On the tunneling time of arbitrary continuous potentials and the Hartman effect*

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This paper obtains a generalized tunneling time of one-dimensional potentials via time reversal invariance. It also proposes a simple explanation for the Hartman effect using the useful concept of the scattered subwaves.

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1. Introduction

Tunneling is one of the most important and basic phenomena of quantum mechanics, which is already well understood and the physical basis for many useful devices such as tunneling diode and scanning tunneling microsope. On the contrary, the duration of the tunneling process, i.e., the tunneling time issue is still a very attractive field which generates debates and controversy. Despite the long history of interest in the subject, there is still no general valid theory since numerous theoretical predictions seem to contradict each other, and some approaches to the tunneling time problem even lead to complex times.[1−4] Among all the candidates, the asymptotic phase time, which calculates how long it takes for the peak of transmitted wave packet to emerge from the exit of the barrier, agrees with the experimental measured results in several key experiments on optical, microwave, and acoustic tunneling times.[4−6] However, the fact that the phase time may be less than the free space transit time has led a lot of researchers to question this approach. Using the method of stationary phase, one can easily find that the time delay will saturate with the width of the barrier and the relevant group velocity may be faster than c, also known as the Hartman effect.[4] As a result, some states that the phase time 'is just not a useful concept'[7] and that 'physics has no law about a peak turning into a peak.'[8] Very recently, Winful[4] proposed a new explanation that the barrier is in fact an evanescent mode resonator, and the phase time should be reinterpreted as the cavity lifetime, which is proportional to the stored energy in the barrier. Thus the Hartman effect can be explained by the fact that the phase time saturates with barrier length because of the exponential decay of the energy density with distance.[4,9] As seen from what mentioned above, we emphasis that the phase time concept is well established.

Since that the time-independent Schrödinger equation and the Helmholtz equation are identical in form, tunneling through a one-dimensional potential barrier in solid-state physics is analogous to two-dimensional tunneling (e.g., frustrated total internal reflection) in electromagnetism.[10] As a result, the research on tunneling time is also intimately related to various optical phenomena which are of great current interest, such as the Goos–Hänchen effect and slow light effect.

Recently, we proposed a generalized phase time expression for reflection which follows directly from the Schrödinger equation and can be applied to arbitrary continuous potentials including complex absorbing potentials.[11] It was also pointed out that quantum reflection is just the reflection of subwaves, which originates inherently from the inhomogeneity of the fields.[12] The concept of subwaves has also been used to explain why the supersymmetric Wentzel–Kramers–Brillouio (SWKB) quantization condition is exact for all the shape invariant potentials successfully.[13] In this paper, we present a generalized phase time for tunneling via time reversal invariance,

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which can be applied to arbitrary lossless potentials. Also it is found that the Hartman effect stems from the fact that the scattered subwaves decrease and even disappear when the barrier potentials become smoother as the barrier length approaches infinite. We believe that the connection between the scattered subwaves and the Hartman effect may provide some additional insight into this long-standing paradox.

2. Generalized tunneling time

We start from the generalized expression of the reflection coefficient briefly. Let us consider an arbitrary continuous potential barrier that has its support on the interval $[0, d]$ as shown in Fig. 1 and particles of mass m incident from the left with energy $E = \hbar^2 k^2/2m$. The reflection coefficient at the point $x = 0$ can be written as[11]

$$r(0) = r(d) \exp\left(2\,\mathrm{i}\int_0^d (\kappa + \Delta\kappa)\,\mathrm{d}x\right), \qquad (1)$$

where $\kappa(x) = \sqrt{2m(E - V(x))}/\hbar$ denotes the wavenumber of the main waves, and the wavenumber of the subwaves $\Delta\kappa$ is given by[14–16]

$$\Delta\kappa = \frac{q\kappa'}{q^2 + \kappa^2} = -\frac{q\kappa'}{q'}, \qquad (2)$$

where $q(x)$ denotes the minus logarithmic derivatives of the wave function and can be completely specified via the analytical transfer matrix (ATM) method.

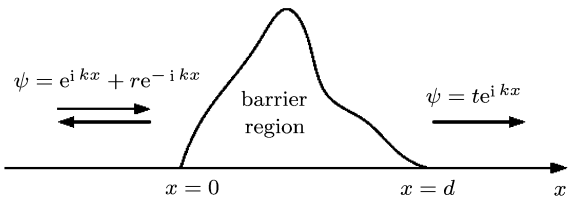


Fig. 1. Potential profile of an arbitrary continuous potential. The plane-wave components of the electron wave function are also indicated.

The Wigner phase time for reflection τ_r depends only on the energy derivative of the phase shift φ_r of the reflection coefficient, i.e., $\tau_r = \hbar(\partial\varphi_r/\partial E)$.[2] Thus we have

$$\tau_r(0) = \tau_r(d) + 2\hbar\,\mathrm{Re}\left(\frac{\partial}{\partial E}\int_0^d (\kappa + \Delta\kappa)\,\mathrm{d}x\right). \quad (3)$$

Strictly speaking, for the continuous potential as shown in Fig. 1, even when the particle has passed the point $x = d$, it may still be reflected by the constant potential. Simply put that if $r(d)$ is equal to zero, then $r(0)$ will also be equal to zero in view of Eq. (1). However the reflection probability beyond point $x = d$ is so small that we usually write the solution of the Schrödinger equation at $x \geq d$ as $\psi(x) = t\mathrm{e}^{\mathrm{i}\,kx}$. Substituting the asymptotic form of the wave function to the reflection time expression Eq. (3), one will have $\tau_r(d) \approx \tau_r(\infty) = 0$. Naturally, we can neglect the term $\tau_r(d)$ in Eq. (3)

$$\tau_r(0) \approx 2\hbar\,\mathrm{Re}\left(\frac{\partial}{\partial E}\int_0^d (\kappa + \Delta\kappa)\,\mathrm{d}x\right). \qquad (4)$$

The reflection coefficient $r(x)$ is always related to $q(x)$ via

$$r(x) = \frac{\kappa(x) + \mathrm{i}\,q(x)}{\kappa(x) - \mathrm{i}\,q(x)}$$

at arbitrary points. However, no such relation exists for the transmission coefficient $t(x)$: it is impossible to write $t(x)$ as a simple function of $q(x)$ and $\kappa(x)$, which makes it difficult to derive the corresponding generalized tunneling time directly from the Schrödinger equation. Fortunately, for arbitrary non-absorbing potentials, time reversal invariance connects the tunneling time with reflection time via a simple relation.

It has been demonstrated that time reversal invariance follows from considerations of energy balance, thus this law holds for all the lossless potentials.[6] Suppose that at time $t = 0$, a wave packet were incident from the left side of the barrier, then at time $t = \tau_r$, we will obtain the reflected wave packet; at time $t = \tau_t$, we will obtain the transmitted wave packet at the right side of the barrier. Now we timely reverse the process as shown in Fig. 2, there is no doubt that the final state will only involve a wave packet exiting from left at $t = 0$.

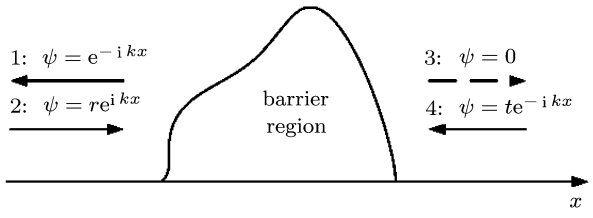


Fig. 2. A lossless barrier potential and the four wave packets concerned in time reversal process: packet (2), the reflected wave packet being time reversed and incident at time $t = -\tau_r$; packet (4), the transmitted wave packet being time reversed at time $t = -\tau_t$; packet (1), the wave packet exit from the left at time $t = 0$ due to time reversal invariance; packet (3), no wave packet ever exit from the right side of the barrier.

Let us consider this reversal process in detail Part of the wave packet (4) incident from right is transmitted through the barrier, while part of the wave packet (2) incident from left is reflected and these two leftward going packets interfere constructively to form wave packet (1). So the transmitted part of wave packet (4) must reach the left side of barrier exactly at time $t = 0$, this implies that tunneling time does not depend on the incident direction of the wave packet, i.e.

$$\tau_t^{\mathrm{L}} = \tau_t^{\mathrm{R}}, \tag{5}$$

where the superscript L(R) denotes the incident direction of the particle.

Similarly, the reflected part of wave packet (4) and the transmitted part of wave packet (2) must interfere destructively, as no wave packet (3) ever exits the barrier region from the right side. Again, this requires that these two parts reach the right side of the barrier simultaneously. So we have

$$-\tau_r^{\mathrm{L}} + \tau_t^{\mathrm{L}} = -\tau_t^{\mathrm{L}} + \tau_r^{\mathrm{R}}, \tag{6}$$

or

$$\tau_t^{\mathrm{L}} = 1/2\left(\tau_r^{\mathrm{L}} + \tau_r^{\mathrm{R}}\right), \tag{7}$$

where the reflection time can easily be obtained via Eq. (4). It is worthwhile to note that Eq. (7) holds for arbitrary continuous barrier potentials. As an illustration of the proposed tunneling time, let us consider a simple asymmetrical barrier potential given as

$$V(x) = \sin(x)\cos(x) + 2\cos(x) + \sin(x) + 2, \quad -\pi < x < \pi. \tag{8}$$

In this illustration atomic units are used, i.e., we set $\hbar = 2m = 1$.

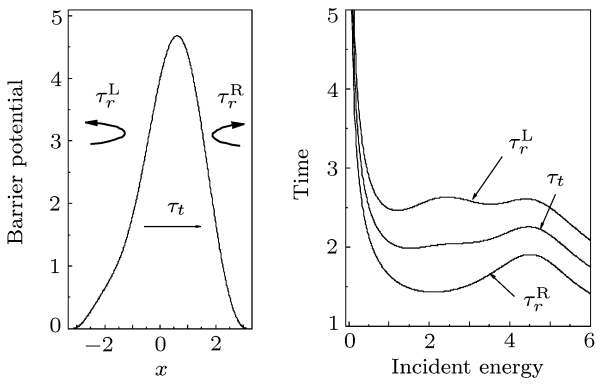


Fig. 3. (a) Potential profile of the illustrated asymmetrical barrier potential Eq. (8). (b) Left (right) incident reflection time and tunneling time as function of incident electron energy.

As can be seen from Fig. 3, the tunneling time approaches infinite for incident energy at the base of the barrier and decreases rapidly for above-barrier incident energy. To the best of our knowledge, this is the first way to calculate the tunneling time of arbitrary asymmetrical continuous barriers.

For a second example, let us consider the tunneling time of an asymmetrical multiplebarrier structure given as

$$V(x) = \left|\sin(x) + \frac{1}{1+x^2}\right|, \quad -3\pi < x < 3\pi. \tag{9}$$

It is interesting to note that the fluctuating behaviour of the tunneling time is rather similar to that of the tunneling probability. This can be easily understood as follows. As a consequence of the Kramers–Kronig relations, the sharp transition usually induces a strong change in its phase, which is responsible for large changes in the relative delay of an electron as it travels through the potential. As can be seen from Fig. 4, longer tunneling time corresponds to narrower bandwidth of transmission peak. This is exactly the same mechanism used in optics to generate slow light in various media and structures.[17]

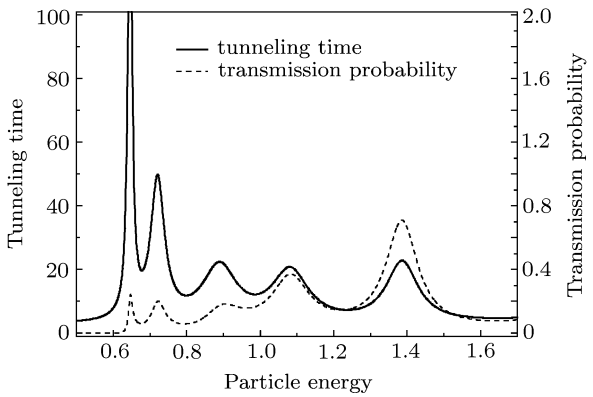


Fig. 4. The tunneling time (solid line) follows the fluctuating behaviour of transmission probability (dash line) as a function of incident electron energy for the asymmetrical multiple-barrier structure.

As mentioned above, this proposed tunneling time may be useful in lots of research areas due to the well-known analogue between the Schrödinger equation and the Helmholtz equation.

3. The Hartman effect

The Hartman effect, which predicts that the tunneling time would be independent of the barrier length for thick enough barriers, ultimately leads to superluminality.[4]

When we extend the proposed tunneling time expression to this issue, its connection with the scattered subwaves immediately becomes apparent. Since in classically forbidden region $V(x) > E$, $\kappa(x)$ is always an imagine quantity, thus Eq. (4) can be further simplified as

$$\tau_r = 2\hbar \,\mathrm{Re}\left(\frac{\partial}{\partial E}\int \Delta\kappa \mathrm{d}x\right).$$

This means that the main waves have no effect on the reflection time and tunneling time in the barrier region. As a result inside the barrier region, the tunneling time is completely determined by the subwaves.

As shown in Fig. 5, when the barrier length becomes large and eventually approaches infinite, then the length of an arbitrary chosen section in the barrier region will also approach infinite and the wave function in this section can be approximated as

$$\begin{aligned}\psi(x) &= c_1 \exp(-\alpha x) + c_2 \exp(\alpha x) \\ &\approx c_1 \exp(-\alpha x),\end{aligned} \tag{10}$$

where c_1, c_2 are constants and $\alpha(x) = \sqrt{2m(V(x)-E)}/\hbar = \mathrm{i}\,\kappa(x)$. After substituting Eq. (10) into the definition of $q(x)$, we obtain $q(x) \approx -\mathrm{i}\,\kappa(x)$.

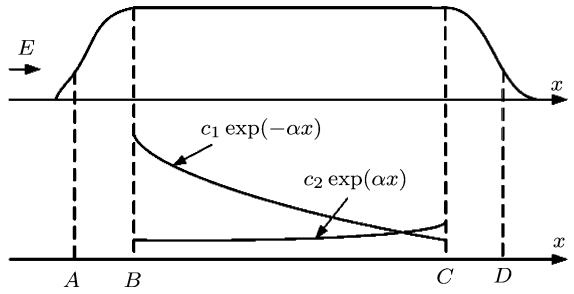


Fig. 5. Interval $[A \;\; D]$ denotes the classically forbidden region, while interval $[B \;\; C]$ is an arbitrary chosen section in the barrier region, which tends to constant as the barrier length approaches infinite.

Thus we obtain the wavenumber of subwaves $\Delta\kappa = -q\kappa'/q' \approx -\kappa(x)$, which implies that the real part of $\Delta\kappa$ at arbitrary point approaches zero for thick enough barrier, so does its energy derivative And numerical simulation confirms that this integral over the whole classically forbidden region approaches a finite quantity with barrier length.

It should be noted that $\Delta\kappa = -\kappa$ can only exist when the field is constant, i.e., when the wave function takes the form $\psi = \exp(\pm\,\mathrm{i}\,\kappa x)$ or $\psi = \exp(\pm\kappa x)$. So we can draw a conclusion that the situation $\Delta\kappa$ approaches $-\kappa$ in fact denotes the case that the scattered subwaves are disappearing.

Clearly from what is mentioned above, the underlying physics is extremely simple: for arbitrary continuous potentials, the tunneling time is only due to phase accumulated by the scattered subwaves; and the Hartman effect stems from the decrease of scattered subwaves as the barrier becomes broader. So the tunneling time saturates instead of approaching infinite large. This is a little more complicated than the simple rectangular potential barrier case which has been already studied extensively. In the barrier region of a rectangular barrier there are no scattered subwaves; the tunneling time is due to phase accumulated at the barrier entrance and exit, which is independent of barrier length.[4]

4. Conclusion

In conclusion, we proposed a generalized tunneling time of arbitrary one-dimensional continuous potentials via time reversal invariance and explored the relation between the Hartman effect and the scattered subwaves.

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