Stochastic systems with cross-correlated Gaussian white noises*

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This paper theoretically investigates three stochastic systems with cross-correlation Gaussian white noises. Both steady state properties of the stochastic nonlinear systems and the nonequilibrium transitions induced by the cross-correlated noises are studied. The stationary solutions of the Fokker–Planck equation for three specific examples are analysed. It is shown explicitly that the cross-correlation of white noises can induce nonequilibrium transitions.

Keywords: phase transitions, fluctuation phenomenon, stochastic processes
PACC: 6400, 0540, 0500

1. Introduction

Recently the interactions of a nonlinear system with a fluctuating environment have attracted extensive studies and noise-induced transition has got wide applications in the field of physics, chemistry, and biology.[1−16] The fluctuating environment is often regarded as a source of external noise with known statistical characteristics. The external noises are considered to be white and Gaussian for the reasons that white noise provides an idealisation of real noise with a small correlation time for a large variety of applications, causes the evolution of the nonlinear system with white noise, and simplifies considerably the analytical study of the statistical behaviour of the system.

In the white noise cases, spatially homogeneous systems with a variable are described by the Langevin equation which represents a valuable tool for modeling a great variety of phenomena and processes involving stochastic resonance,[9] resonant activation,[17] and directed transport.[18] It is commonly known that in the cross-correlated Gaussian white noises describing the fluctuating environment of the system plays an important role in some applications. Denisov and Vitrenko[19] studied the role that the cross-correlation of noises plays in the phenomenon of nonlinear systems, starting from a Langevin equation with two multiplicative cross-correlated white noises. Cao and Wu[20,21] examine the mechanisms of the cross-correlation of white noise in a single-mode laser. Fuliński and Góra[22] investigated the transport of a quantum particle driven by the cross-correlation. Tessone Claudio *et al.*[23] analysed the effects caused by the simultaneous presence of correlated additive and multiplicative noises for stochastic resonance. Zailin and Schimansky-Geier[24] studied the phase transition caused by the cross-correlated Gaussian white noises. Doering and Gadoua[17] considered the problem of thermally activated potential barrier crossing in the presence of thecross-correlated noises.

In these studies only the kinetic form of the stochastic nonlinear system driven by the cross-correlation of noises is shown to be the reason for the transition and much less attention has been paid to the steady state properties. In this paper, we not only study the steady state properties of the stochastic nonlinear systems driven by the cross-correlation of noises, but also nonequilibrium transition induced by the same ones.

This paper is organised as follows. In Section 2, the stationary solution of the Fokker–Planck equation is given which corresponds to the Langevin equation with dimensionless state. In Section 3, both the nonequilibrium transition and the steady state properties of the stochastic nonlinear systems driven by the cross-correlation of noises are obtained through three examples. The results are summarised in Section 4.

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2. The possibility of nonequilibrium transitions

In the white noise approximation, spatially homogeneous systems with a single-state variable $x(t)$ can be described by the Langevin equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)) + G(x(t))\Gamma(t), \tag{1}$$

where $F(x)$ and $G(x)$ are deterministic functions which can depend explicitly on t as well, and $\Gamma(t)$ is Gaussian white noise. In general terms, $G(x)$ is not constant and the effect of the noise depends on the state of the system. As is well known, the systems driven by two white noises are often investigated. Usually, a multiplicative and an additive noise are considered. We are interested in multiplicative Gaussian white noise. When the fluctuation of the system is modeled by two multiplicative Gaussian white noises, we can rewrite Eq. (1), as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t)) + \sum_{i=1}^{2} g_i(x(t))\,\gamma_i(t). \tag{2}$$

To obtain the broadest variety of application, we assume that each Gaussian white noise $\gamma_i(t)$ is characterised by its own parameter λ_i $(0 \leq \lambda_i \leq 1)$. The $\gamma_1(t)$ and $\gamma_2(t)$ are Gaussian white noise with the following statistical properties:

$$\langle \gamma_1(t) \rangle = \langle \gamma_2(t) \rangle = 0, \tag{3}$$

$$\langle \gamma_1(t)\,\gamma_1(t') \rangle = 2\,\Delta_1\,\delta(t-t'), \tag{4}$$

$$\langle \gamma_2(t)\,\gamma_{21}(t') \rangle = 2\,\Delta_2\,\delta(t-t'), \tag{5}$$

$$\langle \gamma_1\,\gamma_2(t') \rangle = \langle \gamma_2(t)\,\gamma_1(t') \rangle = 2\,\Delta_{12}\,\delta(t-t') = 2\,\Delta_{21}\,\delta(t-t'), \tag{6}$$

where $\Delta_1(\geq 0)$ and $\Delta_2(\geq 0)$ are the intensities of the noises $\gamma_1(t)$ and $\gamma_2(t)$, respectively, $\Delta_{12} = \Delta_{21} \equiv r\sqrt{\Delta_1\,\Delta_2}$, r is the coefficient of correlation between $\gamma_1(t)$ and $\gamma_2(t)$, $\delta(t)$ is the Dirac δ function.

We can obtain the Fokker–Planck equation for the evolution of the steady probability distribution function (SPDF) corresponding to Eq. (2) for the system driven by two multiplicative noises. The equation is:[17]

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}[f(x)+h(x)] + \frac{\partial^2}{\partial\,x^2}d(x)p(x,t), \tag{7}$$

where

$$h(x) = 2\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_i\,\Delta_{ij\,g_i'}(x)\,g_j(x), \tag{8}$$

$$d(x) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Delta_{ij}\,g_i(x)\,g_j(x). \tag{9}$$

Equation (2) is stochastically equivalent to Eq. (7), which is also valid in the general case when the functions $f(x)$ and $g_i(x)$ depend explicitly on t. In addition, $h(x) = 2\lambda\Delta\,g'(x)g(x)$ and $2d(x) = 2\Delta\,g^2(x)$ are the noise-induced drift and the diffusion coefficient of the Fokker–Planck equation. We can capture the conclusion that $h(x) \propto d'(x)$ if we assume $d(x) > 0$. The stationary solution of the Fokker–Planck equation (7) is obtained as follows:[17]

$$p_{st}(x) = \frac{N}{d(x)}\exp\left[\int^{x}\frac{f(x')+h(x')}{d(x')}\mathrm{d}x'\right], \tag{10}$$

where N is normalisation constant. The extrema of $p_{\mathrm{st}}(x)$ obey a general equation:

$$f(x) + h(x) - d'(x) = 0. \tag{11}$$

The transitions that are induced by the cross-correlation of the noises are seen from Eq. (11). In other words, Eq. (11) has the same number of roots as equation $f(x) = 0$ for $r = 0$ and that number changes for $r = r_{\mathrm{cr}}$

Using the SPDF we can obtain the moments of the state variable x:

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n\,p_{\mathrm{st}}(x)\mathrm{d}x. \tag{12}$$

The mean and variance of the state variable are given by the numerical integrations of Eq. (12). The mean of the state variable is

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,p_{\mathrm{st}}(x)\mathrm{d}x. \tag{13}$$

The variance of the state variable is

$$\delta x^2 = \langle x^2 \rangle - \langle x \rangle^2. \tag{14}$$

In this paper, simply, we take $g_2(x) = 1$ in three specific examples of Eq. (2), then the stationary solutions of Eq. (10) are obtained and we can study the possibility of nonequilibrium transitions induced by the cross-correlation of two Gaussian white noises.

3. Specific example and results

3.1. First example

In this example, we consider the system with a linear restoring force $f(x) = -kx(k > 0)$ and

$$g_1(x) = l\frac{x^2}{1+x^2}(l > 0). \tag{15}$$

The stationary probability distribution function of the system can be obtained from Eq. (10) as

$$p_{\rm st}(x) = Z\frac{\left[1-2(1+rv)(1+x^2)+(1+2rv+v^2)\left(1+x^2\right)^2\right]^C}{\left[\left(\dfrac{x^2}{1+x^2}\right)^2+2rv\dfrac{x^2}{1+x^2}+v^2\right]^{1-\lambda_1}} \times \exp\left[-m\frac{1+x^2}{1+2rv+v^2}+Dm\mathrm{arctg}\frac{\dfrac{1}{1+x^2}-(1+rv)}{v\sqrt{1-r^2}}\right], \quad (0\le r\langle 1), \tag{16}$$

where

$$C=\frac{-m(1+rv)}{\left(1+2rv+v^2\right)^2}, \quad m=2(1-\lambda_1)/\eta, \tag{17}$$

$$\eta=\frac{4(1-\lambda_1)\,\Delta_1\,l^2}{k}, \quad v=\frac{1}{l}\sqrt{\frac{\Delta_2}{\Delta_1}}, \tag{18}$$

$$D=\frac{\dfrac{2\left(1+rv\right)^2}{\left(1+2rv+v^2\right)^2}-\dfrac{1}{1+2rv+v^2}}{v\sqrt{1-r^2}}. \tag{19}$$

From Eq. (16) we can see that the stationary probability distribution function is an even function. So the mean of state variable of x is zero, and we mainly investigate SPDF and variance of state variable of the system in first example, equally in second example and third example.

As is explicitly described in Fig. 1, the curve of the SPDF is unimodal as the change of state variable x for $r > r_{\rm cr}$, and is bimodal for $r < r_{\rm cr}$ This means that two cross-correlated white noises can induce nonequilibrium transition, while each noise does not. We find that a noise induced transition occurs at $r = r_{\rm cr}$, if the number of local maxima of the SPDF $p_{\rm st}(x)$ is different for $r > r_{\rm cr}$ and $r < r_{\rm cr}$. Therefore, a unimodal-bimodal transition occurs at $r = r_{\rm cr}$.

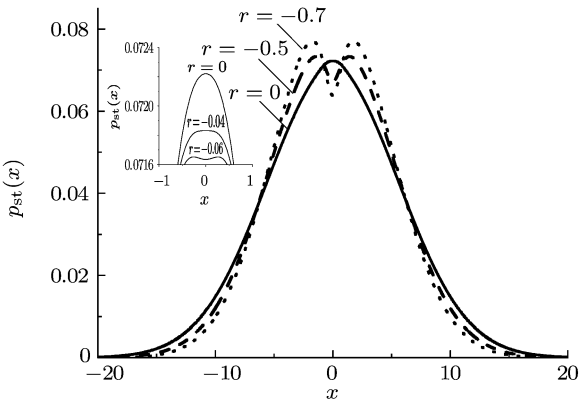


Fig. 1. The SPDF of first example for $r = 0$, $r = -0.5$ and $r = -0.7$ ($r_{\rm cr} = -0.04$). $\eta = v = 5.0$, $\lambda_1 = 0.0$, respectively.

In Fig. 2, we study the relation between variance of state variable and state variable v. The changing trend of variance can response the growth law of state variable x, and its increase means that the growth law of state variable is inhibited, and is promoted when the variance of state variable decreases. The curve steadily rapidly increases as the values of v changes. It shows that the growth law of state variable x is inhibited as v increases.

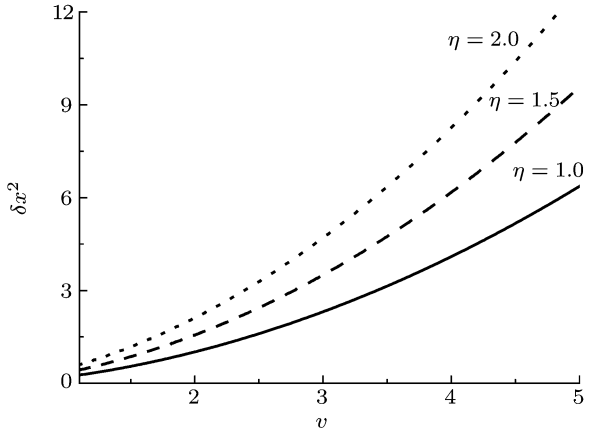


Fig. 2. The variance of the state variable δx^2 as a function of v with $\lambda_1 = 0$, $r = 0$ for first example, for $\eta = 1.0$, $\eta = 1.5$, $\eta = 2.0$, respectively.

In Fig. 3, we can find that the curve increases slowly in little values of λ_1 and fast in large values. That means the growth law of state variable x is greatly inhibited in large values of λ_1, but weakly in little λ_1. Figures 2 and 3 show that the parameters v and λ_1 have different effects on variance of state variable.

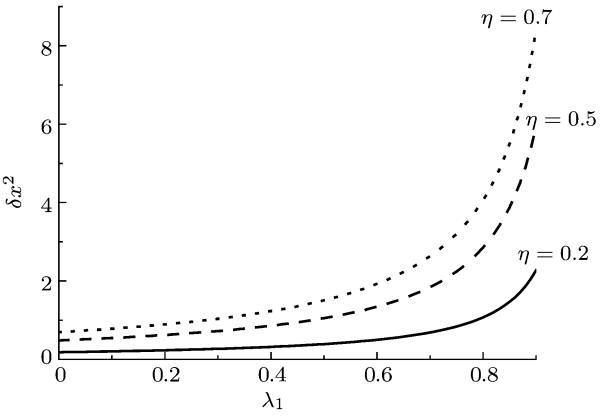


Fig. 3. The variance of the state variable δx^2 as a function of λ_1 with $v = 3.0$, $r = 0$ for first example, for $\eta = 0.2$, $\eta = 0.5$, $\eta = 0.7$, respectively.

3.2. Second example

Next, we consider a system with a cubic restoring force $f(x) = -2k\,x^3 (k > 0)$, and for convenience we take

$$g_1(x) = l\frac{x^4}{1+x^4} (l > 0). \tag{20}$$

And the same as first example, we can obtain the SPDF from Eq. (10) as

$$p_{\rm st}(x) = Z\frac{\left[1-2(1+rv)(1+x^4)+(1+2rv+v^2)\left(1+x^4\right)^2\right]^C}{\left[\left(\dfrac{x^4}{1+x^4}\right)^2+2rv\dfrac{x^4}{1+x^4}+v^2\right]^{1-\lambda_1}} \times \exp\left[-m\frac{1+x^4}{1+2rv+v^2}+Dm\,\mathrm{arctg}\frac{\dfrac{1}{1+x^4}-(1+rv)}{v\sqrt{1-r^2}}\right], \quad (0 \le r < 1). \tag{21}$$

Equations (16) and (21) have the same parameter values. In this example, we also find noise-induced nonequilibrium phase transition in Fig. 4 when the value of r changes from $r > r_{\rm cr}$ to $r < r_{\rm cr}$. In other words, at $r = r_{\rm cr}$, a unimodal-bimodal transition occurs.

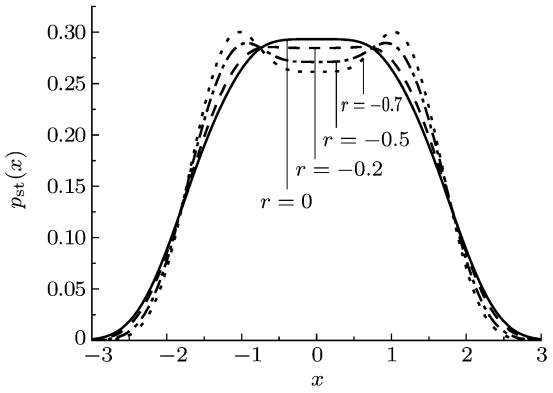


Fig. 4. The SPDF of second example for $r = 0$, $r = -0.2$, $r = -0.5$, $r = -0.7$ ($r_{\rm cr} \approx -0.11$), $\eta = v = 3.0$, $\lambda_1 = 0$, respectively.

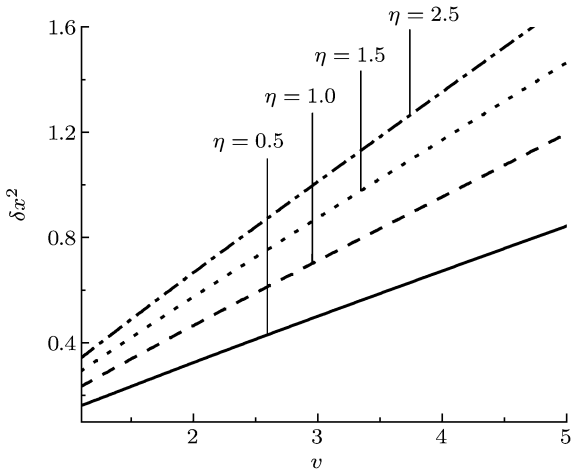


Fig. 5. The variance of the state variable δx^2 as a function of v with $\lambda_1 = 0$, $r = 0$ for second example, for $\eta = 0.5$, $\eta = 1.0$, $\eta = 1.5$, $\eta = 2.5$, respectively.

Figure 5 shows the relation between variance of state variable and the parameter v. The values of variance change quasi-linearly as v increases. Figure 6 shows the variances of state variable x increase in a small range as λ_1 increases, but rapidly in larger values of λ_1.

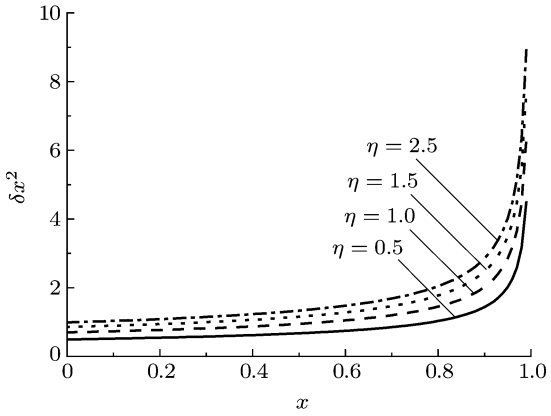


Fig. 6. The variance of the state variable δx^2 as a function of λ_1 with $v = 3.0$, $r = -0.5$ for second example, for $\eta = 0.5$, $\eta = 1.0$, $\eta = 1.5$, $\eta = 2.0$, respectively.

3.3. Third example

We consider a system with a linear restoring force and with $g_1(x)$ given by Eq. (20), similarly, we can obtain the stationary solution of Fokker–Planck equation as

$$p_{\rm st}(x) = \frac{Z}{[d(x)]^{1-\lambda_1}}\exp\left\{-m\left[A\,U_1(x) + \frac{2B}{\sqrt{v-1}}\,U_2(x) + \frac{C}{v}\,U_3(x) + \frac{2C\sqrt{v}}{(v-1)^{3/2}}\,U_4(x)\right]\right\}, \quad (r = -1). \tag{22}$$

Where

$$d(x) = \left(\frac{x^2}{1+x^2} - v\right)^2, \tag{23}$$

$$U_1(x) = x^2\left\{1 + \mathrm{tg}\left[\frac{\pi}{4} + \frac{1}{2}\mathrm{arctg}(x^2)\right]\mathrm{tg}\left[\frac{\pi}{4} - \frac{1}{2}\mathrm{arctg}(x^2)\right]\right\}, \tag{24}$$

$$U_2(x) = \mathrm{arctg}\frac{1+\sqrt{v}\mathrm{tg}\left[\frac{1}{2}\mathrm{arctg}(x^2)\right]}{\sqrt{v-1}} + \mathrm{arctg}\frac{-1+\sqrt{v}\mathrm{tg}\left[\frac{1}{2}\mathrm{arctg}(x^2)\right]}{\sqrt{v-1}}, \tag{25}$$

$$U_3(x) = -\frac{\sin[2\mathrm{arctg}(x^2)]}{(v-1)\left\{1 - \frac{\sin^2[\mathrm{arctg}(x^2)]}{v}\right\}}, \tag{26}$$

$$U_4(x) = \mathrm{arctg}\frac{1+\sqrt{v}\mathrm{tg}\left[\frac{1}{2}\mathrm{arctg}(x^2)\right]}{\sqrt{v-1}} + \mathrm{arctg}\frac{-1+\sqrt{v}\mathrm{tg}\left[\frac{1}{2}\mathrm{arctg}(x^2)\right]}{\sqrt{v-1}}, \tag{27}$$

$$A = \frac{1}{2\,(v-1)^2}; \quad B = \frac{1-3v}{4\,\nu^{3/2}\,(v-1)^2}; \quad C = \frac{1}{4v(1-v)}. \tag{28}$$

When we change the value of the correlation coefficient for $r = 1.0$, the stationary solution of Eq. (10) is obtained as

$$p_{\mathrm{st}}(x) = \frac{Z}{[d\,(x)']^{1-\lambda_1}}\exp\{-m[A'\,U_1\,(x)' + B'\,U_2\,(x)' + C'\,U_3\,(x)']\}, \tag{29}$$

where

$$d(x) = \left(\frac{x^2}{1+x^2} + v\right)^2, \tag{30}$$

$$U_1(x)' = \mathrm{tg}\left(\frac{\pi}{4} + \frac{1}{2}\mathrm{arctg}(x^2)\right) - \mathrm{tg}\left(\frac{\pi}{4} - \frac{1}{2}\mathrm{arctg}(x^2)\right), \tag{31}$$

$$U_2(x)' = \frac{\sin(2\mathrm{arctg}(x^2))}{(v + \sin^2(\mathrm{arctg}(x^2)))}, \tag{32}$$

$$U_3(x)' = \mathrm{arctg}\left(\sqrt{1+\frac{1}{v}}\,x^2\right), \tag{33}$$

$$A' = \frac{1}{2\,(1+v)^2}, \quad B' = \frac{1}{4v\,(1+v)^2}, \quad C' = \frac{1}{\sqrt{v\,(1+v)^5}} + \frac{1+2v}{2\sqrt{v^3\,(1+v)^5}}, \quad (r = 1.0). \tag{34}$$

Figure 7 shows that $p_{\mathrm{st}}(x)$ has a unimodal-trimodal transition when the correlation coefficient r between noises takes different values, for $r = 1$ is unimodal and for $r = -1$ is trimodal.

The relations between variance of state variable x and the parameter v are studied in Fig. 8. Every curve increases rapidly as v increases. In Fig. 9, it is explicitly shown that the values of δx^2 are almost unchanged as λ_1 increases in small values of λ_1, but rapidly increase in large values of λ_1. Figures 8 and 9 show that parameters v and λ_1 have different effects on variance of state variable x.

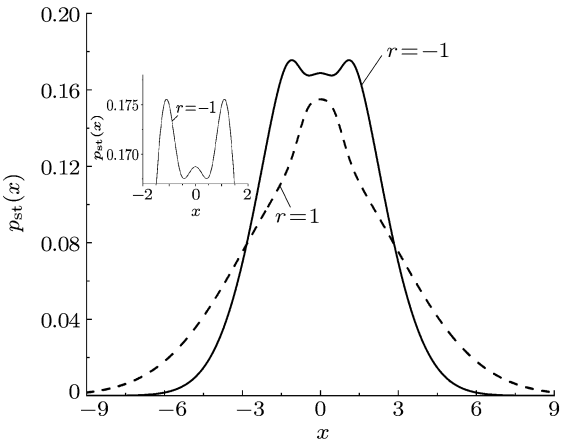


Fig. 7. The SPDF of third example for $r = -1$ and $r = 1$ ($r_{\mathrm{cr}} \approx -0.11$), $\eta = 0.7$, $v = 4.2$, $\lambda_1 = 0.5$, respectively.

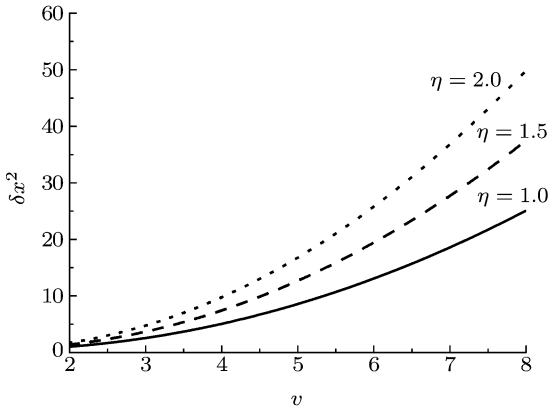


Fig. 8. The variance of the state variable δx^2 as a function of v with $\lambda_1 = 0.5$, $r = -1$ for third example, for $\eta = 1.0$, $\eta = 1.5$, $\eta = 2.0$, respectively.

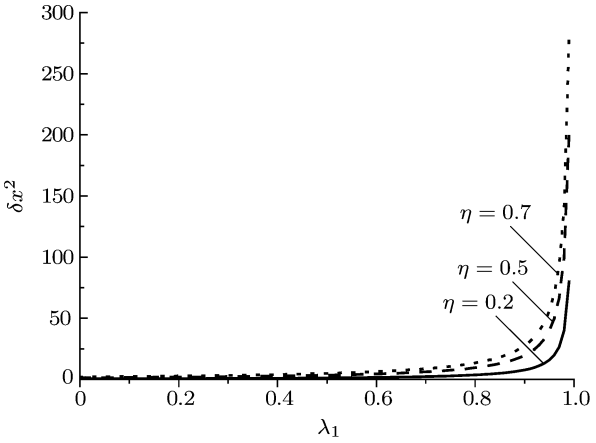


Fig. 9. The variance of the state variable δx^2 as a function of λ_1 with $r = 1$ for third example, for $\eta = 0.2$, $\eta = 0.5$, $\eta = 0.7$, respectively.

4. Conclusion

To sum up, in this paper, we have studied the effects of cross-correlation of two multiplicative Gaussian white noises on the steady state properties of three nonlinear systems with a fluctuating environment, and we find that noise-induced nonequilibrium phase transitions occur when the degree of correlation of noises changes. We obtain the Fokker–Planck equation according to the general Langevin equation, and the stationary solution for the corresponding Fokker–Planck equation is obtained. We can use the stationary probability distribution function to study the moments of the state variable x of systems driven by two cross-correlated Gaussian white noises.

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