Lagrange equations of nonholonomic systems with fractional derivatives*

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This paper obtains Lagrange equations of nonholonomic systems with fractional derivatives. First, the exchanging relationships between the isochronous variation and the fractional derivatives are derived. Secondly, based on these exchanging relationships, the Hamilton's principle is presented for non-conservative systems with fractional derivatives. Thirdly, Lagrange equations of the systems are obtained. Furthermore, the d'Alembert–Lagrange principle with fractional derivatives is presented, and the Lagrange equations of nonholonomic systems with fractional derivatives are studied. An example is designed to illustrate these results.

Keywords: fractional derivative, d'Alembert–Lagrange principle, Lagrange equation, nonholonomic system

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1. Introduction

Nowadays fractional analysis has proved to be useful in physics and mathematics.[1−5] Its origin goes back more than 300 years, when in 1695 L Hospital asked Leibniz the meaning of $\mathrm{d}^n y/\mathrm{d}x^n$ for $n = \frac{1}{2}$. After that, many famous mathematicians, like Fourier, Riemann, among others, contributed to the development of the fractional calculus.[6,7] The respective Euler–Lagrange-type equations have been researched in many fields. Riewe[8,9] has developed a new approach to mechanics, obtained a version of the Euler–Lagrange equations for problems of the calculus of variations with fractional derivatives for the dynamical systems. Agrawal[10] proved a Larangian formulation for variation problems with right and left fractional derivatives in the Riemann–Liouville sense. The problem of having non-conservative dissipative system's equations of motion arisen from the use of a variation principle is re-examined by Dreisigmeyer and Young.[11] It is worth while to mention that the Lagrange equations of nonholonomic systems are not uniquely defined.

In this paper, we have derived the exchanging relationships between the isochronous variation and the fractional derivatives, and presented the Lagrange equations of non-conservative systems with fractional derivatives based on the Hamilton's principle. Furthermore, the Lagrange equations of nonholonomic systems with fractional derivatives are derived from the d'Alembert–Lagrange principle.

This paper is organized as follows. In Section 2, we provide a brief review of fractional derivatives. In Section 3, we examine the exchange relationship between the isochronous variation and the fractional derivatives. In Section 4, we derive the Hamilton's principle for non-conservative systems with fractional derivatives, and then obtain the Lagrange equations of the systems. In Section 5, we present the Lagrange equations of nonholonomic systems. In Section 6, an example is considered to demonstrate the application of the formalism. A short conclusion is given in Section 7.

2. Fractional derivatives

In this section we collect the definitions of right and left Riemann–Liouville fractional derivatives and their main properties.

Definition 1 Let f be a continuous and integrable function in the interval $[a, b]$. For all $t \in [a, b]$, the left Riemann–Liouville fractional derivative ${}_aD_t^\alpha$, and the right Riemann–Liouville fractional derivative ${}_tD_b^\alpha$, of order α, are defined in the following way:

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$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta)\mathrm{d}\theta, \tag{1} $$

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (t-\theta)^{n-\alpha-1} f(\theta)\mathrm{d}\theta, \tag{2} $$

where $n \in N$, $n-1 \le \alpha < n$, and Γ is the Euler gamma function.

Remark 1 If α is an integer, from Eqs. (1) and (2) we obtain the standard derivatives, that is,

$$ {}_aD_t^\alpha f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t), \quad {}_tD_b^\alpha f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t). $$

Theorem 1 Let f and g be two continuous functions on $[a, b]$. Then, for all $t \in [a, b]$, the following properties hold:

$$ \begin{aligned} &\text{(i) for } \ m>0, {}_aD_t^m[f(t)+g(t)] = {}_aD_t^m f(t) + {}_aD_t^m g(t);\\ &\text{(ii) for } \ m \ge n \ge 0, {}_aD_t^m({}_aD_t^{-n} f(t)) = {}_aD_t^{m-n} f(t);\\ &\text{(iii) for } \ m>0, {}_aD_t^m({}_aD_t^{-m} f(t)) = f(t);\\ &\text{(iv) for } \ m>0, \int_a^b ({}_aD_t^m f(t))g(t)\mathrm{d}t = \int_a^b f(t)({}_tD_b^m g(t))\mathrm{d}t. \end{aligned} \tag{3} $$

3. Exchange relationship between isochronous variation and fractional derivatives

Consider two infinitely closed orbits: γ and $\gamma+\mathrm{d}\gamma$. We denote the generalized coordinates by $q = q(t, \gamma)$, $q' = q(t, \gamma + \mathrm{d}\gamma)$, corresponding to the two infinitely closed orbits individually in giving time, where $q = \{q_1, q_2, \ldots, q_n\}$.

We define the isochronous variation as

$$ \delta q = q(t, \gamma + \mathrm{d}\gamma) - q(t, \gamma). \tag{4} $$

Extending $q(t, \gamma + \mathrm{d}\gamma)$ to the linear terms of $\mathrm{d}\gamma$, we obtain

$$ q(t, \gamma + \mathrm{d}\gamma) = q(t, \gamma) + \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma, \tag{5} $$

substituting Eq. (5) into Eq. (4), we have

$$ \delta q = \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma, $$

similarly we have

$$ {}_aD_t^\alpha \delta q = {}_aD_t^\alpha \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma. \tag{6} $$

Using Eq. (5), one has

$$ \begin{aligned} \delta {}_aD_t^\alpha q &= {}_aD_t^\alpha q(t, \gamma + \mathrm{d}\gamma) - {}_aD_t^\alpha q(t, \gamma)\\ &= {}_aD_t^\alpha \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma, \end{aligned} \tag{7} $$

comparing Eqs. (6) and (7), we obtain

$$ \delta {}_aD_t^\alpha q = {}_aD_t^\alpha \delta q, \tag{8} $$

similarly we have

$$ \delta {}_tD_b^\beta q = {}_tD_b^\beta \delta q. \tag{9} $$

We call Eqs. (8) and (9) the exchanging relationships with respect to the isochronous variation and the fractional derivatives.

4. Hamilton's principle and Lagrange equations for nonconservative systems with fractional derivatives

Assuming that the kinetic energy function of the system is $T = T(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t)$.

Hamilton's principle states that the actual pace exists when the Hamiltonian action has determining value. Therefore, the Hamilton's principle for nonconservative systems with fractional derivatives can be written in the form

$$ \int_a^b (\delta T + \delta' A)\mathrm{d}t = 0, \tag{10} $$

where $\delta' A = Q\delta q$ is the virtual work of generalized force.

Taking total variation for function T we have

$$\delta T = \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial_a D_t^\alpha q}\delta_a D_t^\alpha q + \frac{\partial T}{\partial_t D_b^\beta q}\delta_t D_b^\beta q = \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial_a D_t^\alpha q}{}_a D_t^\alpha \delta q + \frac{\partial T}{\partial_t D_b^\beta q}{}_t D_b^\beta \delta q, \tag{11}$$

substituting Eq. (11) into Eq. (10), we have

$$\begin{aligned}
&\int_a^b \left(\frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial_a D_t^\alpha q}{}_a D_t^\alpha \delta q + \frac{\partial T}{\partial_t D_b^\beta q}{}_t D_b^\beta \delta q + Q\delta q\right)\mathrm{d}t \\
&= \int_a^b \left(\frac{\partial T}{\partial q}\delta q + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q}\delta q + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q}\delta q + Q\delta q\right)\mathrm{d}t \\
&= \int_a^b \left(\frac{\partial T}{\partial q} + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q} + Q\right)\delta q \mathrm{d}t = 0,
\end{aligned} \tag{12}$$

hence

$$\left(\frac{\partial T}{\partial q} + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q} + Q\right)\delta q = 0. \tag{13}$$

By virtue of the independence of δq, we obtain

$$\frac{\partial T}{\partial q} + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q} + Q = 0. \tag{14}$$

When Q contains conservative force Q' and non-conservative force Q'', and Q' satisfies the following conditions:

If Q' is potential, i.e., there exists a function $V = V(q,t)$ such that

$$Q' = -\frac{\partial V}{\partial q}, \tag{15}$$

substituting Eq. (15) into Eq. (14), we obtain

$$\frac{\partial T}{\partial q} + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q} - \frac{\partial V}{\partial q} + Q'' = 0, \tag{16}$$

since the function only depends on the generalized coordinates, therefore

$$\frac{\partial V}{\partial_a D_t^\alpha q} = 0, \quad \frac{\partial V}{\partial_t D_b^\beta q} = 0, \tag{17}$$

using Eq. (17), from Eq. (16) we have

$$\frac{\partial (T-V)}{\partial q} + {}_t D_b^\alpha \frac{\partial (T-V)}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial (T-V)}{\partial_t D_b^\beta q} = -Q''. \tag{18}$$

Introducing function $L = T - V$, we can write Eq. (18) as

$$\frac{\partial L}{\partial q} + {}_t D_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial L}{\partial_t D_b^\beta q} = -Q''. \tag{19}$$

If Q' has generalized potential, i.e., there exists a function $U(q, {}_a D_t^\alpha q, {}_t D_b^\beta q, t)$, such that

$$Q' = \frac{\partial U}{\partial q} + {}_t D_b^\alpha \frac{\partial U}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial U}{\partial_t D_b^\beta q},$$

where

$$U = A(t,q)_a D_t^\alpha q + B(t,q)_t D_b^\beta q + C(t,q),$$

then equation (14) can be written as

$$\frac{\partial (T+U)}{\partial q} + {}_t D_b^\alpha \frac{\partial (T+U)}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial (T+U)}{\partial_t D_b^\beta q} = -Q''.$$

Introducing function $L = T + U = T - V$, we can obtain the Lagrange equations of non-conservative systems with fractional derivatives

$$\frac{\partial L}{\partial q} + {}_t D_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial L}{\partial_t D_b^\beta q} = -Q''. \tag{20}$$

5. Lagrange equations of nonholonomic systems with fractional derivatives

Let us consider a constrained mechanical system whose configuration is determined by the generalized coordinate q. The motion of the system is subjected to the ideal bilateral nonholonomic constraint of Appell–Chetaev type

$$f(q, {}_a D_t^\alpha q, {}_t D_b^\beta q, t) = 0, \tag{21}$$

the Appell–Chetaev condition gives

$$\frac{\partial f}{\partial_a D_t^\alpha q}\delta q = 0, \tag{22}$$

$$\frac{\partial f}{\partial_t D_b^\beta q}\delta q = 0. \tag{23}$$

Then, we consider the d'Alembert–Lagrange principle with fractional derivatives.[12]

By virtue of Eq. (14), it is straightforward to show that the universal d'Alembert–Lagrange principle with fractional derivatives can be written as

$$\left(\frac{\partial T}{\partial q} + {}_t D_b^\alpha \frac{\partial T}{\partial_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial T}{\partial_t D_b^\beta q} + Q\right)\cdot \delta q = 0. \tag{24}$$

Introducing the Lagrange multiplier λ, and multiplying λ on both sides of Eqs. (22) and (23), we obtain

$$\lambda\frac{\partial f}{\partial_a D_t^\alpha q}\delta q = 0, \tag{25}$$

$$\lambda\frac{\partial f}{\partial_t D_b^\beta q}\delta q = 0, \tag{26}$$

plus Eqs. (24), (25) and (26), one has

$$\left(\frac{\partial T}{\partial q} + {}_tD_b^\alpha\frac{\partial T}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial T}{\partial_t D_b^\beta q} + Q\right)\cdot\delta q + \lambda\frac{\partial f}{\partial_a D_t^\alpha q}\delta q + \lambda\frac{\partial f}{\partial_t D_b^\beta q}\delta q = 0, \tag{27}$$

it can be presented as

$$\left(\frac{\partial T}{\partial q} + {}_tD_b^\alpha\frac{\partial T}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial T}{\partial_t D_b^\beta q} + Q + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q}\right)\cdot\delta q = 0. \tag{28}$$

The value of λ we take is to make the expression in parentheses be zero for independent variations, hence equation (28) is equivalent to

$$\frac{\partial T}{\partial q} + {}_tD_b^\alpha\frac{\partial T}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial T}{\partial_t D_b^\beta q} + Q + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q} = 0. \tag{29}$$

When Q contains conservative force Q' and nonconservative force Q'', and Q' satisfies the following conditions.

If Q' is potential, i.e., there exists a function $V = V(q,t)$ such that

$$Q' = -\frac{\partial V}{\partial q}, \tag{30}$$

substituting Eq. (30) into Eq. (29), we have

$$\frac{\partial T}{\partial q} + {}_tD_b^\alpha\frac{\partial T}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial T}{\partial_t D_b^\beta q} + Q'' - \frac{\partial V}{\partial q} + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q} = 0, \tag{31}$$

since the function only depends on generalized coordinates, therefore

$$\frac{\partial V}{\partial_a D_t^\alpha q} = 0, \quad \frac{\partial V}{\partial_t D_b^\beta q} = 0, \tag{32}$$

using Eq. (32), we change Eq. (31) into

$$\frac{\partial(T-V)}{\partial q} + {}_tD_b^\alpha\frac{\partial(T-V)}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial(T-V)}{\partial_t D_b^\beta q} + Q'' + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q} = 0. \tag{33}$$

Introducing function $L = T - V$, we can write Eq. (33) as

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta q} + Q'' + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q} = 0.$$

If Q' has generalized potential, i.e., there exists a function $U(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t)$, such that

$$Q' = \frac{\partial U}{\partial q} + {}_tD_b^\alpha\frac{\partial U}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial U}{\partial_t D_b^\beta q},$$

where

$$U = A(t,q)_aD_t^\alpha q + B(t,q)_tD_b^\beta q + C(t,q),$$

then equation (29) can be written as

$$\frac{\partial(T+U)}{\partial q} + {}_tD_b^\alpha\frac{\partial(T+U)}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial(T+U)}{\partial_t D_b^\beta q} + Q'' + \lambda\frac{\partial f}{\partial_a D_t^\alpha q} + \lambda\frac{\partial f}{\partial_t D_b^\beta q} = 0.$$

Introducing function $L = T + U = T - V$, we obtain the Lagrange equations of nonholonomic systems with fractional derivatives,

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q} + {}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta q} = -Q'' - \lambda\frac{\partial f}{\partial_a D_t^\alpha q} - \lambda\frac{\partial f}{\partial_t D_b^\beta q}. \tag{34}$$

6. Example

Suppose that the Lagrangian and non-potential force of the system with fractional derivatives are

$$L = \frac{1}{2}m({}_aD_t^\alpha q)^2 - mgq, \quad Q'' = 0, \tag{35}$$

and the nonholonomic constraint is

$$f = ({}_aD_t^\alpha q)^2 + q = 0, \tag{36}$$

we can easily obtain

$$\frac{\partial L}{\partial q} = -mg, \quad \frac{\partial L}{\partial_a D_t^\alpha q} = m_aD_t^\alpha q,$$
$$\frac{\partial L}{\partial_t D_b^\beta q} = 0, \quad \frac{\partial f}{\partial_a D_t^\alpha q} = 2_aD_t^\alpha q,$$
$$\dot f = 2_aD_t^\alpha q\cdot{}_aD_t^{\alpha+1}q + \dot q = 0. \tag{37}$$

By virtue of Eq. (34), we obtain

$$m_tD_b^\alpha({}_aD_t^\alpha q) - mg = -2\lambda_aD_t^\alpha q, \tag{38}$$

using Eqs. (37) and (38), we have

$$\lambda = \frac{m[\,{}_tD_b^\alpha({}_aD_t^\alpha q) - g]}{\dot q}({}_aD_t^\alpha q). \tag{39}$$

7. Conclusion

In this paper, we have presented the relationships between the isochronous variation and the fractional derivatives, and established the Lagrange equations of general holonomic systems with fractional derivatives. Further, the Lagrange equations of nonholonomic systems with fractional derivatives are derived from the d'Alembert–Lagrange principle with fractional derivatives. Our results represent a significant approach to profound research of the constraint dynamical systems.

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