Exact solution of phantom dark energy model

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We investigate the phantom dark energy model derived from the scalar field with a negative kinetic term. By assuming a particular relation between the time derivative of the phantom field and the Hubble function, an exact solution of the model is constructed. Absence of the 'big rip' singularity is shown explicitly. We then derive special features of phantom dark energy model and show that its predictions are consistent with all astrophysical observations.

Keywords: cosmology, dark energy, phantom field, exact solution

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1. Introduction

Recent observations of large scale structure, the Hubble diagram of type Ia supernovae (SN1a) and the angular power spectrum of the comic microwave background (CMB) etc, strongly indicate that the universe is in a stage of accelerated expansion and spatially flat.[1−3] One immediate conclusion is that the universe is dominated by a form of matter with negative pressure which is widely referred to dark energy now. The simplest model of the dark energy may be Einstein's cosmological constant with equation of state $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. Although there are still some problems the cosmological constant models are consistent with the current observations. A reliable model should explain why the present amount of the dark energy is so small compared with the fundamental scale (fine-tuning problem) and why it is comparable with the critical density today (coincidence problem). To overcome the coincidence problem, the dynamic dark energy models, such as quintessence, are proposed as alternative candidates of dark energy, which is a rolling scalar field φ and may have a potential. The quintessence is proposed to satisfy an equation of state of the form $p_\varphi = \omega_\varphi \rho_\varphi$, with an equation of state $\omega_\varphi > -1$.[3] However, recent observational data indicates that the equation of state parameter ω lies in a narrow strip around $\omega = -1$ and is quite consistent with being below this value.[4] The region where the equation of state is less than –1 is typically referred to as a being due to some form of phantom dark energy.[5] It seems that the contribution of this phantom field leads the universe to reach the critical energy and to accelerate its expansion. The simplest explanation for the phantom dark energy is provided by a scalar field with a negative kinetic energy density. Such a field may be motivated from S-brane constructions in string theory.[6] The phantom energy has some strange properties. For example, the energy density of phantom field increases with time. It also violates the dominant-energy condition.[7] The scale factor a, the Hubble parameter H and its time-derivative $\dot{H}$ reach infinity in finite time and the universe ends in an explosive 'big rip', a final singularity in which the universe is destroyed in a finite proper time by excessive expansion.[8]

It is, however, necessary to construct models of phantom energy in which a future singularity is avoided. In this paper, we study the dynamical behaviour of a simple cosmological model in presence of both nonrelativistic matter and phantom field with a negative kinetic term. By assuming a particular relation between the time derivative of the phantom field and the Hubble function, an exact solution of the model is constructed. This model can lead to classical evolution free of the big rip. We use information on the shift parameter S from the 3-years WMAP (Wilkinson Microwave Anisotropy Probe) data and the quantity A for the baryon acoustic oscillation peak from SDSS (Sloan Digital Sky Survey). Comparing such model with the current cosmological observations shows that its predictions are consistent with all astrophysical observations.

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2. Exact solution of phantom dark energy model

We will consider a phantom field with a potential energy density $V(\varphi)$ evolving in a spatially-flat Friedmann–Robertson–Walker (FRW) universe containing the energy density of the ordinary matter ρ_m (cold dark matter plus baryons matter). We use the system of units in which $8\pi G = c = 1$.

The evolution equations for the flat FRW cosmology read

$$3H^2 = \rho, \tag{1}$$

$$\dot{\rho} = -3H(\rho + p), \tag{2}$$

where overdot denotes time derivative and $H = \dot{a}/a$ is the Hubble parameter. In Eqs. (1) and (2), ρ and p represent the total energy density and pressure of the matter and phantom fields:

$$\rho = \rho_\varphi + \rho_m, \tag{3}$$

$$p = p_\varphi + p_m. \tag{4}$$

In the case of dust, $p_m = 0$ and the energy density of ordinary matter is related with the scale factor through

$$\rho_m = \rho_{m0}(a_0/a)^3, \tag{5}$$

where the lower index '0' indicating present day values of the corresponding quantity. It is worth stressing that this model cannot be used from the very beginning of the universe, but only since decoupling of radiation and dust.

In a phantom model with negative kinetic term, the energy density and pressure of the phantom are given by

$$\rho_\varphi = -\dot{\varphi}^2/2 + V(\varphi), \tag{6}$$

and

$$p_\varphi = -\dot{\varphi}^2/2 - V(\varphi). \tag{7}$$

The evolution equation for this field is

$$\ddot{\varphi} + 3H\dot{\varphi} = V'(\varphi), \tag{8}$$

where prime denotes the derivative with respect to φ.

Equations (1), (2) and (8) are the evolution equations for the flat FRW cosmology. Clearly, according to the Bianchi identity, there are only two independent field equations. From Eqs. (1), (2) and (6), (7) we find

$$2\dot{H} + 3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi). \tag{9}$$

An implicit symmetry of the left hand side (LHS) of Eqs. (8) and (9) is evident under the change $H \to \alpha\dot{\varphi}$, i.e., instead of assuming a particular potential for the phantom field or a particular form of the scale factor,[9] if we assume a linear relation between the time derivative of the phantom field and the Hubble function

$$\dot{\varphi} = -\alpha H, \tag{10}$$

where α is a positive constant, the LHS of Eqs. (8) and (9) coincide up to the factor α.[10] Equation (10) is a basic assumption in this paper. The assumption is valid. Firstly, Eq. (14) obtained from it is a common double-exponential form for the self-interaction potential. Secondly, its predictions are consistent with all astrophysical observations.

Using Eqs. (8) and (9) one can get

$$(2H'' - 3H)\dot{\varphi}^2 + 2(H' - \dot{\varphi})\ddot{\varphi} + 6HH'\dot{\varphi} = 0. \tag{11}$$

Substitute Eq. (10) to Eq. (11), we obtain a differential equation for determining the functional form of the Hubble expansion parameter

$$2H'^2 + 2HH'' + 2\left(\alpha - \frac{3}{\alpha}\right)HH' - 3H^2 = 0. \tag{12}$$

From Eq. (12) the Hubble parameter $H(\varphi)$ is found as

$$H^2(\varphi) = B\,\mathrm{e}^{3\varphi/\alpha} + C\,\mathrm{e}^{-\alpha\varphi}, \tag{13}$$

where B and C are positive integration constants.

Using Eqs. (9), (10), and (13), one can get the form of the potential which is given by

$$V(\varphi) = \left(3 + \frac{\alpha^2}{2}\right)C\,\mathrm{e}^{-\alpha\varphi} - \frac{\alpha^2}{2}B\,\mathrm{e}^{3\varphi/\alpha}. \tag{14}$$

An exponential dependence upon the scalar field is a common functional form for the self-interaction potential.[11] It is to be found in higher dimension theories, supergravity and superstring models.[12]

In Ref. [8] it was shown that the future singularities can in principle appear in a phantom energy universe. When the equation of state of dark energy is less than –1, the density of the dark energy increases with increasing scalar factor, and both the scalar factor and the phantom energy density can become infinite at a finite time, i. e. the universe reaches a Big Rip singularity within a finite time. However, as we show below, for the model under consideration there exists a cosmological solution without any singularity. From Eq. (10) the evolution of the scale factor is

$$a = a_0\,\mathrm{e}^{(\varphi-\varphi_0)/\alpha}. \tag{15}$$

It is shown that the scale factor is finite at a finite time. From Eq. (15), one can get

$$\varphi = \varphi_0 + \alpha \ln(1+z), \tag{16}$$

where the redshift z is defined by $1+z = a_0/a$. Clearly, Eq. (16) tells us that the phantom field value is finite and decreasing monotonously as the age of the universe increases. This ensures that there is no future singularity. Using Eqs. (6), (10) and (13), (14) the expression of the energy density is given by

$$\rho_\varphi = 3C\,\mathrm{e}^{-\alpha\varphi} - \alpha^2 B\,\mathrm{e}^{3\varphi/\alpha}, \tag{17}$$

Obviously, ρ_φ is finite at a finite time as the universe expands. Thus this model can lead to classical evolution free of the big rip.

The pressure of phantom field is

$$p_\varphi = -(3+\alpha^2)C\,\mathrm{e}^{-\alpha\varphi}. \tag{18}$$

As for the case of phantom energy, in which $\omega < -1$, the sum of the pressure and energy density is negative.

The energy density of the phantom field in units of the critical density read

$$\Omega_\varphi = \frac{\rho_\varphi}{3H^2} = 1 - \Omega_m, \tag{19}$$

where

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{(3+\alpha^2)B\,\mathrm{e}^{3\varphi/\alpha}}{3B\,\mathrm{e}^{3\varphi/\alpha} + 3C\,\mathrm{e}^{-\alpha\varphi}}. \tag{20}$$

From Eqs. (19) and (20) the integration constants B and C can be written in the following way:

$$B\,\mathrm{e}^{3\varphi_0/\alpha} = \frac{3H_0^2(1-\Omega_{\varphi 0})}{3+\alpha^2}, \tag{21}$$

$$C\,\mathrm{e}^{-\alpha\varphi_0} = \frac{3\Omega_{\varphi 0}+\alpha^2}{3+\alpha^2}H_0^2. \tag{22}$$

3. Comparing with supernovae data

The present expansion rate of the universe is measured by the Hubble parameter, often parameterised in terms of the undimensional quantity h as $H_0 = 100h\ \mathrm{km\cdot s^{-1}\cdot Mpc^{-1}}$. The value derived from WMAP 3-year observations of CMB anisotropy under the assumption that the universe is flat is in remarkable agreement with this: $H_0 = 73.4^{+2.8}_{-3.8}\ \mathrm{km\cdot s^{-1}\cdot Mpc^{-1}}$.[13] To be more quantitative in all the calculations we choose the following parameters: $\Omega_{m0} = 0.26$,[14] $h = 0.70$ and $\alpha^2 = 0.01$.

The model can provide a reasonable explanation to the transition from matter to dark energy regime and the late-time cosmic acceleration. In terms of redshift z, the deceleration parameter $q = -\ddot{a}/aH^2$ is determined by

$$q = \frac{1}{2} - \frac{3+\alpha^2}{2}\left[\frac{3\Omega_{m0}}{3\Omega_{\varphi 0}+\alpha^2}(1+z)^{3+\alpha^2} + 1\right]^{-1}. \tag{23}$$

From Eq. (23) one can get that the deceleration parameter today is $q_0 = -0.61$. The transition from deceleration to acceleration happens when the deceleration parameter $q = 0$. In this model the universe began accelerating only recently, at redshifts about 0.79. The acceleration of the universe is therefore a very recent phenomenon.

In the absence of compelling theoretical guidance, there is a simple way to parameterise dark energy, by its equation of state ω_φ. The dark energy equation of state parameter is

$$\omega_\varphi = \frac{p_\varphi}{\rho_\varphi} = -1 - \frac{\alpha^2}{3\Omega_\varphi}. \tag{24}$$

Then we find that the dark energy equation of state parameter today is $\omega_{\varphi 0} \approx -1.01$. This value is reasonable and in good agreement with the recent estimation $\omega_{\varphi 0} = -1.01^{+0.37}_{-0.40}$.[15] The state parameter of the model under consideration is always less than –1 and tends to –1 when $t \to \infty$. Such a behaviour of the state parameter leads to an absence of the big rip singularity in our model.[16]

In terms of Eqs. (5) and (17) the radio of the dark energy density to the matter density is

$$\frac{\rho_\varphi}{\rho_m} = \frac{3\Omega_{\varphi 0}+\alpha^2}{(3+\alpha^2)\Omega_{m0}}(1+z)^{-(3+\alpha^2)} - \frac{\alpha^2}{3+\alpha^2}. \tag{25}$$

It is shown that the dark energy density evolves rapidly and is dynamically unimportant even at moderate redshift (e.g. $0.35 \geq \rho_\varphi/\rho_m \geq 0.04$ at $1 \leq z \leq 3$). Thus, the early structure ($z > 1$) is unaffected, but structure will stop growing sooner.[17] It is obviously that the negative pressure of the phantom field can on the one hand, drive the accelerated expansion now and, on the other hand, be only weakly so that the structure formation scenario does not get spoiled in recent past.

The central current question in cosmology research is what theoretical models are consistent with the currently detected form of $H(z)$. The simplest model consistent with the currently detected form of $H(z)$ is the flat cosmological constant model. In this model, it predicts an expansion history of the universe

which is described by a Hubble parameter $H(z)$ as a function of the redshift z given by

$$H(z) = H_0[\Omega_{m0}(1+z)^3 + \Omega_\Lambda]^{1/2}, \tag{26}$$

where Ω_Λ is a constant density due to the cosmological constant Λ and H_0 is the present value of the Hubble parameter. Its disadvantages are lack of theoretical motivation and fine tuning. In terms of redshift, the expression of the Hubble parameter is given by

$$H(z) = H_0 \left[\frac{3\Omega_{m0}}{3+\alpha^2}(1+z)^3 + \left(\frac{3\Omega_{\varphi 0}+\alpha^2}{3+\alpha^2}\right)(1+z)^{-\alpha^2}\right]^{1/2}. \tag{27}$$

It is similar to Eq. (26). So our model is going to be in agreement with the observed value of any quantity depending only on the expansion history.

The fundamental test of the background dynamics of a cosmological model is the SNe magnitude-redshift, based on the luminosity distance.[18] For a flat universe the luminosity distance $d_{\rm L}$ as a function of the redshift is given by

$$d_{\rm L} = (1+z)\int_0^z \frac{{\rm d}z}{H(z)}, \tag{28}$$

The redshift distance $r(z_{\rm dec})$ is

$$r(z_{\rm dec}) = \int_0^{z_{\rm dec}} \frac{{\rm d}z}{H(z)}, \tag{29}$$

where $z_{\rm dec} = 1100$. The currently measured value is $r(z_{\rm dec}) = 13.7\pm0.5$ Gpc.[18] Using Eq. (27) one obtains $r(z_{\rm dec}) = 14.3$ Gpc.

The CMB shift parameter S is given by

$$S = \sqrt{\Omega_{m0}} H_0 \frac{d_{\rm L}(z_r)}{1+z_r}\ , \tag{30}$$

Wang and Mukherjee[19] have used the WMAP 3-year data[13] to derive $S = 1.70 \pm 0.03$ with $z_r = 1090$. Using Eqs. (27), (28) and (30) one obtains $S = 1.73$.

A parameter A that is constrained by the observed scale of the first acoustic peak in baryon oscillations is given by

$$A = \sqrt{\Omega_{m0}}\left(\frac{H_0^3 d_{\rm L}^2(z_1)}{H_1 z_1^2 (1+z_1)^2}\right)^{1/3}, \tag{31}$$

and Eisenstein find[20] that $A = 0.469 \pm 0.017$ where $z_1 = 0.35$. Using Eqs. (27), (28) and (31) one obtains $A = 0.460$.

The age of universe is another observable parameter that can be used to constrain the parameters of dark energy models. The inflationary and radiation-dominated eras is so tiny compared with the matter-dominated period that we could assume that $z \to \infty$ when $t = 0$. The age of the universe is directly related to the Hubble parameter by an integral over the redshift z. The predicted time of expansion from the very early universe to redshift z is

$$t_z = \int \frac{{\rm d}a}{\dot{a}} = \int_z^\infty \frac{{\rm d}z}{(1+z)H(z)}, \tag{32}$$

where $H(z)$ is given in Eq. (27). At the present epoch, the redshift z is zero. Using Eq. (32), the present age of the universe is $t_0 = 14$ Gyr. It is fully consistent with the recent estimation of $13.73^{+0.13}_{-0.17}$ Gyr reported WMAP team.[13] From Eq. (32) we find that the age of the universe when it starts to accelerate is $t_{\rm T} = 7.1$ Gyr, i.e. the Universe had undergone the transition from deceleration phase to acceleration phase 6.9 Gyrs ago. These values are reasonable and in good agreement with the recent estimation of reported WMAP team.

4. Conclusions

We study the dynamical behaviour of a simple cosmological model in the presence of both nonrelativistic matter and phantom field with a negative kinetic term. The exact solution of the Einstein's equations is derived by assuming a particular relation between the time derivative of the phantom field and the Hubble function and then determining the evolution of the expansion scale factor and the potential from it. Absence of the 'big rip' singularity is shown explicitly. In our model, there merely exist two free independent parameters, namely the present Hubble parameter H_0 and Ω_{m0}. Such a solution makes it possible to study the detailed transition from deceleration phase to acceleration phase. The analysis results explicitly show that the universe had undergone the transition from deceleration phase to acceleration phase 6.9 Gyr ago. In terms of H_0 and Ω_{m0}, the present age of the universe is derived as $t_0 = 14$ Gyr and the present values of observational interest are also derived as $\omega_{\varphi 0} = -1.01$ and $q_0 = -0.61$. Such results are in agreement with astrophysical observations.

We use information on the shift parameter S from the 3-years WMAP data and the quantity A for the baryon acoustic oscillation peak from SDSS. Comparing such model with the current cosmological observations shows that its predictions are consistent with all astrophysical observations. Our detailed analysis shows that in the model under consideration the phantom field can be an excellent candidate for dark energy.

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