Two mutually conjugated tripartite entangled states and their fractional Fourier transformation kernel*

Lü Cui-Hong(吕翠红)[a], Fan Hong-Yi(范洪义)[a], and Jiang Nian-Quan(姜年权)[b]†

[a] *Department of Material Science and Engineering, University of Science and Technology of China, Hefei* 230026, *China*
[b] *College of Physics and Electronic Information, Wenzhou University, Wenzhou* 325035, *China*

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We newly construct two mutually-conjugate tripartite entangled state representations, based on which we propose the formulation of three-mode entangled fractional Fourier transformation (EFFT) and derive the transformation kernel. The EFFT's additivity property is proved and the eigenmode of EFFT is derived. As an application, we calculate the EFFT of the three-mode squeezed vacuum state.

Keywords: tripartite entangled state, three-mode entangled fractional Fourier transformation, the technique of integral within an ordered product of operators

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1. Introduction

Fourier transformation is often employed to solve miscellaneous mathematical physical problems. In 1980, the one-dimensional (1D) fractional Fourier transformation (FFT) was first introduced by Namias as a mathematical tool for solving theoretical physical problems,[1,2] but did not attract much attention until its reintroduction to optics in 1993 by Mendlovic, Ozaktas and Lohmann.[3,4] Since then a series of works have been done on its properties, optical implementations and applications.[5–7] The FFT has been widely used in signal processing,[5] beam shaping etc. The connection between the integration kernel of FFT and the coordinate–momentum representation has been discussed. The eigenmodes of FFT, the single-variable Hermite–Gauss function, has also been derived.[3,8,9]

In quantum mechanics theory, the realization of Fourier transform is the mutual transform of the coordinate–momentum representation. To be concerned with quantum entanglement, which has been the subject of much research since the publication of the famous paper of Einstein, Podolsky and Rosen (EPR)[10] arguing the incompleteness of quantum mechanics, an interesting question naturally arises: Is there any entangled fractional Fourier transformation in the context of quantum mechanics? The answer is affirmative, in Ref. [11], Fan has established the corresponding entangled fractional Fourier transformation in the bipartite EPR entangled state representation and investigated its special properties. Later, Dragoman has shown that its kernel can be classically produced with rotated astigmatic optical systems that mimic the quantum entanglement.[12]

In this paper, we try to propose entangled fractional Fourier transformation in three-mode case. For this aim we shall construct tripartite entangled state $|\rho, \chi_1, \chi_2\rangle$, which is the common eigenvector of $(Q_1 - Q_3)$, $(Q_1 + Q_2 + Q_3)$ and $(P_1 - 2P_2 + P_3)$, and its conjugate state representations. We shall derive the corresponding entangled fractional Fourier transform (EFFT) in these two representations. As one can see shortly later, though the EFFT is not a direct product of three 1D FFTs, it possesses all the properties the FFT should have.

Our work is arranged as follows: In Sections 2 and 3 we briefly introduce the two mutually conjugate three-mode entangled states of continuum variables. We devote Section 4 to the derivation of the three-mode EFFT in terms of the tripartite entangled states, and the EFFT's additivity property is discussed too. The eigenmode of EFFT is obtained in Section 4. As an application, in Section 5 we calculate the EFFT of the three-mode squeezed vacuum state.

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†Corresponding author. E-mail: jiangnq@wzu.edu.cn

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2. Common eigenstate of the three compatible observable

The orthogonal transformation

$$R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{1}$$

which relates $\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ and $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ to $Q_1 - Q_2$ and $P_1 + P_2$, can be generalized to the 3×3 case

$$R_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \tag{2}$$

which yields

$$R_3 \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}(Q_1 - 2Q_2 + Q_3) \\ -\frac{1}{\sqrt{2}}(Q_1 - Q_3) \\ \frac{1}{\sqrt{3}}(Q_1 + Q_2 + Q_3) \end{pmatrix} \tag{3}$$

and

$$R_3 \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}(P_1 - 2P_2 + P_3) \\ -\frac{1}{\sqrt{2}}(P_1 - P_3) \\ \frac{1}{\sqrt{3}}(P_1 + P_2 + P_3) \end{pmatrix}. \tag{4}$$

Thus we can search for the common eigenvector, denoted by $|\rho, \chi_1, \chi_2\rangle$, of the three compatible operators $Q_1 - Q_3$, $Q_1 + Q_2 + Q_3$ and $P_1 - 2P_2 + P_3$.

After some effort, we find that $|\rho, \chi_1, \chi_2\rangle$ in the three-mode Fock space has the form

$$\begin{aligned} |\rho, \chi_1, \chi_2\rangle = & \frac{1}{\sqrt{6}\pi^{3/4}} \exp\left[-\frac{1}{12}\left(\rho^2 + 3\chi_1^2 + 2\chi_2^2\right) + \frac{\sqrt{2}}{6}\,\mathrm{i}\,\rho\left(a_1^\dagger - 2a_2^\dagger + a_3^\dagger\right) \right. \\ & + \frac{\sqrt{2}}{2}\chi_1\left(a_1^\dagger - a_3^\dagger\right) + \frac{\sqrt{2}}{3}\chi_2 \sum_{j=1}^{3} a_j^\dagger \\ & \left. - \frac{1}{6}\left(2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}\right) - \frac{1}{3}\left(2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger\right) \right] |000\rangle, \end{aligned} \tag{5}$$

where ρ, χ_1, and χ_2 are real numbers, $a_j^\dagger$, and a_j $(j = 1, 2)$ are the Bose operators satisfying $\left[a_j, a_k^\dagger\right] = \delta_{jk}$, and $|000\rangle$ is the three-mode vacuum state. In fact, using $\left[a_j, f(a_j, a_j^\dagger)\right] = \partial f(a_j, a_j^\dagger)/\partial a_j^\dagger$, $a_j\,|0\rangle = 0$, and respectively operating a_j on $|\rho, \chi_1, \chi_2\rangle$ leads to

$$a_1\,|\rho, \chi_1, \chi_2\rangle = \frac{1}{3}\left[\frac{\sqrt{2}}{2}\left(\mathrm{i}\,\rho + 3\chi_1 + 2\chi_2\right) - \left(2a_1^\dagger + 2a_2^\dagger - a_3^\dagger\right)\right] |\rho, \chi_1, \chi_2\rangle, \tag{6}$$

$$a_2\,|\rho, \chi_1, \chi_2\rangle = \frac{1}{3}\left[\sqrt{2}\left(-\,\mathrm{i}\,\rho + \chi_2\right) + \left(a_2^\dagger - 2a_1^\dagger - 2a_3^\dagger\right)\right] |\rho, \chi_1, \chi_2\rangle, \tag{7}$$

and

$$a_3\,|\rho, \chi_1, \chi_2\rangle = \frac{1}{3}\left[\frac{\sqrt{2}}{2}\left(\mathrm{i}\,\rho - 3\chi_1 + 2\chi_2\right) + \left(a_1^\dagger - 2a_2^\dagger - 2a_3^\dagger\right)\right] |\rho, \chi_1, \chi_2\rangle. \tag{8}$$

It then follows from $Q_j = (a_j + a_j^\dagger)/\sqrt{2}$ and $P_j = (a_j - a_j^\dagger)/\mathrm{i}\sqrt{2}$ that

$$(Q_1 - Q_3)\,|\rho, \chi_1, \chi_2\rangle = \chi_1\,|\rho, \chi_1, \chi_2\rangle, \tag{9}$$

$$(Q_1 + Q_2 + Q_3)\,|\rho, \chi_1, \chi_2\rangle = \chi_2\,|\rho, \chi_1, \chi_2\rangle, \tag{10}$$

and

$$(P_1 - 2P_2 + P_3)\,|\rho, \chi_1, \chi_2\rangle = \rho\,|\rho, \chi_1, \chi_2\rangle. \tag{11}$$

By virtue of the technique of integral within an ordered product (IWOP) of operators[13−18] and the normal product form of the three-mode vacuum state projector

$$|000\rangle\langle 000| = :\,\exp\left(-\sum_{j=1}^{3} a_j^\dagger a_j\right):, \tag{12}$$

where $:\ :$ stands for the normal ordering, it is easily proved that $|\rho, \chi_1, \chi_2\rangle$ spans a complete set, i.e.,

$$\int \mathrm{d}\rho\mathrm{d}\chi_1\mathrm{d}\chi_2\,|\rho,\chi_1,\chi_2\rangle\langle\rho,\chi_1,\chi_2| = \frac{1}{6\pi^{3/2}}\int \mathrm{d}\rho\mathrm{d}\chi_1\mathrm{d}\chi_2{:}\ \exp\Bigg[-\frac{1}{6}\left(\rho^2+3\chi_1^2+2\chi_2^2\right) + \frac{\mathrm{i}\sqrt{2}}{6}\rho\left(a_1^\dagger-2a_2^\dagger+a_3^\dagger-a_1+2a_2-a_3\right)+\frac{\sqrt{2}}{2}\chi_1\left(a_1^\dagger-a_3^\dagger+a_1-a_3\right) + \frac{\sqrt{2}}{3}\chi_2\sum_{j=1}^{3}\left(a_j+a_j^\dagger\right)-\frac{1}{6}\left(2a_1^{\dagger2}-a_2^{\dagger2}+2a_3^{\dagger2}+2a_1^2-a_2^2+2a_3^2\right) - \frac{1}{3}\left(2a_1^\dagger a_2^\dagger-a_1^\dagger a_3^\dagger+2a_2^\dagger a_3^\dagger+2a_1a_2-a_1a_3+2a_2a_3\right)-\sum_{j=1}^{3}a_j^\dagger a_j\Bigg]{:} = :\mathrm{e}^0{:}\ = 1, \tag{13}$$

where we have used $\int_{-\infty}^{\infty}\mathrm{e}^{-\lambda x^2+bx}\mathrm{d}x=\sqrt{\pi/\lambda}\,\mathrm{e}^{b^2/4\lambda}$. Using Eqs. (9)–(11), we can also calculate

$$\langle\rho',\chi_1',\chi_2'|\,(Q_1-Q_3)\,|\rho,\chi_1,\chi_2\rangle = \chi_1'\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle = \chi_1\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle, \tag{14}$$

$$\langle\rho',\chi_1',\chi_2'|\,(Q_1+Q_2+Q_3)\,|\rho,\chi_1,\chi_2\rangle = \chi_2'\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle = \chi_2\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle, \tag{15}$$

and

$$\langle\rho',\chi_1',\chi_2'|\,(P_1-2P_2+P_3)\,|\rho,\chi_1,\chi_2\rangle = \rho'\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle = \rho\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle, \tag{16}$$

implying that $|\rho,\chi_1,\chi_2\rangle$ is orthogonal

$$\langle\rho',\chi_1',\chi_2'\,|\rho,\chi_1,\chi_2\rangle = \delta(\rho'-\rho)\delta(\chi_1'-\chi_1)\delta(\chi_2'-\chi_2). \tag{17}$$

Thus $|\rho,\chi_1,\chi_2\rangle$ is able to make up a quantum mechanical representation. The Schmidt decomposition of $|\rho,\chi_1,\chi_2\rangle$ in coordinate representation $|q\rangle_j$ $(j=1,2,3)$ is

$$|\rho,\chi_1,\chi_2\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}\,\rho\left(\frac{1}{2}\chi_1+\frac{1}{3}\chi_2\right)}\int \mathrm{d}q\mathrm{e}^{\mathrm{i}\,q\rho}\,|q\rangle_1\otimes|-2q+\chi_1+\chi_2\rangle_2\otimes|q-\chi_1\rangle_3, \tag{18}$$

which exhibits entangled property of $|\rho,\chi_1,\chi_2\rangle$. This may be used in analysing quantum teleportation of quantum states with continuous variables.

3. The conjugate state of $|\rho,\chi_1,\chi_2\rangle$

On the other hand, by observing Eqs. (3) and (4) we search for the common eigenvector of P_1-P_3, $P_1+P_2+P_3$ and $Q_1-2Q_2+Q_3$; the result is

$$|\chi,\rho_1,\rho_2\rangle = \frac{1}{\sqrt{6}\pi^{3/4}}\exp\Bigg[-\frac{1}{12}\left(\chi^2+3\rho_1^2+2\rho_2^2\right)+\frac{\mathrm{i}\sqrt{2}}{2}\rho_1\left(a_1^\dagger-a_3^\dagger\right) + \frac{\mathrm{i}\sqrt{2}}{3}\rho_2\left(\sum_{j=1}^{3}a_j^\dagger\right)+\frac{\sqrt{2}}{6}\chi\left(a_1^\dagger-2a_2^\dagger+a_3^\dagger\right) + \frac{1}{6}\left(2a_1^{\dagger2}-a_2^{\dagger2}+2a_3^{\dagger2}\right)+\frac{1}{3}\left(2a_1^\dagger a_2^\dagger-a_1^\dagger a_3^\dagger+2a_2^\dagger a_3^\dagger\right)\Bigg]|000\rangle, \tag{19}$$

which obeys the eigenvector equations

$$(P_1-P_3)\,|\chi,\rho_1,\rho_2\rangle = \rho_1\,|\chi,\rho_1,\rho_2\rangle, \tag{20}$$

$$(P_1+P_2+P_3)\,|\chi,\rho_1,\rho_2\rangle = \rho_2\,|\chi,\rho_1,\rho_2\rangle, \tag{21}$$

and

$$(Q_1-2Q_2+Q_3)\,|\chi,\rho_1,\rho_2\rangle = \chi\,|\chi,\rho_1,\rho_2\rangle. \tag{22}$$

The term "conjugate" originates from the canonical conjugate commutators of the operators appearing in Eqs. (9)–(11) and those in Eqs. (20)–(22), i.e.,

$$[Q_1-Q_3,P_1-P_3] = 2\,\mathrm{i}, \tag{23}$$

$$[Q_1+Q_2+Q_3,P_1+P_2+P_3] = 3\,\mathrm{i}, \tag{24}$$

and

$$[P_1 - 2P_2 + P_3, Q_1 - 2Q_2 + Q_3] = 4\,\mathrm{i}\,. \tag{25}$$

$|\chi, \rho_1, \rho_2\rangle$ is complete and orthogonal too, i.e.,

$$\int \mathrm{d}\chi \mathrm{d}\rho_1 \mathrm{d}\rho_2\, |\chi, \rho_1, \rho_2\rangle \langle \chi, \rho_1, \rho_2| = 1, \tag{26}$$

and

$$\begin{aligned} &\langle \chi', \rho_1', \rho_2' \,|\chi, \rho_1, \rho_2\rangle \\ &= \delta(\chi' - \chi)\delta(\rho_1' - \rho_1)\delta(\rho_2' - \rho_2). \end{aligned} \tag{27}$$

The Schmidt decomposition of $|\chi, \rho_1, \rho_2\rangle$ in momentum representation is

$$\begin{aligned} |\chi, \rho_1, \rho_2\rangle \,=\, & \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\,\chi\left(\frac{1}{2}\rho_1 + \frac{1}{3}\rho_2\right)} \int \mathrm{d}p\, \mathrm{e}^{-\,\mathrm{i}\,\chi p}\, |p\rangle_1 \\ & \otimes |-2p + \rho_1 + \rho_2\rangle_2 \otimes |p - \rho_1\rangle_3\,, \end{aligned} \tag{28}$$

where the momentum eigenvectors $|p\rangle_j$ ($P_j\,|p\rangle_j = p_j\,|p\rangle_j$, $j = 1, 2, 3$) are defined as

$$|p\rangle_j = \pi^{-1/4} \exp\left(-\frac{1}{2}p^2 + \mathrm{i}\,\sqrt{2} p a_j^\dagger + \frac{1}{2}a_j^{\dagger 2}\right) |0\rangle_j\,. \tag{29}$$

Employing the relation

$${}_j\langle q\,|p\rangle_j = (2\pi)^{-1/2}\, \mathrm{e}^{\mathrm{i}\,p_j q_j} \tag{30}$$

and with the help of Eqs. (18) and (28), we obtain the overlap between $\langle \rho, \chi_1, \chi_2|$ and $|\chi, \rho_1, \rho_2\rangle$,

$$\begin{aligned} \langle \rho, \chi_1, \chi_2\,|\chi, \rho_1, \rho_2\rangle \,=\, & \frac{1}{6\,(2\pi)^{3/2}} \exp\Bigg[-\frac{\mathrm{i}}{6}(\rho\chi \\ & - 2\chi_2\rho_2 - 3\chi_1\rho_1)\Bigg], \end{aligned} \tag{31}$$

we name it the kernel of three-dimensional entangled Fourier transformation.

Finally, the overlaps between the three-mode coherent state

$$|\boldsymbol{z}\rangle \equiv |z_1, z_2, z_3\rangle = \exp\left[-\sum_{j=1}^{3}\left(\frac{|z_j|^2}{2} + a_j^\dagger z_i\right)\right] \tag{32}$$

and $|\chi, \rho_1, \rho_2\rangle$, $|\rho, \chi_1, \chi_2\rangle$ are

$$\begin{aligned} \langle \rho, \chi_1, \chi_2|\,\boldsymbol{z}\rangle \,=\, & \frac{1}{\sqrt{6}\pi^{3/4}} \exp\Bigg[-\frac{1}{12}\left(\rho^2 + 3\chi_1^2 + 2\chi_2^2\right) - \frac{\sqrt{2}}{6}\,\mathrm{i}\,\rho\,(z_1 - 2z_2 + z_3) \\ & + \frac{\sqrt{2}}{2}\chi_1\,(z_1 - z_3) + \frac{\sqrt{2}}{3}\chi_2 \sum_{j=1}^{3} z_i - \frac{1}{6}\left(2z_1^2 - z_2^2 + 2z_3^2\right) \\ & - \frac{1}{3}\,(2z_1 z_2 - z_1 z_3 + 2z_2 z_3) - \frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right)\Bigg], \end{aligned} \tag{33}$$

and

$$\begin{aligned} \langle \boldsymbol{z}'|\,\chi, \rho_1, \rho_2\rangle \,=\, & \frac{1}{\sqrt{6}\pi^{3/4}} \exp\Bigg[-\frac{1}{12}\left(\chi^2 + 3\rho_1^2 + 2\rho_2^2\right) + \frac{\sqrt{2}}{2}\,\mathrm{i}\,\rho_1\,(z_1'^* - z_3'^*) \\ & + \frac{\sqrt{2}}{3}\,\mathrm{i}\,\rho_2\Bigg(\sum_{j=1}^{3} z_i'^*\Bigg) + \frac{\sqrt{2}}{6}\chi\,(z_1'^* - 2z_2'^* + z_3'^*) + \frac{1}{6}\left(2z_1'^{*2} - z_2'^{*2} + 2z_3'^{*2}\right) \\ & + \frac{1}{3}\,(2z_1'^* z_2'^* - z_1'^* z_3'^* + 2z_2'^* z_3'^*) - \frac{1}{2}\left(|z_1'|^2 + |z_2'|^2 + |z_3'|^2\right)\Bigg], \end{aligned} \tag{34}$$

respectively.

4. Three-mode EFFT in tripartite entangled state representations

Firstly we calculate the matrix element of $\exp\left[f\sum_{j=1}^{3} a_j^\dagger a_j\right]$ between $\langle \rho, \chi_1, \chi_2|$ and $|\chi, \rho_1, \rho_2\rangle$. Using Eqs. (33) and (34), and the overcompleteness relation of $|\boldsymbol{z}\rangle$,

$$\int \frac{\mathrm{d}^2\boldsymbol{z}}{\pi^3}\, |\boldsymbol{z}\rangle \langle \boldsymbol{z}| = 1, \tag{35}$$

as well as the operator identity

$$\mathrm{e}^{f a_1^\dagger a_1} = :\, \exp\left[\left(\mathrm{e}^f - 1\right) a_1^\dagger a_1\right] :\,, \tag{36}$$

we have

$$
\begin{aligned}
&\langle\rho,\chi_1,\chi_2|\exp\left[f\sum_{j=1}^{3}a_j^{\dagger}a_j\right]|\chi,\rho_1,\rho_2\rangle\\
&=\int\frac{\mathrm{d}^2\boldsymbol{z}}{\pi^3}\frac{\mathrm{d}^2\boldsymbol{z}'}{\pi^3}\langle\rho,\chi_1,\chi_2|\,\boldsymbol{z}\rangle\langle\boldsymbol{z}|:\exp\left[\left(\mathrm{e}^{f}-1\right)\left(a_1^{\dagger}a_1+a_2^{\dagger}a_2+a_3^{\dagger}a_3\right)\right]:|\boldsymbol{z}'\rangle\langle\boldsymbol{z}'|\,\chi,\rho_1,\rho_2\rangle\\
&=\frac{1}{6\pi^{3/2}\left(\mathrm{e}^{2f}+1\right)^{3/2}}\exp\left[\frac{\mathrm{e}^{f}-\mathrm{e}^{-f}}{12\left(\mathrm{e}^{f}+\mathrm{e}^{-f}\right)}\left(\rho^2+3\chi_1^2+2\chi_2^2+\chi^2+3\rho_1^2+2\rho_2^2\right)\right.\\
&\quad\left.+\frac{1}{3\left(\mathrm{e}^{f}+\mathrm{e}^{-f}\right)}(3\,\mathrm{i}\,\chi_1\rho_1-\mathrm{i}\,\chi\rho+2\,\mathrm{i}\,\chi_2\rho_2)\right],
\end{aligned}
\tag{37}
$$

where we have used

$$
\langle z|:f\left(a^{\dagger},a\right):|z'\rangle=f\left(z^{*},z'\right)\langle z|\,z'\rangle\,. \tag{38}
$$

Let $f=\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)$, equation (37) arrives at

$$
\begin{aligned}
&\langle\rho,\chi_1,\chi_2|\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)\left(\sum_{j=1}^{3}a_j^{\dagger}a_j+\frac{3}{2}\right)\right]|\chi,\rho_1,\rho_2\rangle\\
&=\frac{1}{6\left(2\pi\sin\alpha\right)^{3/2}}\exp\left[\frac{\mathrm{i}}{12\tan\alpha}\left(\rho^2+3\chi_1^2+2\chi_2^2+\chi^2+3\rho_1^2+2\rho_2^2\right)+\frac{\mathrm{i}}{6\sin\alpha}\left(-\chi\rho+3\rho_1\chi_1+2\rho_2\chi_2\right)\right]\\
&\equiv A\left(\alpha\right).
\end{aligned}
\tag{39}
$$

Specializing now, in the case of $\alpha=\pi/2$, equation (39) reduces to Eq. (31), as expected. Now we take the right-hand side of Eq. (39) as the kernel of EFFT of the α-th order, i.e., we define the α-th EFFT of function $g\left(\chi,\rho_1,\rho_2\right)$ as $F_{\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)$ via the following relation,

$$
F_{\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)=\mathrm{e}^{\mathrm{i}\,3(\alpha-\pi/2)/2}\int\mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2A\left(\alpha\right)g\left(\chi,\rho_1,\rho_2\right)\equiv G\left(\rho,\chi_1,\chi_2\right). \tag{40}
$$

We will show that this EFFT can help us to reveal some new properties which have been overlooked in the formulation of the direct product of three real FFTs. Now we prove the additivity property of EFFT,

$$
\begin{aligned}
F_{\beta}\left[F_{\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)\right]&=F_{\beta}\left[G\left(\rho,\chi_1,\chi_2\right)\right]=\mathrm{e}^{\mathrm{i}\,3(\beta-\pi/2)/2}\int\mathrm{d}\rho\mathrm{d}\chi_1\mathrm{d}\chi_2A\left(\beta\right)G\left(\rho,\chi_1,\chi_2\right)\\
&=\frac{\mathrm{e}^{\mathrm{i}\,3(\beta-\pi/2)/2}}{6\left(2\pi\sin\beta\right)^{3/2}}\frac{\mathrm{e}^{\mathrm{i}\,3(\alpha-\pi/2)/2}}{6\left(2\pi\sin\alpha\right)^{3/2}}\int\mathrm{d}\rho\mathrm{d}\chi_1\mathrm{d}\chi_2\\
&\quad\times\exp\left[\frac{\mathrm{i}}{12\tan\beta}(\rho^2+3\chi_1^2+2\chi_2^2+\chi'^2+3\rho_1'^2+2\rho_2'^2)+\frac{\mathrm{i}}{6\sin\beta}(-\chi'\rho+3\rho_1'\chi_1+2\rho_2'\chi_2)\right]\\
&\quad\times\int\mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2g\left(\chi,\rho_1,\rho_2\right)\exp\left[\frac{\mathrm{i}}{12\tan\alpha}\left(\rho^2+3\chi_1^2+2\chi_2^2+\chi^2+3\rho_1^2+2\rho_2^2\right)\right.\\
&\quad\left.+\frac{\mathrm{i}}{6\sin\alpha}\left(-\chi\rho+3\rho_1\chi_1+2\rho_2\chi_2\right)\right],
\end{aligned}
\tag{41}
$$

according to the relationship between trigonometric functions

$$
\cot\alpha-\frac{\sin\beta}{\sin\alpha\sin\left(\alpha+\beta\right)}=\cot\beta-\frac{\sin\beta}{\sin\alpha\sin\left(\alpha+\beta\right)}\cot\left(\alpha+\beta\right), \tag{42}
$$

equation (41) becomes

$$
\begin{aligned}
F_{\beta}\left[F_{\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)\right]&=\frac{\mathrm{e}^{\mathrm{i}\,3(\alpha+\beta-\pi)/2}}{6\left(2\pi\right)^{3/2}\left[\sin\left(\alpha+\beta\right)\right]^{3/2}}\exp\left[\frac{\mathrm{i}\left(\chi'^2+3\rho_1'^2+2\rho_2'^2\right)}{12\tan\left(\alpha+\beta\right)}\right]\int\mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2g\left(\chi,\rho_1,\rho_2\right)\\
&\quad\times\exp\left[\frac{\mathrm{i}}{12\tan(\alpha+\beta)}\left(\chi^2+3\rho_1^2+2\rho_2^2\right)-\frac{\mathrm{i}}{6\sin\left(\alpha+\beta\right)}(\chi\chi'-3\rho_1\rho_1'-2\rho_2\rho_2')\right].
\end{aligned}
\tag{43}
$$

By letting $\chi'=-\rho'$, $\rho_1'=-\chi_1'$, $\rho_2'=-\chi_2'$ in Eq. (43) we obtain the additivity property of EFFT as expected

$$
F_{\beta}\left[F_{\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)\right]=F_{\beta+\alpha}\left(g\left(\chi,\rho_1,\rho_2\right)\right)=G_{\beta+\alpha}\left(\rho',\chi_1',\chi_2'\right). \tag{44}
$$

5. Eigenmodes of EFFT

Funcion $g(\chi,\rho_1,\rho_2)$ can be considered as the wave function of the state vector $|g\rangle$ in the $\langle\chi,\rho_1,\rho_2|$ representation, i.e., $\langle\chi,\rho_1,\rho_2\,|g\rangle$, then from Eq. (40) and the completeness relation (26) we see that the EFFT of $\langle\chi,\rho_1,\rho_2\,|g\rangle$ is actually

$$\begin{aligned}
&F_\alpha\left(g\left(\chi,\rho_1,\rho_2\right)\right)\\
&=\mathrm{e}^{\mathrm{i}\,3(\alpha-\pi/2)/2}\int \mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2\,\langle\rho,\chi_1,\chi_2|\\
&\quad\times\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)\left(\sum_{j=1}^{3}a_j^\dagger a_j+\frac{3}{2}\right)\right]\\
&\quad\times|\chi,\rho_1,\rho_2\rangle\,\langle\chi,\rho_1,\rho_2\,|g\rangle\\
&=\langle\rho,\chi_1,\chi_2|\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)\sum_{j=1}^{3}a_j^\dagger a_j\right]|g\rangle\\
&\equiv G\left(\rho,\chi_1,\chi_2\right).
\end{aligned}\tag{45}$$

Furthermore, let $G(\rho,\chi_1,\chi_2)\equiv\langle\rho,\chi_1,\chi_2\,|g\rangle$, equation (45) implies

$$|G\rangle=\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)\sum_{j=1}^{3}a_j^\dagger a_j\right]|g\rangle\,.\tag{46}$$

The formulas (26) and (45) can help us to derive EFFT of some wave functions easily. For example, when $|g\rangle$ is a three-mode number state

$$|n,m,l\rangle=\frac{a_1^{\dagger n}a_2^{\dagger m}a_3^{\dagger l}}{\sqrt{n!m!l!}}\,|000\rangle\,,$$

then the EFFT of the wave function $\langle\chi,\rho_1,\rho_2\,|n,m,l\rangle$ is

$$\begin{aligned}
&F_\alpha\left(\langle\chi,\rho_1,\rho_2\,|n,m,l\rangle\right)\\
&=\langle\rho,\chi_1,\chi_2|\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)\sum_{j=1}^{3}a_j^\dagger a_j\right]|n,m,l\rangle\\
&=\mathrm{i}^{n+m+l}\mathrm{e}^{-\,\mathrm{i}\,\alpha(n+m+l)}\,\langle\rho,\chi_1,\chi_2\,|n,m,l\rangle\,.
\end{aligned}\tag{47}$$

To calculate $\langle\rho,\chi_1,\chi_2\,|n,m,l\rangle$, we define the polynomials $\mathrm{H}_{nml}^{\{R\}}(y_1,y_2,y_3)$ by means of its generating function

$$\begin{aligned}
&\exp\left[-\frac{1}{2}SR\widetilde{S}+SRY\right]\\
&=\sum_{n,m,l=0}^{\infty}\frac{s_1^n s_2^m s_3^l}{n!m!l!}\mathrm{H}_{nml}^{\{R\}}(y_1,y_2,y_3),
\end{aligned}\tag{48}$$

where $S=(s_1,s_2,s_3)$, $\widetilde{Y}=(y_1,y_2,y_3)$, and s_j and y_j are arbitrary complex numbers, and R is a symmetric matrix. From Eq. (48) we see

$$\mathrm{H}_{nml}^{\{R\}}(y_1,y_2,y_3)=\frac{\partial^{n+m+l}}{\partial s_1^n\partial s_2^m\partial s_3^l}\exp\left[-\frac{1}{2}SR\widetilde{S}+SRY\right]_{s_1=s_2=s_3=0}.\tag{49}$$

Then $\langle\rho,\chi_1,\chi_2|$ can be rewritten as

$$\begin{aligned}
\langle\rho,\chi_1,\chi_2|\;=\;&\frac{1}{\sqrt{6}\pi^{3/4}}\,\langle000|\exp\left[-\frac{1}{4}\left|y_j\right|^2+y_1^*\left(\mathrm{i}\,a_1\right)+y_2^*\left(\mathrm{i}\,a_2\right)+y_3^*\left(\mathrm{i}\,a_3\right)\right.\\
&\left.+\frac{1}{6}\left[2\left(\mathrm{i}\,a_1\right)^2-\left(\mathrm{i}\,a_2\right)^2+2\left(\mathrm{i}\,a_3\right)^2\right]+\frac{1}{3}\left[2\left(\mathrm{i}\,a_1\right)\left(\mathrm{i}\,a_2\right)-\left(\mathrm{i}\,a_1\right)\left(\mathrm{i}\,a_3\right)+2\left(\mathrm{i}\,a_2\right)\left(\mathrm{i}\,a_3\right)\right]\right]\\
=\;&\frac{1}{\sqrt{6}\pi^{3/4}}\,\langle000|\exp\left(-\frac{1}{4}\sum_{j=1}^{3}\left|y_j\right|^2\right)\sum_{n,m,l=0}^{\infty}\frac{\left(\mathrm{i}\,a_1\right)^n\left(\mathrm{i}\,a_2\right)^m\left(\mathrm{i}\,a_3\right)^l}{n!m!l!}\mathrm{H}_{nml}^{\{R\}}(y_1,y_2,y_3),
\end{aligned}\tag{50}$$

where

$$R=\frac{1}{3}\begin{pmatrix}2&2&-1\\2&-1&2\\-1&2&2\end{pmatrix},\tag{51}$$

and

$$y_1=\frac{\sqrt{2}}{6}\left[\rho+\mathrm{i}\left(3\chi_1+2\chi_2\right)\right],\qquad y_2=\frac{\sqrt{2}}{3}\left[-\rho+i\chi_2\right],\qquad y_3=\frac{\sqrt{2}}{6}\left[\rho-\mathrm{i}\left(3\chi_1-2\chi_2\right)\right].\tag{52}$$

Similarly, we have

$$\langle\chi,\rho_1,\rho_2|\;=\;\frac{1}{\sqrt{6}\pi^{3/4}}\,\langle000|\exp\left(-\frac{1}{4}\sum_{j=1}^{3}\left|\zeta_j\right|^2\right)\sum_{n,m,l=0}^{\infty}\frac{(a_1)^n(a_2)^m(a_3)^l}{n!m!l!}\mathrm{H}_{nml}^{\{R\}}(\zeta_1,\zeta_2,\zeta_3),\tag{53}$$

where

$$\zeta_1 = \frac{\sqrt{2}}{6}\left[\chi + \mathrm{i}\left(3\rho_1 + 2\rho_2\right)\right], \qquad \zeta_2 = \frac{\sqrt{2}}{3}\left[-\chi + \mathrm{i}\,\rho_2\right], \qquad \zeta_3 = \frac{\sqrt{2}}{6}\left[\chi - \mathrm{i}\left(3\rho_1 - 2\rho_2\right)\right]. \tag{54}$$

It then follows that

$$\langle \rho, \chi_1, \chi_2 \,|n, m, l\rangle = \frac{1}{\sqrt{6}\pi^{3/4}} \exp\left(-\frac{1}{4}\sum_{j=1}^{3}|y_j|^2\right) \sum_{n,m,l=0}^{\infty} \frac{\mathrm{i}^{n+m+l}}{\sqrt{n!m!l!}} \mathrm{H}_{nml}^{\{R\}}(y_1, y_2, y_3), \tag{55}$$

$$\langle \chi, \rho_1, \rho_2 \,|n, m, l\rangle = \frac{1}{\sqrt{6}\pi^{3/4}} \exp\left(-\frac{1}{4}\sum_{j=1}^{3}|\zeta_j|^2\right) \sum_{n,m,l=0}^{\infty} \frac{1}{\sqrt{n!m!l!}} \mathrm{H}_{nml}^{\{R\}}(\zeta_1, \zeta_2, \zeta_3). \tag{56}$$

Using Eqs. (55) and (56) we can see that equation (47) becomes

$$F_\alpha\left\{\exp\left(-\frac{1}{4}\sum_{j=1}^{3}|\zeta_j|^2\right)\mathrm{H}_{nml}^{\{R\}}(\zeta_1, \zeta_2, \zeta_3)\right\} = \left(-\mathrm{e}^{-\mathrm{i}\,\alpha}\right)^{(n+m+l)} \exp\left(-\frac{1}{4}\sum_{j=1}^{3}|y_j|^2\right)\mathrm{H}_{nml}^{\{R\}}(y_1, y_2, y_3). \tag{57}$$

If we consider the operation F_α as an operator, we can say that the eigenfunction of F_α (the eigenmodes of EFFT) is the three-variable polynomials $\mathrm{H}_{nml}^{\{R\}}$ with the eigenvalue $\left(-\mathrm{e}^{-\mathrm{i}\,\alpha}\right)^{(n+m+l)}$. The eigenmode of EFFT is different from the usual Hermite polynomials. This is a new property of EFFT. One can confirm that the eigenmodes of EFFT form an orthogonal and complete basis set, i.e., the function space spanned by $\mathrm{H}_{nml}^{\{R\}}(y_1, y_2, y_3)$ is complete,

$$\begin{aligned} &\iiint \frac{\mathrm{d}^2\zeta_1\mathrm{d}^2\zeta_2\mathrm{d}^2\zeta_3}{\pi^{3/2}} \exp\left(-\frac{1}{4}\sum_{j=1}^{3}|\zeta_j|^2\right)\mathrm{H}_{nml}^{\{R\}}(\zeta_1, \zeta_2, \zeta_3)\left[\mathrm{H}_{n'm'l'}^{\{R\}}(\zeta_1, \zeta_2, \zeta_3)\right]^* \\ &= \sqrt{n!m!l!n'!m'!l'!}\,\delta_{n,n'}\delta_{m,m'}\delta_{l,l'}. \end{aligned} \tag{58}$$

6. Some wave function's EFFT

Equation (45) provides us with a new approach for calculating the EFFT of quantum mechanical wave functions in the $|\chi, \rho_1, \rho_2\rangle$ representation. For instance, when $|g\rangle$ is the three-mode squeezed vacuum state which reads

$$|g\rangle \to S_3\,|000\rangle = \frac{\mu^{3/2}}{\sqrt{6}\pi^{3/4}} \int \mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2\,|\mu\chi, \mu\rho_1, \mu\rho_2\rangle \exp\left[-\frac{1}{12}\left(\chi^2 + 3\rho_1^2 + 2\rho_2^2\right)\right], \tag{59}$$

where S_3 is the squeezing operator, in the $|\chi, \rho_1, \rho_2\rangle$ representation it can be expressed as

$$S_3 = \mu^{3/2} \int \mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2\,|\mu\chi, \mu\rho_1, \mu\rho_2\rangle\,\langle\chi, \rho_1, \rho_2|\,\mu = \mathrm{e}^{\lambda}. \tag{60}$$

Then according to Eq. (45), the EFFT of the wave function of the new squeezed state is

$$\begin{aligned} F_\alpha\left(\langle\chi, \rho_1, \rho_2|\,S_3\,|000\rangle\right) &= \langle\rho, \chi_1, \chi_2|\exp\left[\mathrm{i}\left(\frac{\pi}{2} - \alpha\right)\sum_{j=1}^{3} a_j^\dagger a_j\right] S_3\,|000\rangle \\ &= \frac{\mu^{3/2}}{\sqrt{6}\pi^{3/4}} \int \mathrm{d}\chi\mathrm{d}\rho_1\mathrm{d}\rho_2\,\langle\rho, \chi_1, \chi_2|\exp\left[\mathrm{i}\left(\frac{\pi}{2} - \alpha\right)\sum_{j=1}^{3} a_j^\dagger a_j\right]|\mu\chi, \mu\rho_1, \mu\rho_2\rangle \\ &\quad \times \exp\left[-\frac{1}{12}\left(\chi^2 + 3\rho_1^2 + 2\rho_2^2\right)\right] \\ &= \left(\frac{\mu\mathrm{e}^{\mathrm{i}\,\alpha}}{(\mathrm{i}\sin\alpha + \mu^2\cos\alpha)}\right)^{3/2} \frac{1}{\sqrt{6}\pi^{3/4}} \exp\left[\frac{\mathrm{i} - \mu^2\tan\alpha}{12\,(\tan\alpha - \mathrm{i}\,\mu^2)}\left(\rho^2 + 3\chi_1^2 + 2\chi_2^2\right)\right]. \end{aligned} \tag{61}$$

In summary, for the first time we introduce three-mode entangled fractional Fourier transform in two mutually conjugate entangled state representation, which is different from that in Ref. [19] where the two

representations are actually not conjugate to each other, this can be seen from their overlap (see Eq. (22) in Ref. [19]). To be more concrete, we have employed the two mutually conjugate three-mode entangled state representations ($|\chi,\rho_1,\rho_2\rangle - |\rho,\chi_1,\chi_2\rangle$) in order to establish the formulation of EFFT, which means that we have linked the fractional Fourier transformation to appropriate operator transformation between the two kinds of entangled state representations. In doing so, the EFFT of squeezed wave functions can be derived more directly and concisely. Because multimode squeezed states are simultaneously entangled states,[20−24] the introduction of EFFT is a necessity.

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