Completeness of the system of eigenvectors of off-diagonal operator matrices and its applications in elasticity theory*

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This paper deals with off-diagonal operator matrices and their applications in elasticity theory. Two kinds of completeness of the system of eigenvectors are proven, in terms of those of the compositions of two block operators in the off-diagonal operator matrices. Using these results, the double eigenfunction expansion method for solving upper triangular matrix differential systems is proposed. Moreover, we apply the method to the two-dimensional elasticity problem and the problem of bending of rectangular thin plates on elastic foundation.

Keywords: off-diagonal operator matrix, completeness, double eigenfunction expansion method, elasticity theory

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1. Introduction

Partial differential equations (PDEs) have wide applications in science, engineering and technology. Many PDEs, especially many of those resulting from practical applications, can be written as various matrix differential systems $u' = Mu$, where u is a function of a single variable t and taking values in $X \times X$ with X being a Hilbert space, and M is an operator matrix. Note that a PDE usually covers infinitely many matrix differential systems (corresponding to M), depending on different types of transformations.

For four-block operator matrices, there is no general result concerning spectral problems such as essential spectrum, eigenvector expansion theorem, etc. So, it is necessary to introduce some useful tools and develop some effective methods. In fact, off-diagonal operator matrices might be one of the appropriate tools in the above research. For example, a four-block operator matrix in general can be decomposed into the sum of an off-diagonal operator matrix and an operator matrix with certain good properties.[1] In 2000 and 2001, V. Hardt, A. Konstantinov and R. Mennicken (see Refs. [1] and [2]) investigated the spectral properties of off-diagonal operator matrices, and proved the commutation relation $\overline{A(BA-\lambda I)^{-1}B} - \lambda(AB - \lambda I)^{-1} = I$ for $\lambda \in \rho(AB)\backslash\{0\}$. In Ref. [3], the authors studied the super-symmetric operator and their applications in the characterization of self-adjoint extension of a closed symmetric operator and the strong commutativity of two unbounded self-adjoint operators. Recently, the spectrum of off-diagonal Hamiltonian operators was considered in Ref. [4]. For more publications involving the spectral problems of Hamiltonian operators, see Refs. [5] and [6] and the references therein. On the other hand, in Refs. [7]–[9] the authors studied the completeness of certain operator matrices including Hamiltonian operators. In Refs. [10]–[13], the authors considered the analytical solving of problems of applied mechanics based on Hamiltonian operators.

In this paper, we study the eigenvector expansion theorem of off-diagonal operator matrices. To be precise, a sufficient and necessary condition for the orthogonal system of eigenvectors of super-symmetric operators to be complete in the sense of Cauchy princi-

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pal value is obtained (not necessarily complete in general); and for general off-diagonal operator matrices, a sufficient and necessary condition of the symplectic orthogonal system of eigenvectors being complete in the sense of Cauchy principal value is given. Based on these results, the double eigenfunction expansion method is introduced to solve upper triangular matrix differential systems.

2. Theoretical aspects

Note that for the eigenvalue λ of an operator T, the set $E(\lambda;T)$ consists of all associated eigenvectors; where $\mathscr{D}(T)$, $\mathscr{N}(T)$ and $\sigma_{\rm p}(T)$ denote the domain, null space and point spectrum (collection of all eigenvalues) of the operator T, respectively; and the symbol Λ represents a countable index set. We begin with the following definition.

Definition 1 Let Z be a Hilbert space over $\mathbb{C}$, and let the function $(\,\cdot\,,\,\cdot\,):(Z\times Z)\times(Z\times Z)\to\mathbb{C}$ be an inner product on the product space $Z\times Z$. Then, the function $(\,\cdot\,,\,\cdot\,)$ is called a symplectic form induced by the inner product $(\,\cdot\,,\,\cdot\,)$, where

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

with I being the identity operator on Z.

A linear space equipped with a symplectic form is called a symplectic space. In the present paper, the symplectic form on the symplectic space $Z\times Z$ is always defined by the function $(\,\cdot\,,\,\cdot\,)$.

Definition 2 Let X be a Hilbert space. Then, the (symplectic) orthogonal system of vectors $\{u_{k1},u_{k2},\ldots,u_{ks_k}\,|\,k\in\Lambda,s_k\in\mathbb{N}\}$ is said to be complete in the sense of Cauchy principal value in $X\times X$, if for each $\Delta\in X\times X$ there exists a unique constant sequence $\{c_{k1},c_{k2},\ldots,c_{ks_k}\,|\,k\in\Lambda,s_k\in\mathbb{N}\}$ such that

$$\Delta=\sum_{k\in\Lambda}c_{k1}u_{k1}+\cdots+c_{ks_k}u_{ks_k}. \tag{1}$$

Lemma 1 Let X be a Hilbert space and $A:\mathscr{D}(Q)\times\mathscr{D}(P)\to X\times X$,

$$A=\begin{pmatrix}0 & P\\ Q & 0\end{pmatrix}. \tag{2}$$

Then, the following statements hold.

(i) $\lambda\in\sigma_{\rm p}(A)\Leftrightarrow-\lambda\in\sigma_{\rm p}(A)$, and $(f\ \ g)^{\rm T}\in E(\lambda;A)$ if and only if $(-f\ \ g)^{\rm T}\in E(-\lambda;A)$.

(ii) $0\neq\lambda\in\sigma_{\rm p}(A)\Leftrightarrow\lambda^2\in\sigma_{\rm p}(PQ)$, and $(f\ \ g)^{\rm T}\in E(\lambda;A)$ if and only if $f\in E(\lambda^2;PQ)$, where $g=Qf/\lambda$.

(iii) If P is a closed operator with dense domain, and $Q=P^*$ in Eq. (1), then

$$A=\begin{pmatrix}0 & P\\ P^* & 0\end{pmatrix} \tag{3}$$

is called the super-symmetric operator. Further, the operator A is self-adjoint, and $\sigma_{\rm p}(A)\subseteq\mathbb{R}$.

(iv) Assume that the operator A possesses countable simple eigenvalues. By (i), we set

$$\sigma_{\rm p}(A)=\{\lambda_k,\omega_k\in\mathbb{C}\,|\,k\in\Lambda\}$$

with $\omega_k=-\lambda_k$. Then, for $(f_k\ \ g_k)^{\rm T}\in E(\lambda_k;A)$ ($\lambda_k\neq0$, $k\in\Lambda$), we have that $(-f_k\ \ g_k)^{\rm T}\in E(\omega_k;A)$. In particular, if $Q=P^*$, then

$$(f_k,f_j)=\left(\frac{P^*f_k}{\lambda_k},\frac{P^*f_j}{\lambda_j}\right)\begin{cases}\neq0, & k=j,\\ =0, & k\neq j.\end{cases} \tag{4}$$

Moreover, for the eigenvalue 0, we have that

(v) $0\in\sigma_{\rm p}(A)\Leftrightarrow0\in\sigma_{\rm p}(P)\cup\sigma_{\rm p}(Q)$. Thus $(f\ \ g)^{\rm T}\in E(0;A)$ if and only if $f\in E(0;Q)$ and $g\in\mathscr{N}(P)$, or $g\in E(0;P)$ and $f\in\mathscr{N}(Q)$.

(vi) If $0\notin\sigma_{\rm p}(PQ)$, then $0\notin\sigma_{\rm p}(Q)$, and hence $0\in\sigma_{\rm p}(A)$ if and only if $0\in\sigma_{\rm p}(P)$.

(vii) If $0\in\sigma_{\rm p}(PQ)$, then $0\in\sigma_{\rm p}(P)\cup\sigma_{\rm p}(Q)$, i.e., $0\in\sigma_{\rm p}(A)$ by the statement (v).

Proof The statements (iii) and (v) are obvious, and the proofs are omitted here. Let $\lambda\in\sigma_{\rm p}(A)$ and $(f\ \ g)^{\rm T}\in E(\lambda;A)$. Then, $(f\ \ g)^{\rm T}\neq(0\ 0)^{\rm T}$ and

$$\lambda f-Pg=0, \tag{5}$$

$$\lambda g-Qf=0. \tag{6}$$

Since

$$(-\lambda I-A)\begin{pmatrix}-f\\ g\end{pmatrix}=\begin{pmatrix}\lambda f-Pg\\ Qf-\lambda g\end{pmatrix},$$

it follows that $-\lambda\in\sigma_{\rm p}(A)$ and $(-f\ \ g)^{\rm T}\in E(-\lambda;A)$. The converse implications are proved similarly. Thus, the assertion (i) is proved.

Let $0\neq\lambda\in\sigma_{\rm p}(A)$ and $(f\ \ g)^{\rm T}\in E(\lambda;A)$. Then, $f\neq0$ (otherwise $f=g=0$ by Eq. (6), a contradiction occurs). By Eq. (6), we have that $g=Qf/\lambda$ and substituting it into Eq. (5) yields

$$\lambda^2f-PQf=0,$$

which implies that $\lambda^2 \in \sigma_{\rm p}(PQ)$ and $f \in E(\lambda^2; PQ)$. Conversely, let $\lambda^2 \in \sigma_{\rm p}(PQ)$. Then, there exists a vector $f \neq 0$ such that $PQf = \lambda^2 f$. Setting $g = Qf/\lambda$, we have $Pg = \lambda f$. Thus, $\lambda \in \sigma_{\rm p}(A)$. This proves the assertion (ii).

To prove the assertion (iv), it suffices to verify the relation (4). Since A is self-adjoint, we have

$$(\lambda_k - \lambda_j)\left(\begin{pmatrix} f_k \\ g_k \end{pmatrix}, \begin{pmatrix} f_j \\ g_j \end{pmatrix}\right) = 0, \tag{7}$$

$$(\lambda_k - \omega_j)\left(\begin{pmatrix} f_k \\ g_k \end{pmatrix}, \begin{pmatrix} -f_j \\ g_j \end{pmatrix}\right) = 0,\ k, j \in \Lambda. \tag{8}$$

Note that $g_k = (1/\lambda_k)P^* f_k$. Thus, for $\lambda_k \neq \omega_j$, we obtain by relation (8)

$$-(f_k, f_j) + \left(\frac{P^* f_k}{\lambda_k}, \frac{P^* f_j}{\lambda_j}\right) = 0. \tag{9}$$

In particular, taking $j = k$ yields

$$(f_k, f_k) = \left(\frac{P^* f_k}{\lambda_k}, \frac{P^* f_k}{\lambda_k}\right) \neq 0,$$

since $f_k \neq 0$. For $\lambda_k \neq \lambda_j$, i.e., $k \neq j$, we see by relation (7) that

$$(f_k, f_j) + \left(\frac{P^* f_k}{\lambda_k}, \frac{P^* f_j}{\lambda_j}\right) = 0, \tag{10}$$

which together with Eq. (9) implies that

$$(f_k, f_j) = \left(\frac{P^* f_k}{\lambda_k}, \frac{P^* f_j}{\lambda_j}\right) = 0, \quad k \neq j.$$

So, relation (4) is valid.

Now we prove assertion (vi). Suppose that $0 \in \sigma_{\rm p}(Q)$ and $h \in E(0; Q)$. Then, $Qh = 0$, and so $PQh = 0$. This shows $0 \in \sigma_{\rm p}(PQ)$, which is a contradiction to the assumption $0 \notin \sigma_{\rm p}(PQ)$. Further, we see that $0 \in \sigma_{\rm p}(A)$ if and only if $0 \in \sigma_{\rm p}(P)$, since $0 \in \sigma_{\rm p}(A) \Leftrightarrow 0 \in \sigma_{\rm p}(P) \cup \sigma_{\rm p}(Q)$ in assertion (v). The proof of the assertion (vi) is finished. The assertion (vii) can be proved in the same way. □

Next, we give complete descriptions of the completeness of the system of eigenvectors of super-symmetric operators, i.e., we obtain an eigenvector expansion theorem of such operators.

Theorem 1 Let the operator PP^* possesses countable simple non-zero eigenvalues $\{\nu_k\}_{k\in\Lambda}$ with $f_k \in E(\nu_k; PP^*)$ and $y_0 \in E(0; P)$. Then, the orthogonal system of eigenvectors of the super-symmetric operator A defined in Eq. (3) is complete in the sense of Cauchy principal value if and only if the systems $\{f_k\}_{k\in\Lambda}$ and $\{y_0, P^* f_k/\lambda_k\}_{k\in\Lambda}$ of vectors are both complete in X, where $\nu_k = \lambda_k^2$ and $0 \notin \Lambda$.

Proof Since the operator PP^* possesses countable eigenvalues, the super-symmetric operator

$$A = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}$$

does so by Lemma 1 statement (ii). Obviously, $y_0 \in E(0; P)$ implies $0 \in \sigma_{\rm p}(A)$. Write $\lambda_0 = 0$, then $u_{01} = (0, y_0)^{\rm T} \in E(\lambda_0; A)$. By Lemma 1 statement (iv), it follows that the point spectrum and the system of eigenvectors of A can be given by $\{\lambda_0, \lambda_k, \omega_k |\, k \in \Lambda\}$ and $\{u_{01}, u_{k1}, u_{k2} |\, k \in \Lambda\}$, respectively, where

$$u_{01} = \begin{pmatrix} 0 \\ y_0 \end{pmatrix},\ u_{k1} = \begin{pmatrix} f_k \\ \dfrac{P^* f_k}{\lambda_k} \end{pmatrix},\ u_{k2} = \begin{pmatrix} -f_k \\ \dfrac{P^* f_k}{\lambda_k} \end{pmatrix},$$

and $\omega_k = -\lambda_k$ with

$$(\lambda_k)^2 = (\omega_k)^2 = \nu_k, \quad 0 \neq k \in \Lambda.$$

Sufficiency By Lemma 1 statement (iv) and the self-adjointness of A, it can be readily seen that for $k, j \in \Lambda$,

$$\begin{cases} (u_{k1}, u_{j2}) = (u_{01}, u_{j1}) = (u_{01}, u_{j2}) = 0, \quad (u_{01}, u_{01}) = (y_0, y_0), \\ (u_{k1}, u_{j1}) = \begin{cases} 2(f_k, f_k),\ k = j, \\ 0, \qquad\ k \neq j, \end{cases} \quad (u_{k2}, u_{j2}) = \begin{cases} 2(f_k, f_k),\ k = j, \\ 0, \qquad\ k \neq j. \end{cases} \end{cases}$$

For each $\Delta = (x\ \ y)^{\rm T} \in X \times X$, set

$$c_{01} = \frac{(\Delta, u_{01})}{(u_{01}, u_{01})}, \quad c_{k1} = \frac{(\Delta, u_{k1})}{(u_{k1}, u_{k1})}, \quad c_{k2} = \frac{(\Delta, u_{k2})}{(u_{k2}, u_{k2})}, \quad k \in \Lambda. \tag{11}$$

Then, we see that

$$c_{01}u_{01}+\sum_{k\in\Lambda}c_{k1}u_{k1}+c_{k2}u_{k2}=\begin{pmatrix}\displaystyle\sum_{k\in\Lambda}\frac{(x,f_k)}{(f_k,f_k)}f_k\\ \displaystyle\frac{(y,y_0)}{(y_0,y_0)}y_0+\sum_{k\in\Lambda}\frac{\left(y,\dfrac{P^*f_k}{\lambda_k}\right)}{(f_k,f_k)}\frac{P^*f_k}{\lambda_k}\end{pmatrix}. \tag{12}$$

Note that $\{f_k\}_{k\in\Lambda}$ and $\{y_0,P^*f_k/\lambda_k\}_{k\in\Lambda}$ are both complete in X, and the two components of the vector on the right-hand side of Eq. (12) are Fourier series expansions associated with x and y in terms of $\{f_k\}_{k\in\Lambda}$ and $\{y_0,P^*f_k/\lambda_k\}_{k\in\Lambda}$, respectively. Thus,

$$\Delta=c_{01}u_{01}+\sum_{k\in\Lambda}c_{k1}u_{k1}+c_{k2}u_{k2}. \tag{13}$$

On the other hand, we assume that there is another constant sequence $\{\hat{c}_{01},\hat{c}_{k1},\hat{c}_{k2}|\,k\in\Lambda\}$ such that the expansion (13) is valid. Then, we have

$$\begin{aligned}(c_{01}-\hat{c}_{01})u_{01}+\sum_{k\in\Lambda}(c_{k1}-\hat{c}_{k1})u_{k1}\\ +\ (c_{k2}-\hat{c}_{k2})u_{k2}=0,\end{aligned}$$

which clearly implies that $c_{01}=\hat{c}_{01}$, $c_{k1}=\hat{c}_{k1}$ and $c_{k2}=\hat{c}_{k2}$ $(k\in\Lambda)$. Therefore, the system $\{u_{01},u_{k1},u_{k2}|\,k\in\Lambda\}$ of eigenvectors of the super-symmetric operator A is complete in the sense of Cauchy principal value by Definition 2.

Necessity Assume that the system $\{u_{01},u_{k1},u_{k2}|\,k\in\Lambda\}$ of eigenvectors of the operator A is complete in the sense of Cauchy principal value, i.e., for each $\Delta=(x\quad y)^{\mathrm{T}}\in X\times X$, there exists a unique constant sequence $\{c_{01},c_{k1},c_{k2}|\,k\in\Lambda\}$ such that the equality (13) holds. Then, taking the inner product of the relation (13) by u_{01}, u_{k1} and u_{k2} on the right, respectively, we deduce that c_{01}, c_{k1} and c_{k2} $(k\in\Lambda)$ are determined by Eqs. (11). Thus,

$$\begin{aligned}x&=\sum_{k\in\Lambda}(c_{k1}-c_{k2})f_k=\sum_{k\in\Lambda}\frac{(x,f_k)}{(f_k,f_k)}f_k,\\ y&=c_{01}y_0+\sum_{k\in\Lambda}\frac{(c_{k1}+c_{k2})P^*f_k}{\lambda_k}\\ &=\frac{(y,y_0)}{(y_0,y_0)}y_0+\sum_{k\in\Lambda}\frac{\left(y,\dfrac{P^*f_k}{\lambda_k}\right)}{(f_k,f_k)}\frac{P^*f_k}{\lambda_k}.\end{aligned}$$

Therefore, by relation (4) and the arbitrariness of x and y, the systems $\{f_k\}_{k\in\Lambda}$ and $\{y_0,P^*f_k/\lambda_k\}_{k\in\Lambda}$ of vectors are both complete in X. The proof is completed. □

By Lemma 1 statement (vi), we know that $0\notin\sigma_{\mathrm{p}}(P)$ if and only if $0\notin\sigma_{\mathrm{p}}(A)$, when $0\notin\sigma_{\mathrm{p}}(PP^*)$. So, in fact, the case when $0\notin\sigma_{\mathrm{p}}(PP^*)$ and $0\in\sigma_{\mathrm{p}}(A)$ is considered in Theorem 1. We can similarly prove the following corollary, when $0\notin\sigma_{\mathrm{p}}(PP^*)$ and $0\notin\sigma_{\mathrm{p}}(A)$.

Corollary 1 Let the operator PP^* possesses countable simple non-zero eigenvalues $\{\nu_k\}_{k\in\Lambda}$ with $f_k\in E(\nu_k;PP^*)$ and $0\notin\sigma_{\mathrm{p}}(P)$. Then, the orthogonal system of eigenvectors of the super-symmetric operator A defined in Eq. (3) is complete in the sense of Cauchy principal value if and only if the systems $\{f_k\}_{k\in\Lambda}$ and $\{P^*f_k/\lambda_k\}_{k\in\Lambda}$ of vectors are both complete in X, where $\nu_k=\lambda_k^2$.

Remark 1 Combining Lemma 1 statements (v) and (vii) with the previous results, one can prove and obtain the eigenvector expansion theorem when $0\in\sigma_{\mathrm{p}}(PP^*)$ in the same way.

Corollary 2 Let the operator PP^* possesses countable simple eigenvalues $\{\nu_k\}_{k\in\Lambda}$ with $f_k\in E(\nu_k;PP^*)$, where $0\in\Lambda$, $\nu_0=0$ and $P^*f_0=0$. Then,

(I) if $0\notin\sigma_{\mathrm{p}}(P)$, then the orthogonal system $\{u_{02},u_{k1},u_{k2}|\,k\in\Lambda\setminus\{0\}\}$ of eigenvectors of the super-symmetric operator A is complete in the sense of Cauchy principal value if and only if the systems $\{f_k\}_{k\in\Lambda}$ and $\{P^*f_k/\lambda_k\}_{k\in\Lambda\setminus\{0\}}$ of vectors are both complete in X, where $\nu_k=\lambda_k^2$, $u_{02}=(f_0,0)^{\mathrm{T}}$ and $\lambda_0=0$.

(II) If $0\in\sigma_{\mathrm{p}}(P)$ and $y_0\in E(0;P)$, then the orthogonal system $\{u_{k1},u_{k2}|\,k\in\Lambda,0\in\Lambda\}$ of eigenvectors of the super-symmetric operator A is complete in the sense of Cauchy principal value if and only if the systems $\{f_k\}_{k\in\Lambda}$ and $\{y_0,P^*f_k/\lambda_k\}_{k\in\Lambda\setminus\{0\}}$ of vectors are both complete in X, where $\nu_k=\lambda_k^2$, $u_{01}=(0,y_0)^{\mathrm{T}}$, $u_{02}=(f_0,0)^{\mathrm{T}}$ and $\lambda_0=0$.

Remark 2 In Corollary 2, if $0\notin\sigma_{\mathrm{p}}(P^*)$, then $0\in\sigma_{\mathrm{p}}(P)$. Further, $0\in\sigma_{\mathrm{p}}(A)$. Thus, the complete-

ness of eigenvectors of the super-symmetric operator A can be reduced to Theorem 1.

In the following, we consider the completeness of the general off-diagonal operator matrices.

Theorem 2 Let the operator PQ possesses countable simple non-zero eigenvalues $\{\nu_k\}_{k\in\Lambda}$ with $f_k \in E(\nu_k; PQ)$ and $0 \notin \sigma_{\rm p}(P)$. Assume that $(Qf_k, f_k) \neq 0$ and $(Qf_k, f_j) = 0\,, k \neq j \in \Lambda$. Then, the symplectic orthogonal system of eigenvectors of the off-diagonal operator matrix A defined in Eq. (1) is complete in the sense of Cauchy principal value if and only if the system $\{f_k\}_{k\in\Lambda}$ of vectors is a Schauder base in X.

Proof By Lemma 1 statement (vi), $0 \notin \sigma_{\rm p}(PQ)$ and $0 \notin \sigma_{\rm p}(P)$ implies that $0 \notin \sigma_{\rm p}(A)$. Similar to the discussions in Theorem 1, we see that the off-diagonal operator matrix $A = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ also possesses countable non-zero eigenvalues $\{\lambda_k, \omega_k |\, k \in \Lambda\}$ with $\omega_k = -\lambda_k$, and the associated eigenvectors are

$$u_{k1} = \begin{pmatrix} f_k \\ Qf_k/\lambda_k \end{pmatrix}, \quad u_{k2} = \begin{pmatrix} -f_k \\ Qf_k/\lambda_k \end{pmatrix},$$

respectively, where

$$(\lambda_k)^2 = (\omega_k)^2 = \nu_k.$$

By $(Qf_k, f_j) = 0$ $(k \neq j)$, we may prove the following symplectic orthogonal relations

$$(u_{k1}, Ju_{j1}) = \begin{cases} -\left(\dfrac{Qf_k}{\lambda_k}, f_k\right) + \left(f_k, \dfrac{Qf_k}{\lambda_k}\right), & k = j, \\ 0, & k \neq j, \end{cases}$$

$$(u_{k1}, Ju_{j2}) = \begin{cases} \left(\dfrac{Qf_k}{\lambda_k}, f_k\right) + \left(f_k, \dfrac{Qf_k}{\lambda_k}\right), & k = j, \\ 0, & k \neq j, \end{cases}$$

$$(u_{k2}, Ju_{j2}) = \begin{cases} \left(\dfrac{Qf_k}{\lambda_k}, f_k\right) - \left(f_k, \dfrac{Qf_k}{\lambda_k}\right), & k = j, \\ 0, & k \neq j. \end{cases}$$

Now, we prove that the system $\{u_{k1}, u_{k2} |\, k \in \Lambda\}$ of eigenvectors is complete in the sense of Cauchy principal value. For each $\Delta = (x \;\; y)^{\rm T} \in X$, set

$$\begin{aligned} c_{k1} &= \frac{(\Delta, Ju_{k1})(u_{k2}, Ju_{k2}) - (\Delta, Ju_{k2})(u_{k2}, Ju_{k1})}{(u_{k1}, Ju_{k1})(u_{k2}, Ju_{k2}) - (u_{k1}, Ju_{k2})(u_{k2}, Ju_{k1})}, \\ c_{k2} &= \frac{(\Delta, Ju_{k2})(u_{k1}, Ju_{k1}) - (\Delta, Ju_{k1})(u_{k1}, Ju_{k2})}{(u_{k1}, Ju_{k1})(u_{k2}, Ju_{k2}) - (u_{k1}, Ju_{k2})(u_{k2}, Ju_{k1})}, \quad k \in \Lambda. \end{aligned} \tag{14}$$

Then, we have

$$\sum_{k\in\Lambda} c_{k1}u_{k1} + c_{k2}u_{k2} = \sum_{k\in\Lambda} \begin{pmatrix} \dfrac{(x, Qf_k)}{(f_k, Qf_k)} f_k \\ \dfrac{(y, f_k)}{(Qf_k, f_k)} Qf_k \end{pmatrix}. \tag{15}$$

Since $\{f_k\}_{k\in\Lambda}$ is a Schauder base in X, there exists a unique constant sequence $\{d_k(x)\}_{k\in\Lambda}$ such that

$$x = \sum_{k\in\Lambda} d_k(x) f_k$$

for each $x \in X$. Taking the inner product of the above relation by Qf_k on the right, we clearly obtain $d_k(x) = (x, Qf_k)/(f_k, Qf_k)$, which shows that the first component of the right-hand side of Eq. (15) is exactly the expression of x in terms of the base $\{f_k\}_{k\in\Lambda}$, i.e.,

$$x = \sum_{k\in\Lambda} \frac{(x, Qf_k)}{(f_k, Qf_k)} f_k.$$

On the other hand, for each $j \in \Lambda$, we see that

$$\left(y - \sum_{k\in\Lambda} \frac{(y, f_k)}{(Qf_k, f_k)} Qf_k, f_j\right) = (y, f_j) - (y, f_j) = 0,$$

which implies that

$$y = \sum_{k\in\Lambda} \frac{(y, f_k)}{(Qf_k, f_k)} Qf_k$$

since $\{f_j\}_{j\in\Lambda}$ is a Schauder base. Thus,

$$\Delta = \sum_{k\in\Lambda} c_{k1}u_{k1} + c_{k2}u_{k2}. \tag{16}$$

In addition, the uniqueness of such constant sequence $\{c_{k1}, c_{k2}|\, k \in \Lambda\}$ can be proved similar to that of Theorem 1. By Definition 2, the proof of sufficiency is finished.

Conversely, suppose that the system $\{u_{k1}, u_{k2}|\, k \in \Lambda\}$ of eigenvectors of the off-diagonal operator matrix A defined in Eq. (1) is complete in the sense of the Cauchy principal value, i.e., for each $\Delta = (x \;\; y)^{\mathrm{T}} \in X \times X$, there exists a unique constant sequence $\{c_{k1}, c_{k2}|\, k \in \Lambda\}$ such that the equality (16) holds. Then, taking the inner product of the relation (16) by Ju_{k1} and Ju_{k2} on the right, respectively, we see that c_{k1} and c_{k2} $(k \in \Lambda)$ are determined by relation (14). Thus,

$$x = \sum_{k\in\Lambda}(c_{k1} - c_{k2})f_k = \sum_{k\in\Lambda}\frac{(x, Qf_k)}{(f_k, Qf_k)}f_k.$$

Write $d_k(x) = (x, Qf_k)/(f_k, Qf_k)$. Note that

$$|d_k(x)| = \left|\frac{(x, Qf_k)}{(f_k, Qf_k)}\right| \leq \frac{\|Qf_k\|}{|(f_k, Qf_k)|}\|x\|.$$

Therefore, by the arbitrariness of x, the system $\{f_k\}_{k\in\Lambda}$ of vectors is a Schauder base in X. □

Remark 3 Similar to the case of supersymmetric operator, we can obtain the complete characterization of the eigenvector expansion theorem for the off-diagonal operator matrix with $(Qf_k, f_k) \neq 0$ and $(Qf_k, f_j) = 0\,, k \neq j \in \Lambda$.

3. Double eigenfunction expansion method

Consider the upper triangular matrix differential system in $Z \times Z$ with Z being a Hilbert space of functions:

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}, \tag{17}$$

where

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(M) \to Z \times Z \tag{18}$$

is the corresponding upper triangular differential operator matrix. Clearly, the matrix differential system (17) can be equivalently written as the following system of differential equations

$$\frac{\partial p}{\partial y} = Ap + Cq, \tag{19}$$

$$\frac{\partial q}{\partial y} = Bq. \tag{20}$$

Assume that both diagonal block operators A and B of the upper triangular operator matrix M have complete normed (symplectic) orthogonal systems of eigenfunctions. Then, the equation (20) can be solved by the (symplectic) eigenfunction expansion method based on the completeness of the system of eigenfunctions of the operator B; further, substituting the obtained q into Eq. (19), one can similarly solve the unknown function p. Thus, we obtain the general solutions of the problem (17). We call this method of finding analytical solution of matrix differential system double eigenfunction expansion method.

The reason for introducing the double eigenfunction expansion method is that many problems in elasticity and related fields can be described by upper triangular matrix differential systems (for instance, see the examples in Section 4).

4. Applications

In this section, the previous conclusions will be applied to solve the upper triangular matrix differential system, mainly in elasticity theory. Note that the diagonal block operators A and B appearing from upper triangular matrix differential systems in elasticity theory usually possess the off-diagonal form, i.e.

$$A = \begin{pmatrix} 0 & P_1 \\ Q_1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & P_2 \\ Q_2 & 0 \end{pmatrix}$$

with P_i and Q_i $(i = 1, 2)$ being the differential operators in a Hilbert space X of functions. Hereafter, we always let $X = L^2(0, 1)$, and write $V = X \times X \times X \times X$. We start with the following example.

Example 1 Consider that the boundary value problem of harmonic equation is

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0, & 0<x<1, 0<y<b, \\ u(0,y)=u(1,y)=0, & 0\le y\le b, \\ u(x,0)=\varphi_1(x), \quad u(x,b)=\varphi_2(x), & 0\le x\le 1. \end{cases} \tag{21}$$

Let $p=-\partial u/\partial y, q=-\partial u/\partial x$, then the matrix differential system of Eq. (21) is

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} 0 & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}, \tag{22}$$

and the supersymmetric operator matrix is

$$A=\begin{pmatrix} 0 & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{pmatrix}. \tag{23}$$

By the the boundary conditions $u(0,y)=u(1,y)=0$, we have $p(0,y)=p(1,y)=0$, so

$$\mathscr{D}(A)=\left\{\begin{pmatrix} p \\ q \end{pmatrix}\in X\times X \,\middle|\, \begin{array}{l} p,q \text{ are absolutely continuous,} \\ p(0)=p(1)=0, p',q'\in X \end{array}\right\}.$$

Direct calculations show that eigenvalues and associated eigenfunctions of the operator PP^* are $\nu_k=(k\pi)^2$, $f_k=\sin k\pi x$, $k\in\Lambda=\{1,2,\dots\}$. Also, $1\in E(0;P)$, $\{f_k=\sin k\pi x|\, k\in\Lambda\}$ and $\{y_0, P^*f_k/\lambda_k|\, k\in\Lambda\}=\{1,\cos k\pi x|\, k\in\Lambda\}$ are both complete in X, where $\lambda_k=k\pi$. So, by Theorem 1, the system of eigenfunctions of the operator A is complete in the sense of Cauchy principal value in $X\times X$.

Example 2 The two-dimensional elasticity problem based on stress is given by

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}=0, \\ \dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{xy}}{\partial x}=0, \\ \left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0, \end{cases} \tag{24}$$

where the boundary conditions for $x=0$ and $x=1$ are

$$\sigma_x=0, \quad v=0, \quad \text{for } x=0,1. \tag{25}$$

Introducing the state functions $p=\partial\sigma_x/\partial y+\partial\sigma_y/\partial y$, $q=\sigma_x+\sigma_y$, we have

$$\frac{\partial}{\partial y}\begin{pmatrix} \sigma_y \\ \tau_{xy} \\ p \\ q \end{pmatrix}=\begin{pmatrix} 0 & -\dfrac{\partial}{\partial x} & 0 & 0 \\ \dfrac{\partial}{\partial x} & 0 & 0 & -\dfrac{\partial}{\partial x} \\ 0 & 0 & 0 & -\dfrac{\partial^2}{\partial x^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_y \\ \tau_{xy} \\ p \\ q \end{pmatrix}, \tag{26}$$

which is an upper triangular matrix differential system derived from Eq. (24), and the corresponding operator matrix is

$$M=\begin{pmatrix} 0 & -\dfrac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 & -\dfrac{\mathrm{d}}{\mathrm{d}x} \\ 0 & 0 & 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{27}$$

By the stress-displacement relation $\partial v/\partial y=(1/E)(\sigma_y-\nu\sigma_x)$, it can readily be shown from the boundary conditions (25) that

$$\sigma_y=0, \quad q=0, \quad \text{for } x=0,1. \tag{28}$$

By conditions (28), the domain of the operator M is described as

$$\mathscr{D}(M)=\left\{(\sigma_y \;\; \tau_{xy} \;\; p \;\; q)^{\mathrm{T}}\in V \,\middle|\, \begin{array}{c} \sigma_y(0)=\sigma_y(1)=0,\ q(0)=q(1)=0,\ \tau_{xy},\sigma_y,q,q' \text{ are} \\ \text{absolutely continuous, } \tau_{xy}',\sigma_y',q',q''\in X \end{array}\right\}.$$

Note that

$$A = \begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ 1 & 0 \end{pmatrix}, \tag{29}$$

where

$$\mathscr{D}(A) = \left\{ \begin{pmatrix} \sigma_y \\ \tau_{xy} \end{pmatrix} \in X \times X \,\middle|\, \begin{array}{c} \sigma_y(0) = \sigma_y(1) = 0, \sigma_y, \tau_{xy} \text{ is} \\ \text{absolutely continuous}, \tau'_{xy}, {\sigma_y}' \in X \end{array} \right\},$$

$$\mathscr{D}(B) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in X \times X \,\middle|\, \begin{array}{c} q(0) = q(1) = 0, q, q' \text{ is} \\ \text{absolutely continuous}, q', q'' \in X \end{array} \right\}.$$

By Example 1, we see that the system of eigenfunctions of the operator A is complete in the sense of Cauchy principal value. On the other hand, the eigenvalues and associated eigenfunctions of the operator P_2Q_2 are

$$\nu_k = (k\pi)^2, \quad f_k = \sin k\pi x, \quad k \in \Lambda = \{1, 2, \dots\},$$

and $(Q_2 f_k, f_k) = 1/2$, and $(Q_2 f_k, f_j) = 0, k \neq j$. Then, by Theorem 2, the system of eigenfunctions of the operator B is complete in the sense of Cauchy principal value since $\{f_k = \sin k\pi y |\ k \in \Lambda\}$ is an orthogonal base in X, which is clearly a Schauder base. Therefore, we can apply double symplectic eigenfunction expansion method to solve the upper triangular matrix differential system (26).

Example 3 The bending problem of rectangular thin plates on elastic foundation with two opposite edges simply supported is

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \omega + \kappa\omega = f(x, y) \tag{30}$$

with $f(x, y)$ being the laterally load in the region $\{(x, y) \mid 0 \leq x \leq h, 0 \leq y \leq 1\}$, and the boundary conditions for simply supported edges $y = 0$ and $y = 1$ are

$$\omega = 0, \quad \frac{\partial^2 \omega}{\partial y^2} = 0, \quad \text{for } y = 0 \text{ and } y = 1. \tag{31}$$

Let

$$\theta = \frac{\partial \omega}{\partial x}, \ q = D\left(\frac{\partial^3 \omega}{\partial x^3} + \frac{\partial^3 \omega}{\partial x \partial y^2} + \mathrm{i}\sqrt{\kappa/D}\frac{\partial \omega}{\partial x}\right), \quad m = 2D\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \mathrm{i}\sqrt{\kappa/D}\omega\right).$$

Then, the upper triangular matrix differential system derived from Eq. (30) is

$$\frac{\partial}{\partial x}\begin{pmatrix} \omega \\ \theta \\ q \\ m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} - \mathrm{i}\sqrt{\frac{\kappa}{D}} & 0 & 0 & \frac{1}{2D} \\ 0 & 0 & 0 & \frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{\mathrm{i}}{2}\sqrt{\frac{\kappa}{D}} \\ 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} \omega \\ \theta \\ q \\ m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f \\ 0 \end{pmatrix}, \tag{32}$$

where the associated operator matrix is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \mathrm{i}\sqrt{\frac{\kappa}{D}} & 0 & 0 & \frac{1}{2D} \\ 0 & 0 & 0 & \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{\mathrm{i}}{2}\sqrt{\frac{\kappa}{D}} \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{33}$$

According to the boundary conditions (31), we have

$$\mathscr{D}(M)=\left\{(\omega\ \ \theta\ \ q\ \ m)^{\mathrm{T}}\in V\middle|\begin{array}{c}\omega(0)=\omega(1)=0,\ m(0)=m(1)=0,\ \omega',m'\text{ are}\\ \text{absolutely continuous},\ \omega'',m''\in X\end{array}\right\}.$$

Here,

$$A=\begin{pmatrix}0 & 1\\ -\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}-\mathrm{i}\sqrt{\dfrac{\kappa}{D}} & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & \dfrac{1}{2}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}-\dfrac{\mathrm{i}}{2}\sqrt{\dfrac{\kappa}{D}}\\ 2 & 0\end{pmatrix}$$

with

$$\mathscr{D}(A)=\left\{\begin{pmatrix}\omega\\ \theta\end{pmatrix}\in X\times X\middle|\begin{array}{l}\omega'\text{ is absolutely continuous,}\\ \omega(0)=\omega(1)=0,\omega''\in L^2[0,1]\end{array}\right\},$$

$$\mathscr{D}(B)=\left\{\begin{pmatrix}q\\ m\end{pmatrix}\in X\times X\middle|\begin{array}{l}m'\text{ is absolutely continuous,}\\ m(0)=m(1)=0,m''\in L^2[0,1]\end{array}\right\}.$$

Direct calculations show that the eigenvalues and associated eigenfunctions of the operator P_1Q_1 are

$$\nu_j=(j\pi)^2-\mathrm{i}\sqrt{\kappa/D},\quad f_j=\sin j\pi y\quad j\in\Lambda=\{1,2,\dots\}.$$

We see that $(Q_1f_k,f_k)=(1/2)((k\pi)^2-\mathrm{i}\sqrt{\kappa/D})$, and $(Q_1f_k,f_j)=0,k\neq j$. Then, the assumptions of Theorem 2 are satisfied. Therefore, the system of eigenfunctions of the operator A is complete in the sense of Cauchy principal value since $\{f_k=\sin k\pi y|\ k\in\Lambda\}$ is an orthogonal base in X. Similarly, we can check that the system of eigenfunctions of the operator B is complete in the sense of Cauchy principal value, where the eigenvalues and associated eigenfunctions of the operator P_2Q_2 are

$$\nu_j=(j\pi)^2+\mathrm{i}\sqrt{\kappa/D},\quad f_j=\sin j\pi y,\quad j\in\Lambda.$$

Therefore, the upper triangular matrix differential system (33) can be solved by the double symplectic eigenfunction expansion method.

References

[1] Hardt V and Mennicken R 2001 *Operator Theory: Advances and Applications* (Birkhäuser Verlag) **124** 243

[2] Hardt V, Konstantinov A and Mennicken R 2000 *Math. Nachr.* **215** 91

[3] Schôichi Ôta and Schmüdgen K 2003 *Integr. Equ. Oper. Theory* **45** 475

[4] Hou G L and Alatancang 2009 *Commun. Theor. Phys.* **51** 200

[5] Huang J J and Alatancang 2006 *Mongolian Mathematical Journal* **10** 1

[6] Alatancang, Huang J J and Fan X Y 2008 *Sci. Chin.* Ser. A: *Mathematics* **51** 915

[7] Zhong W X 2004 *Duality System in Applied Mechanics and Optimal Control* (Dordrecht: Kluwer Academic Publishers)

[8] Xu X, Chu H and Lim C 2008 *International Journal of Structural Stability and Dynamics* **8** 487

[9] Zhang W 2009 *Arch. Appl. Mech.* **79** 793

[10] Huang J J, Alatancang and Chen A 2008 *Acta Mathematicae Applicatae Sinica* Chinese Series **31** 457

[11] Alatancang and Wu D Y 2009 *Sci. Chin.* Ser. A: *Mathematics* **52** 173

[12] Huang J J, Alatancang and Wang H 2009 *Chin. Phys.* B **18** 3616

[13] Wang H, Alatancang and Huang J J 2009 *Commun. Theor. Phys.* **52** 1087