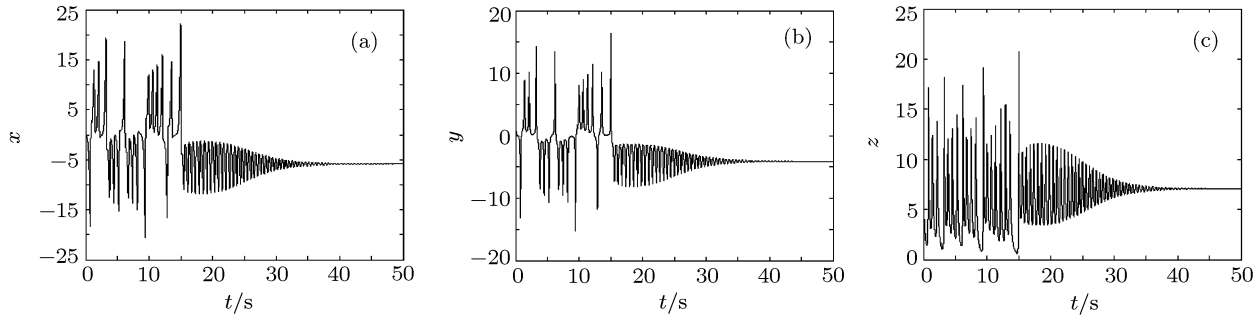


Fig. 7. When $q_1 = 0.81$, $q_2 = 0.81$, and $q_3 = 0.81$, system (4) converges to one of its equilibriums.

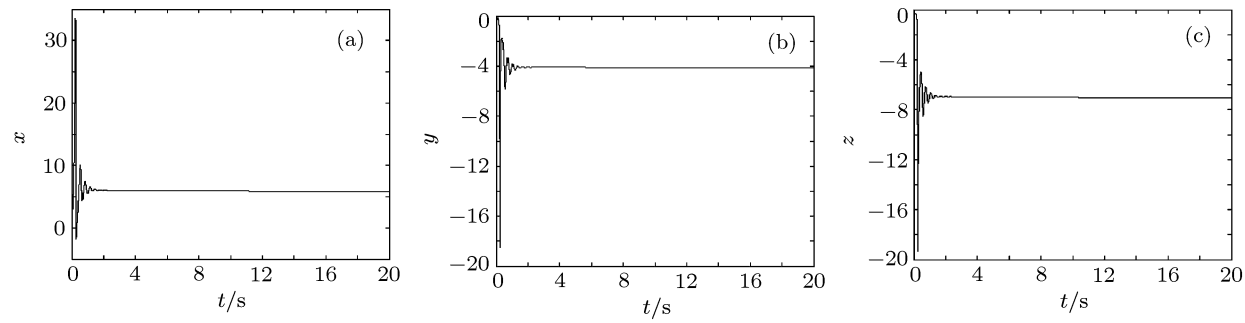


Fig. 8. When $q_1 = 0.6$, $q_2 = 0.7$, and $q_3 = 0.8$, system (4) converges to one of its equilibriums.

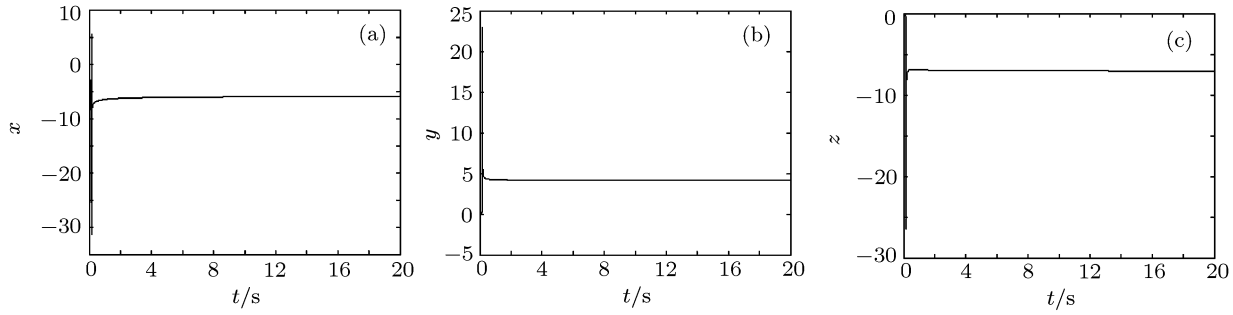


Fig. 9. When $q_1 = 0.5$, $q_2 = 0.5$, and $q_3 = 0.5$, system (4) converges to one of its equilibriums.

4. Generalized projective synchronization of fractional order chaotic system

The system (4) is defined as the drive system of fractional order generalized projective synchronization, then the fractional order response system can be written as

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x'}{\mathrm{d}t^{q_1}} = ax' - y'z' + u_1, \\ \dfrac{\mathrm{d}^{q_2}y'}{\mathrm{d}t^{q_2}} = by' + x'z' + u_2, \\ \dfrac{\mathrm{d}^{q_3}z'}{\mathrm{d}t^{q_3}} = cz' + x'y' + u_3, \end{cases} \tag{8}$$

where x', y', and z' are state variables and $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is a nonlinear control vector. The generalized projective synchronization errors between systems (4) and (8) are defined as $e_1 = x' - \beta x$, $e_2 = y' - \beta y$, and $e_3 = z' - \beta z$, where β is a nonzero constant, and is a scaling factor that defines a proportional relation between the synchronized systems. Hence, complete synchronizations and anti-synchronizations are the special cases of generalized projective synchronization when the scaling factor $\beta = 1$ and $\beta = -1$, respectively.

Theorem 2 Consider the following nonlinear feedback controller

$$\begin{cases} u_1 = -ke_1 + \beta z e_2 + \beta y e_3 + e_2 e_3 + (\beta^2 - \beta)yz, \\ u_2 = -\beta z e_1 - \beta x e_3 - e_1 e_3 + (\beta - \beta^2)xz, \\ u_3 = (\beta - \beta^2)xy, \end{cases} \tag{9}$$

where k ($k > a$) is a coupling constant, then, the generalized projective synchronization of the fractional order chaotic systems (4) and (8) can be accomplished, through changing the value of the scaling factor β.

Proof Substituting Eq. (9) into Eq. (8) and subtracting Eq. (4) from Eq. (8), we obtain the following error dynamical system

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}e_1}{\mathrm{d}t^{q_1}} = -(k-a)e_1, \\ \dfrac{\mathrm{d}^{q_2}e_2}{\mathrm{d}t^{q_2}} = be_2, \\ \dfrac{\mathrm{d}^{q_3}e_3}{\mathrm{d}t^{q_3}} = ce_3 + e_1e_2 + \beta y e_1 + \beta x e_2. \end{cases} \tag{10}$$

Taking Laplace transform[32] on both sides of Eq. (10), letting $E_i(s) = L(e_i(t))$, $(i = 1, 2, 3)$ and applying $L(\mathrm{d}^q e_i/\mathrm{d}t^q) = s^q E_i(s) - s^{q-1}e_i(0)$ $(i = 1, 2, 3)$, we obtain

$$\begin{cases} s^{q_1}E_1(s) - s^{q_1-1}e_1(0) = -(k-a)E_1(s), \\ s^{q_2}E_2(s) - s^{q_2-1}e_2(0) = bE_2(s), \\ s^{q_3}E_3(s) - s^{q_3-1}e_3(0) = cE_3(s) + E_1(s)E_2(s) + \beta(L(ye_1) + L(xe_2)). \end{cases} \tag{11}$$

Equation (11) can be rewritten as follows:

$$\begin{cases} E_1(s) = \dfrac{s^{q_1-1}e_1(0)}{s^{q_1} + (k-a)}, \\ E_2(s) = \dfrac{s^{q_2-1}e_2(0)}{s^{q_2} - b}, \\ E_3(s) = \dfrac{s^{q_3-1}e_3(0) + E_1(s)E_2(s)}{s^{q_3} - c} + \dfrac{\beta(L(ye_1) + L(xe_2))}{s^{q_3} - c}. \end{cases} \tag{12}$$

By the final-value theorem of the Laplace transformation,[33] we obtain:

$$\lim_{t\to\infty} e_1(t) = \lim_{s\to 0} sE_1(s) = \lim_{s\to 0} \frac{s^{q_1} e_1(0)}{s^{q_1} + (k-a)} = 0, \quad (13)$$

$$\lim_{t\to\infty} e_2(t) = \lim_{s\to 0} sE_2(s) = \lim_{s\to 0} \frac{s^{q_2} e_2(0)}{s^{q_2} - b} = 0, \quad (14)$$

$$\lim_{t\to\infty} e_3(t) = \lim_{s\to 0} sE_3(s) = \lim_{s\to 0} \frac{s^{q_3} e_3(0) + sE_1(s)E_2(s)}{s^{q_3} - c} + \frac{\beta(sL(ye_1) + sL(xe_2))}{s^{q_3} - c}. \quad (15)$$

Because of the boundedness of the chaos attractor, there exists $M > 0$ such that $|x| \le M < \infty$, $|y| \le M < \infty$. So, we have

$$|L(xe_2)| \le M|E_2(s)|, \quad (16)$$

$$|L(ye_1)| \le M|E_1(s)|. \quad (17)$$

According to Eqs. (13) and (14), we obtain $\lim_{t\to\infty} e_3(t) = 0$. Hence the proof is completed.

According to the above analyses, with nonlinear feedback controller (9), the error system (10) is asymptotically stable at the origin. This indicates that the generalized projective synchronization of the drive system (4) and the response system (8) with different initial values is achieved, as time goes to infinity.

In the following, the fractional order $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$ are chosen as examples in this paper. The scaling factor $\beta = 2$ and $\beta = -1.5$ are selected, respectively. The above discretization scheme in Section 2 is used to solve the differential equations of the systems (4) and (8). It is supposed that $k = 7 > a$. The initial values of the drive system (4) and response system (8) are $(x(0), y(0), x(0)) = (1, 3, 9)$ and $(x'(0), y'(0), z'(0)) = (-1, 0, 1)$, respectively. Simulation results of generalized projective synchronization are shown in Figs. 10–12.

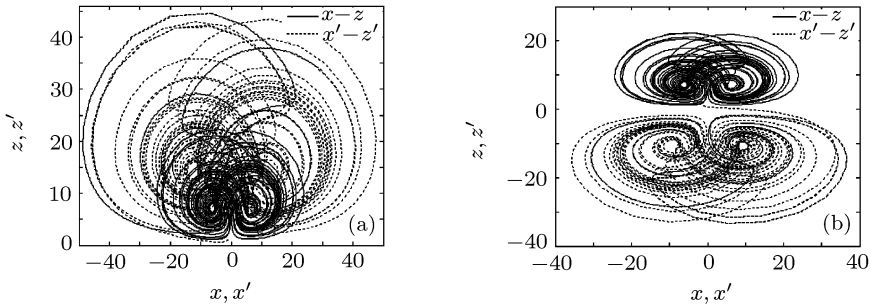


Fig. 10. The projection of attractors of system (4) and system (8) on the two-dimensional plane: (a) $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$, $\beta = 2$; (b) $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$, $\beta = -1.5$.

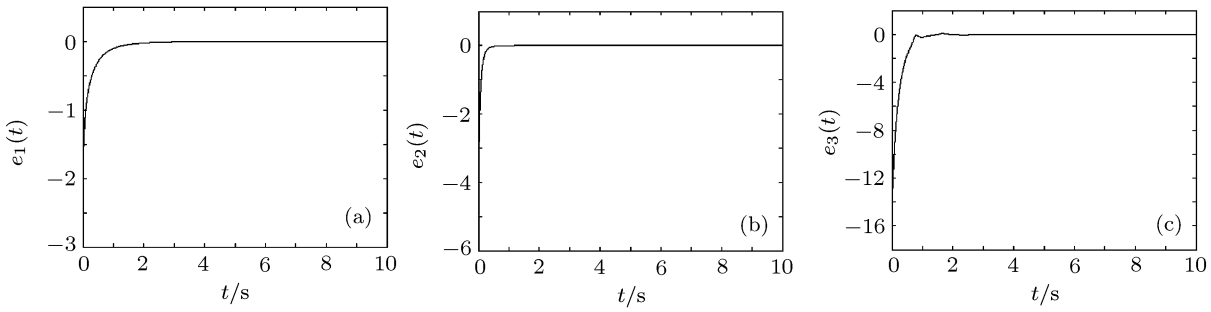


Fig. 11. The history of $e_1(t)$, $e_2(t)$, and $e_3(t)$ for $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$, $\beta = 2$.

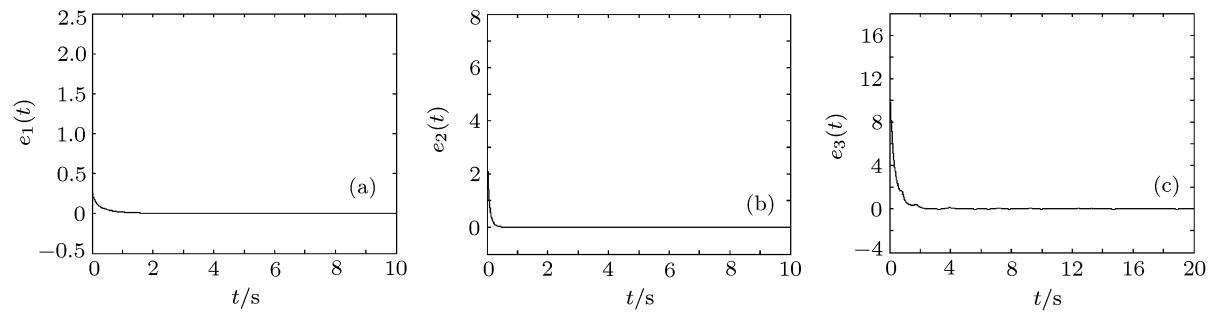


Fig. 12. The history of $e_1(t)$, $e_2(t)$, and $e_3(t)$ for $q_1 = 0.85$, $q_2 = 0.9$, $q_3 = 0.95$, $\beta = -1.5$.

From Fig. 10, it can be seen that state variable amplitudes of system (8) are respectively 2 times and 1.5 times of those of system (4), whereas the phases of systems (8) and (4) are in-phase and anti-phase, respectively. Figures 11 and 12 show that the errors $e_1(t)$, $e_2(t)$, and $e_3(t)$ have been exponentially stabilized to zero, respectively, namely the attractors of systems (8) and (4) are synchronized up to the given scaling factor β.

5. Conclusions

In this paper, we have studied the dynamic behaviours of the new fractional order system using the stability theory of fractional order systems. We found that chaos exists in the fractional order system with an order as low as 2.46. Periodic routes to chaos in the new fractional order system are also found. The numerical simulations have verified the theoretical analysis.

Applying our analysis methods to the fractional order Lorenz system[14] and the fractional order Chen system[15] respectively, we have obtained some theoretical results: the fractional order Lorenz system is chaotic when q_1, q_2, $q_3 \geq 0.99$, and chaos exists in the fractional order Chen system when q_1, q_2, $q_3 \geq 0.69$. These results are consistent with the simulation results in Refs. [14] and [15]. Moreover, in Refs. [13]–[21], the dynamic behaviours of a fractional order system can only be obtained by numerical simulations and computation of the largest Lyapunov exponent, but now we can analyse them theoretically.

Furthermore, in this paper, suitable conditions for achieving the generalized projective synchronization of two identical fractional order systems are derived by using Laplace transform theory. A nonlinear feedback controller is designed to complete the generalized projective synchronization of the new fractional order chaotic systems.

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