Solving nonlinear master equation describing quantum damping by virtue of the entangled state representation*

Fan Hong-Yi(范洪义)[a], Ren Gang(任 刚)[a], Hu Li-Yun(胡利云)[b], and Jiang Nian-Quan(姜年权)[c]†

[a] Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China
[b] College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, China
[c] College of Physics and Electric Information, Wenzhou University, Wenzhou 325035, China

(Received 21 February 2010; revised manuscript received 29 March 2010)

This paper solves the newly constructed nonlinear master equation $\mathrm{d}\rho/\mathrm{d}t = \kappa[2f(N)a\rho(1/f(N-1))a^\dagger - a^\dagger a\rho - \rho a^\dagger a]$, where $f(N)$ is an operator-valued function of $N = a^\dagger a$, for describing amplitude damping channel, and derives the infinite operator sum representation of quasi-Kraus operators for the density operator. It also shows that in this nonlinear process the initial pure number state density operator will evolve into the binomial field (a mixed state) when $f(N) = 1/\sqrt{N+1}$.

Keywords: nonlinear master equation, operator sum representation, Kraus operator, binomial state
PACC: 4250, 0365

1. Introduction

One of the fundamental tasks in quantum optics is studying amplitude decay (dissipation in a lossy cavity) for various initial density operators describing optical fields. The amplitude decay model is governed by the following master equation:[1]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right), \tag{1}$$

where κ is the rate of decay. On the other hand, in recent years nonlinear coherent states (NCS) have been paid much attention because they exhibit nonclassical features[2−12] and many quantum optical states, such as squeezed state, phase states, negative binomial state, can be viewed as a sort of NCS. Physically, a class of NCS can be realised as the stationary states of center-of-mass motion of a trapped ion.[2] In this work we extend Eq. (1) to nonlinear case by introducing the nonlinear operators $(1/f(N-1))a^\dagger$ and $f(N)a$,[13] where $a^\dagger$ and a are Bose creation and annihilation operators respectively, $\left[a, a^\dagger\right] = 1, N = a^\dagger a$ is the number operator. Consequently, we construct the nonlinear master equation for amplitude decay model

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\left[2f(N)a\rho\frac{1}{f(N-1)}a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right], \tag{2}$$

where $f(N)$ is an operator-valued function of N. Such a master equation may describe some nonlinear dissipation process in trapped ion. Especially, when $f(N) = 1$, Eq. (2) reduces to Eq. (1). Owing to the commutative relation

$$\begin{aligned} &[N, a] = -a, \quad \left[N, a^\dagger\right] = a^\dagger, \\ &Na = a(N-1), \quad Na^\dagger = a^\dagger(N+1), \end{aligned} \tag{3}$$

we see

$$f(N)a = af(N-1), \quad \frac{1}{f(N-1)}a^\dagger = a^\dagger\frac{1}{f(N)}, \tag{4}$$

so

$$\left[f(N)a, \frac{1}{f(N-1)}a^\dagger\right] = 1, \tag{5}$$

and the NCS $|z\rangle_f$ is defined as eigenstate of $f(N)a$,

$$\begin{aligned} f(N)a|z\rangle_f &= z|z\rangle_f, \\ |z\rangle_f &= \exp\left(\frac{z}{f(N-1)}a^\dagger\right)|0\rangle, \end{aligned} \tag{6}$$

this is why we introduced the new master equation (2) in that way. Our work is arranged as follows: In Section 2 we solve the new master equation by using the entangled state representation and obtain the infinite operator sum representation of quasi-Kraus operators for the density operator. In Section 3 we consider the dissipation of the pure number state when $f(N) = 1/\sqrt{N+1}$, we find that the pure number state will evolve into the mixed binomial state.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10775097 and 10874174) and the Research Foundation of the Education Department of Jiangxi Province of China (Grant No. GJJ10097).
†Corresponding author. E-mail: jiangnq@wzu.edu.cn
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2. Solving Eq. (2) by using the entangled state representation

In order to directly solve Eq. (2), we construct the two-mode entangled state[14,15]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right)|0\tilde{0}\rangle, \quad (7)$$

where the tilde operator $\tilde{a}^\dagger$ is a fictitious mode accompanying the real photon mode $a^\dagger$, $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $\left[\tilde{a}, \tilde{a}^\dagger\right] = 1$. The state $|\eta = 0\rangle = \exp(a^\dagger \tilde{a}^\dagger)|0\tilde{0}\rangle$ obeys

$$a|\eta = 0\rangle = \tilde{a}^\dagger|\eta = 0\rangle, \quad a^\dagger|\eta = 0\rangle = \tilde{a}|\eta = 0\rangle, \quad (8)$$

$$(a^\dagger a)^n|\eta = 0\rangle = (\tilde{a}^\dagger \tilde{a})^n|\eta = 0\rangle. \quad (9)$$

Acting both sides of Eq. (2) on the state $|I\rangle \equiv |\eta = 0\rangle$, and noticing $|\rho\rangle = \rho\,|I\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle = \kappa\left(2f(N)\,a\rho\frac{1}{f(N-1)}a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)|I\rangle. \quad (10)$$

Using Eq. (9) and noting that density operators are defined in the original space which are commutative with operators in the tilde space, we find that Eq. (10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle = \kappa\left(2f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)} - a^\dagger a - \tilde{a}^\dagger \tilde{a}\right)|\rho\rangle, \quad (11)$$

so its formal solution is

$$|\rho\rangle = \exp\left\{\kappa t\left[2f(N)\,a\tilde{a}\frac{1}{f(\tilde{N}-1)} - a^\dagger a - \tilde{a}^\dagger \tilde{a}\right]\right\}|\rho_0\rangle, \quad (12)$$

where $|\rho_0\rangle \equiv \rho_0\,|I\rangle$, ρ_0 is the initial density operator. Thus we see that by introducing a fictitious mode to be a counterpart mode of the system mode under review we can arrange the nonlinear master equations in quantum statistics as state-vector evolution equations.

Noticing that the operators in Eq. (12) obey the following commutative relation,

$$\left[f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)}, a^\dagger a\right] = \left[f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)}, \tilde{a}^\dagger \tilde{a}\right] = f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)}, \quad (13)$$

$$\left[\frac{a^\dagger a + \tilde{a}^\dagger \tilde{a}}{2}, f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)}\right] = -f(N)\,a\tilde{a}\frac{1}{f\left(\tilde{N}-1\right)}, \quad (14)$$

and using the operator identity[16]

$$\mathrm{e}^{\lambda(A+\sigma B)} = \mathrm{e}^{\lambda A}\exp\left[\sigma\left(1 - \mathrm{e}^{-\lambda\tau}\right)B/\tau\right], \quad (15)$$

(which is valid for $[A, B] = \tau B$), we have

$$\exp\left[-2\kappa t\left(\frac{a^\dagger a + \tilde{a}^\dagger \tilde{a}}{2} - f(N)\,a\tilde{a}\frac{1}{f(\tilde{N}-1)}\right)\right] = \exp\left[-\kappa t\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right)\right] \times \exp\left[Tf(N)\,a\tilde{a}\frac{1}{f(\tilde{N}-1)}\right], \quad (16)$$

where

$$T = 1 - \mathrm{e}^{-2\kappa t}. \quad (17)$$

Then substituting Eq. (16) into Eq. (12) yields

$$\begin{aligned}|\rho\rangle &= \exp\left[-\kappa t\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right)\right]\sum_{n=0}^{\infty}\frac{T^n}{n!} \times \left(f(N)\,a\tilde{a}\frac{1}{f(\tilde{N}-1)}\right)^n|\rho_0\rangle \\ &= \mathrm{e}^{-\kappa t a^\dagger a}\sum_{n=0}^{\infty}\frac{T^n}{n!}\left(f(N)\,a\right)^n\rho_0 \times \left(\frac{1}{f(N-1)}a^\dagger\right)^n \mathrm{e}^{-\kappa t \tilde{a}^\dagger \tilde{a}}|I\rangle \\ &= \sum_{n=0}^{\infty}\frac{T^n}{n!}\,\mathrm{e}^{-\kappa t a^\dagger a}\left(f(N)\,a\right)^n\rho_0 \times \left(\frac{1}{f(N-1)}a^\dagger\right)^n \mathrm{e}^{-\kappa t a^\dagger a}|I\rangle. \end{aligned} \quad (18)$$

Therefore, for any given initial state ρ_0, the density operator ρ can be easily calculated from Eq. (18).

$$\rho = \sum_{n=0}^{\infty}\frac{T^n}{n!}\,\mathrm{e}^{-\kappa t a^\dagger a}\left[f(N)\,a\right]^n\rho_0 \times \left(\frac{1}{f(N-1)}a^\dagger\right)^n \mathrm{e}^{-\kappa t a^\dagger a}. \quad (19)$$

Let

$$\mathcal{M}_n = \sqrt{\frac{T^n}{n!}}\,\mathrm{e}^{-\kappa t a^\dagger a}\left[f(N)\,a\right]^n, \quad (20)$$

and

$$\mathfrak{M}_n^\dagger = \sqrt{\frac{T^n}{n!}}\left(\frac{1}{f(N-1)}a^\dagger\right)^n \mathrm{e}^{-\kappa t a^\dagger a}, \quad (21)$$

such that

$$\rho(t)=\sum_{n=0}^{\infty}\mathcal{M}_n\rho_0\mathfrak{M}_n^{\dagger}, \tag{22}$$

and using the operator identities $\mathrm{e}^{\lambda a^{\dagger}a}=:\exp\left[\left(\mathrm{e}^{\lambda}-1\right)a^{\dagger}a\right]:$ and $\mathrm{e}^{\lambda a^{\dagger}a}a\,\mathrm{e}^{-\lambda a^{\dagger}a}=a\,\mathrm{e}^{-\lambda}$, as well as Eq. (4), we can prove

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{M}_n^{\dagger}\mathcal{M}_n\\
&=\sum_{n=0}^{\infty}\frac{T^n}{n!}\left(\frac{1}{f(N-1)}a^{\dagger}\right)^n\mathrm{e}^{-2\kappa t a^{\dagger}a}\left[f(N)\,a\right]^n\\
&=\sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{2n\kappa t}\left(\frac{1}{f(N-1)}a^{\dagger}\right)^n\left[f(N)\,a\right]^n\mathrm{e}^{-2\kappa t a^{\dagger}a}\\
&=:\mathrm{e}^{T\mathrm{e}^{2\kappa t}a^{\dagger}a}:\ \mathrm{e}^{-2\kappa t a^{\dagger}a}\\
&=:\mathrm{e}^{\left(\mathrm{e}^{2\kappa t}-1\right)a^{\dagger}a}:\ \mathrm{e}^{-2\kappa t a^{\dagger}a}\\
&=1,
\end{aligned} \tag{23}$$

so

$$\mathtt{Tr}\rho(t)=\mathtt{Tr}\sum_{n=0}^{\infty}\mathcal{M}_n\rho_0\mathfrak{M}_n^{\dagger}=\mathtt{Tr}\rho_0, \tag{24}$$

and $\mathcal{M}_n$ is not the Hermite conjugate of $\mathfrak{M}_n^{\dagger}$, they may be named quasi-Kraus operator. The operator ρ in Eq. (22) is qualified to be a density operator. Therefore, for any given initial state ρ_0, the density operator ρ can be easily calculated from Eq. (22).

3. The dissipation of the pure number state into the binomial state

Now we consider the special case, i.e.,

$$f(N)=\frac{1}{\sqrt{N+1}}, \tag{25}$$

and the initial density operator $\rho_0=|m\rangle\langle m|$, a pure number state, we want to see how does it evolve in the dissipative channel. We shall demonstrate that ρ_0 will evolves into a density operator in binomial distribution. For this purpose, let us briefly review the optical binomial state which is defined as a superposition of number states in the following way:

$$|\sigma,M\rangle=\sum_{n=0}^{M}\beta_n^M|n\rangle,\ 0<\sigma<1, \tag{26}$$

where

$$\beta_n^M\equiv\sqrt{\binom{M}{n}\sigma^n(1-\sigma)^{M-n}}, \tag{27}$$

σ is named as the binomial parameter. Due to $\sum_{n=0}^{M}\left(\beta_n^M\right)^2=1$, it is a binomial distribution, and $|\sigma,M\rangle$ is normalised. From the knowledge of mathematical statistics we know that when $\sigma\to 0$, while $M\to\infty$, the binomial distribution goes to the Poisson distribution, so the binomial state is interpolated between the coherent state and the number state. The binomial state exhibits the antibunching property and the sub-Poisson distribution, since

$$\langle\sigma,M|\,a^{\dagger}a\,|\sigma,M\rangle=\sigma M, \tag{28}$$

$$\langle\sigma,M|\left(a^{\dagger}a\right)^2|\sigma,M\rangle=(\sigma M)^2+\sigma(1-\sigma)M, \tag{29}$$

$$\begin{aligned}
&\frac{\langle\sigma,M|\left(a^{\dagger}a\right)^2|\sigma,M\rangle-\left(\langle\sigma,M|\,a^{\dagger}a\,|\sigma,M\rangle\right)^2}{\langle\sigma,M|\,a^{\dagger}a\,|\sigma,M\rangle}\\
&=1-\sigma<1,
\end{aligned} \tag{30}$$

so its second-order coherence

$$\begin{aligned}
&\frac{\langle\sigma,M|\left(a^{\dagger}a\right)^2|\sigma,M\rangle}{\langle\sigma,M|\,a^{\dagger}a\,|\sigma,M\rangle^2}-\frac{1}{\langle\sigma,M|\,a^{\dagger}a\,|\sigma,M\rangle}\\
&=1-\frac{1}{M}<1,
\end{aligned} \tag{31}$$

which is less than that of the coherent state. Correspondingly, the density operator of binomial field is

$$\sum_{n=0}^{M}\binom{M}{n}\sigma^n(1-\sigma)^{M-n}|n\rangle\langle n|. \tag{32}$$

In the following we shall demonstrate that $\rho_0=|m\rangle\langle m|$ will become a binomial field when it undergoes the amplitude dissipative channel. Substituting $\rho_0=|m\rangle\langle m|$ into Eq. (22) we have

$$\begin{aligned}
\rho=&\sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-\kappa t a^{\dagger}a}\left(\frac{1}{\sqrt{N+1}}a\right)^n\\
&\times|m\rangle\langle m|\left(\sqrt{N}a^{\dagger}\right)^n\mathrm{e}^{-\kappa t a^{\dagger}a}.
\end{aligned} \tag{33}$$

Using $a|m\rangle=\sqrt{m}\,|m-1\rangle$, $\frac{1}{\sqrt{N+1}}a|m\rangle=|m-1\rangle$, and

$$\begin{aligned}
\left(\frac{1}{\sqrt{N+1}}a\right)^n|m\rangle&=\left(\frac{1}{\sqrt{N+1}}a\right)^{n-1}|m-1\rangle\\
&=|m-n\rangle,
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&\langle m|\left(\sqrt{N}a^{\dagger}\right)\left(\sqrt{N}a^{\dagger}\right)^{n-1}\\
&=\langle m-1|\,m\left(\sqrt{N}a^{\dagger}\right)^{n-1}\\
&=\langle m-n|\frac{m!}{(m-n)!},
\end{aligned} \tag{35}$$

as well as noticing $T = 1 - \mathrm{e}^{-2\kappa t}$ we have

$$\begin{aligned}\rho(t) &= \sum_{n=0}^{\infty} \frac{T^n}{n!}\, \mathrm{e}^{-kta^\dagger a} \frac{m!}{(m-n)!} |m-n\rangle \\ &\quad \times \langle m-n|\, \mathrm{e}^{-kta^\dagger a} \\ &= \sum_{n=0}^{\infty} \frac{T^n}{n!}\, \mathrm{e}^{-2kt(m-n)} \frac{m!}{(m-n)!} |m-n\rangle \langle m-n| \\ &= \sum_{n=0}^{\infty} \left(1-\mathrm{e}^{-2kt}\right)^n \binom{m}{n} \left(\mathrm{e}^{-2kt}\right)^{m-n} \\ &\quad \times |m-n\rangle \langle m-n| \\ &= \sum_{n'=0}^{\infty} \binom{m}{m-n'} \left(\mathrm{e}^{-2kt}\right)^{n'} \left(1-\mathrm{e}^{-2kt}\right)^{m-n'} \\ &\quad \times |n'\rangle \langle n'|\,. \end{aligned} \tag{36}$$

Noting $\binom{m}{m-n'} = \binom{m}{n'}$, and comparing Eq. (36) with Eq. (26) we conclude that $\rho(t)$ is a binomial optical field (a mixed state) with e^{-2kt} being the binomial parameter, therefore after passing through the channel the number of photons (amplitude) decays, i.e.,

$$\begin{aligned}&\mathtt{Tr}\left[\rho(t)\, a^\dagger a\right] \\ &= \sum_{n=0}^{\infty} \binom{m}{m-n} \left(\mathrm{e}^{-2kt}\right)^n \left(1-\mathrm{e}^{-2kt}\right)^{m-n} \langle n|\, a^\dagger a\, |n\rangle \\ &= m\, \mathrm{e}^{-2kt}, \end{aligned} \tag{37}$$

where κ is the rate of decay of the channel. When t is long enough, $m\, \mathrm{e}^{-2kt} \to 0$, and

$$\begin{aligned}&\mathtt{Tr}\left[\rho(t) \left(a^\dagger a\right)^2\right] \\ &= \left(m\, \mathrm{e}^{-2kt}\right)^2 + m\, \mathrm{e}^{-2kt} \left(1-\mathrm{e}^{-2kt}\right). \end{aligned} \tag{38}$$

4. Conclusions

In summary, for the first time we constructed the nonlinear master equation for the amplitude dissipative channel, Eq. (2) is a nontrivial generalisation of the usual master Eq. (1), and solved it to obtain the infinite operator sum representation of quasi-Kraus operators for the density operator by virtue of the entangled state representation. About more applications of entangled state representation, one may refer to Refs. [17]–[19].

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