Chaos in a Bose–Einstein condensate*

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It is demonstrated that Smale-horseshoe chaos exists in the time evolution of the one-dimensional Bose–Einstein condensate driven by time-periodic harmonic or inverted-harmonic potential. A formally exact solution of the time-dependent Gross–Pitaevskii equation is constructed, which describes the matter shock waves with chaotic or periodic amplitudes and phases.

Keywords: Bose–Einstein condensate, chaos, Gross–Pitaevskii equation
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1. Introduction

Many phenomena observed in Bose–Einstein condensate (BEC) are well modeled by the Gross–Pitaevskii equation(GPE), which include chaos,[1−3] and other nonlinear phenomena. It is well known that classical Smale-horseshoe chaos can exist in a parametrically driven Duffing system with the cubic nonlinear, harmonic, or inverted-harmonic force.[4−8] The harmonic trapping potential is widely used in the investigations of BEC.[9,10] The expulsive parabolic potential has also been applied to the study of the time evolution of quantum tunneling,[11] deterministic chaos,[12,13] and accelerated bright solutions[14,15] in BEC. The harmonic and inverted-harmonic potentials can be time-dependent.[16−18] The presence of such potentials will break the integrability of GPE and bring difficulties to the exact researches of BEC. Although for different initial and boundary conditions the nonlinear GPE has many solutions, the exact solution of the GPE with harmonic or inverted-harmonic potential has not been reported yet even for the nondrying cases. However, we know that a turbulent flow is a fluid regime characterised by chaotic, stochastic property changes, and as an impotent physical behaviour the unstable and irregular quantum turbulence has been studied.[19] Therefore the chaotic shock wave as a different type of turbulent flow needs further investigation.

In the present paper, we treat the time evolution of a qusi-one-dimensional (1D) BEC created initially in a region near the potential centre and driven by the time-periodic harmonic or inverted-harmonic potential, and seek the formally exact solution of the time-dependent GPE. By using the exact solution we reveal that well-known Smale-horseshoe chaos exists in the matter shock waves with unpredictable amplitudes, phases, and wave fronts. The suppression of chaos to the blast of matter wave is revealed. The results can be observed experimentally.[20,21]

2. Numerical simulations

There are some different methods for the reductions from the original three-dimensional GPE to the quasi-1D GPE.[22] Assuming the transverse wave function to be ground state of a harmonic oscillator of frequency $\omega_{\rm r}$, the governing longitudinal GPE becomes the cubic nonlinear equation

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + [V'_\alpha(x,t) + g'_{1\rm d}\,|\psi|^2]\psi, \qquad (1)$$

where m is atomic mass; $g'_{1\rm d} = m\omega_{\rm r}g_0/(2\pi\hbar) = 2\hbar\omega_{\rm r}a_{\rm s}$ is the quasi-1D atom–atom interaction intensity, with $a_{\rm s}$ being the s-wave scattering length. $V'_\alpha(x,t) = (\alpha\frac{1}{2}m\omega_x^2 + V'_1\cos(\omega t))x^2$ denotes the harmonic ($\alpha = 1$) and inverted-harmonic ($\alpha = -1$) potentials of frequency ω_x with the driving of strength V'_1 and frequency ω.[18] Taking an experimentally suitable frequency ω_0 as the units of frequencies ω_x, ω; and normalising the time, space, and wave function as ω_0^{-1}, $l_0 = \sqrt{\hbar/m\omega_0}$, and $1/\sqrt{l_0}$, the GPE becomes dimensionless

$$\mathrm{i}\psi_t = -\frac{1}{2}\psi_{xx} + [k_\alpha(t)x^2 + g_{1\rm d}\,|\psi|^2]\psi \qquad (2)$$

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with $k_\alpha(t) = (\alpha\frac{1}{2}\omega_x^2 + V_1\cos(\omega t))$. Here the interaction is reduced to $g_{1\mathrm{d}} = 2\omega_\mathrm{r}a_\mathrm{s}/\omega_0 l_0$ and the potential strength V_1 is normalised by $\hbar\omega_0$. Throughout the paper we take $|g_{1\mathrm{d}}| = 0.4$, which means $\hbar\omega_0 = 25m\omega_\mathrm{r}^2 a_\mathrm{s}^2$ with m, ω_r and a_s determined by experiment.

Noticing that Eq. (2) can describe a symmetric BEC system in the transformation $x \to -x$, we consider a real physical process in which a BEC is created experimentally with the symmetry profile between positions $\pm x_0 = \pm L$ at initial time t_0, then the condensed atoms propagate along $\pm x$ directions such that boundary coordinates $\pm x_0(t)$ are time dependent. Letting the number of condensed atoms conserve as N, we face the definite problem with the initial data,[23]

$$\psi(x,t_0) = \begin{cases} \psi(x,t_0), & \text{for} \quad |x| \le L, \\ 0, & \text{for} \quad |x| > L, \end{cases} \tag{3}$$

and the boundary-dependent normalisation condition $\int_{-x_0(t)}^{x_0(t)} |\psi(x,t)|^2\,\mathrm{d}x = N$ for any t. For a practical BEC the boundary density $|\psi(\pm L,t_0)|^2$ could be made nonzero by using Feshbach resonances[24] that results in the steep jumps in atomic density and flow velocity, which characterise the feature of shock wave[23] with the shock front $\pm x_0(t)$ at which the front gradients $\psi_x(\pm x_0,t)$ are discontinuous. Although the initial data of $\psi(\pm L,t_0)$ and $\psi_x(\pm L,t_0)$ cannot be measured accurately, in mathematics, they correspond to a unique solution of Eq. (2) strictly. Hence, if we construct an exact solution $\psi(x,t)$ which obeys Eq. (3) and the normalisation condition experimentally, it will describe the exact shock wave of the system. For other initial and boundary conditions, of course, one could obtain different types of solutions. In fact, the accelerated bright solution of the system has been reported for the initial solution state.[20]

We now give the formally exact solution of Eq. (2) as

$$\psi(x,t) = [A(t)x + \mathrm{i}B(t)]\,\mathrm{e}^{\mathrm{i}[a(t)+b(t)x^2]}, \tag{4}$$

where $A(t)$, $B(t)$, $a(t)$, and $b(t)$ are real functions of time and obey the coupled equations

$$\begin{aligned} &\frac{\mathrm{d}a}{\mathrm{d}t} + g_{1\mathrm{d}}B^2 = 0, \\ &\frac{\mathrm{d}b}{\mathrm{d}t} + 2b^2 = -g_{1\mathrm{d}}A^2 - k_\alpha(t), \\ &\frac{\mathrm{d}A}{\mathrm{d}t} + 3Ab = 0, \\ &\frac{\mathrm{d}B}{\mathrm{d}t} + Bb = 0. \end{aligned} \tag{5}$$

Equation (4) can be easily proved by inserting Eqs. (4) and (5) into Eq. (2) directly. The atomic density and flow velocity are associated with macroscopic wave function $\psi(x,t) = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta}$ through

$$\begin{aligned} &\rho(x,t) = |\psi(x,t)|^2 = A^2x^2 + B^2, \\ &v(x,t) = \frac{\hbar}{m}\theta_x = \frac{\hbar}{m}\left(2bx - \frac{AB}{\rho}\right). \end{aligned} \tag{6}$$

In order to evidence the solution (4), we have to solve Eq. (5) as follows. From the third equation of Eq. (5) we have $b = -(\mathrm{d}A/\mathrm{d}t)/3A$, $\mathrm{d}b/\mathrm{d}t = ((\mathrm{d}A/\mathrm{d}\,t)^2)/3A^2 - (\mathrm{d}^2A/\mathrm{d}t^2)/3A$. Applying these formulas to the second one of Eq. (5) produces the decoupled equation

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2} = 3k_\alpha(t)A + 3g_{1\mathrm{d}}A^3 + \frac{5\left(\dfrac{\mathrm{d}A}{\mathrm{d}t}\right)^2}{3A}. \tag{7}$$

This is just the parametrically driven Duffing equation with a 'quadratic damping' term.[25]

We calculate the maximal Lyapunov exponent λ_max of BEC system using the algorithm presented by Wolf *et al.*[26] through Eq. (7). Figure 1 shows the maximal Lyapunov exponent λ_max as a function of $\alpha\omega_x^2$. The parameters are $\omega = 6$, $V_1 = 20.2/3$, $g = -0.4$. Figure 2 shows the maximal Lyapunov exponent λ_max as a function of ω. The parameters are $\alpha\omega_x^2 = -0.8$, $V_1 = 20.2/3$, $g = -0.4$. We find that in some intervals (e.g. $-3 < \alpha\omega_x^2 < 2.7$, $-8 < \omega < -4$, $-3 < \omega < 3$, and so on), the maximal Lyapunov exponent is positive. If ω or $\alpha\omega_x^2$ takes a value within these ranges, the BEC would be in a chaotic state. We find that the BEC system can enter into chaotic and periodic states.

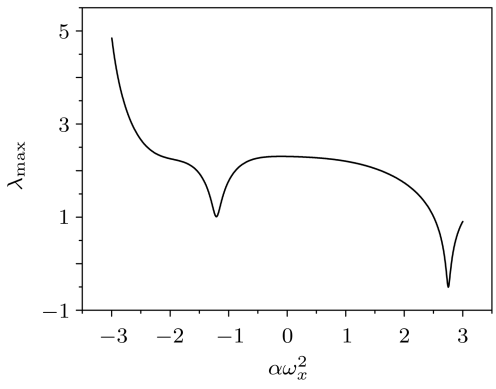


Fig. 1. The maximal Lyapunov exponent λ_max as a function of $\alpha\omega_x^2$ with $V_1 = 20.2/3$, $g = -0.4$, $\omega = 6$.

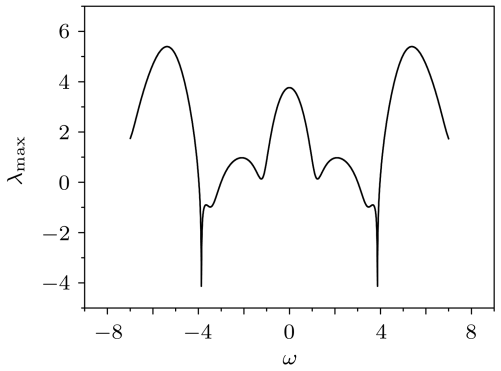


Fig. 2. The maximal Lyapunov exponent $\lambda_{\max}$ as a function of ω with $V_1 = 20.2/3$, $g = -0.4$, $\alpha\omega_x^2 = -0.8$.

Using the fourth Runge–Kutta algorithm, we solve Eq. (7) numerically, and illustrate the attractors in an equivalent phase of $(A, \mathrm{d}A/\mathrm{d}t)$ in Fig. 3 with parameters $V_1 = 20.2/3$, $g = -0.4$. Figure 3(a) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (1, 0.5)$, $\omega = 5.4$, $\alpha\omega_x^2 = -0.8$. Figure 3(b) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (0.9, 0.5)$, $\omega = 6$, $\alpha\omega_x^2 = 0.8$. Figure 3(c) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (1, 0.6)$, $\omega = 5.4$, $\alpha\omega_x^2 = -0.8$. Figure 3(d) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (1.33, 0.55)$, $\omega = 6$, $\alpha\omega_x^2 = 0.8$. Figure 3(e) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (1.26, 0.45)$, $\omega = 6$, $\alpha\omega_x^2 = 0.8$. The three Lyapunov exponents are $\lambda_1 = 1.87638$, $\lambda_2 = 0.0$, and $\lambda_3 = -2.04806$. BEC system is in a chaotic state as the maximal Lyapunov exponent is positive. The chaotic orbit in equivalent phase space $(A, \mathrm{d}A/\mathrm{d}t)$ is localised in a finite region and shows a confused structure. Figure 3(f) for the initial condition: $(A, \mathrm{d}A/\mathrm{d}t) = (0.9, 0.5)$, $\omega = 5.4$, $\alpha\omega_x^2 = -0.8$. The three Lyapunov exponents are $\lambda_1 = 3.47413$, $\lambda_2 = 0.0$, and $\lambda_3 = -3.17834$. BEC system is in a chaotic state as the maximal Lyapunov exponent is positive. One can find that Figs. 3(a), 3(b), 3(c), and 3(d) have periods 1, 2, 3, and 6 respectively; and Figs. 3(e) and 3(f) are in chaotic states.

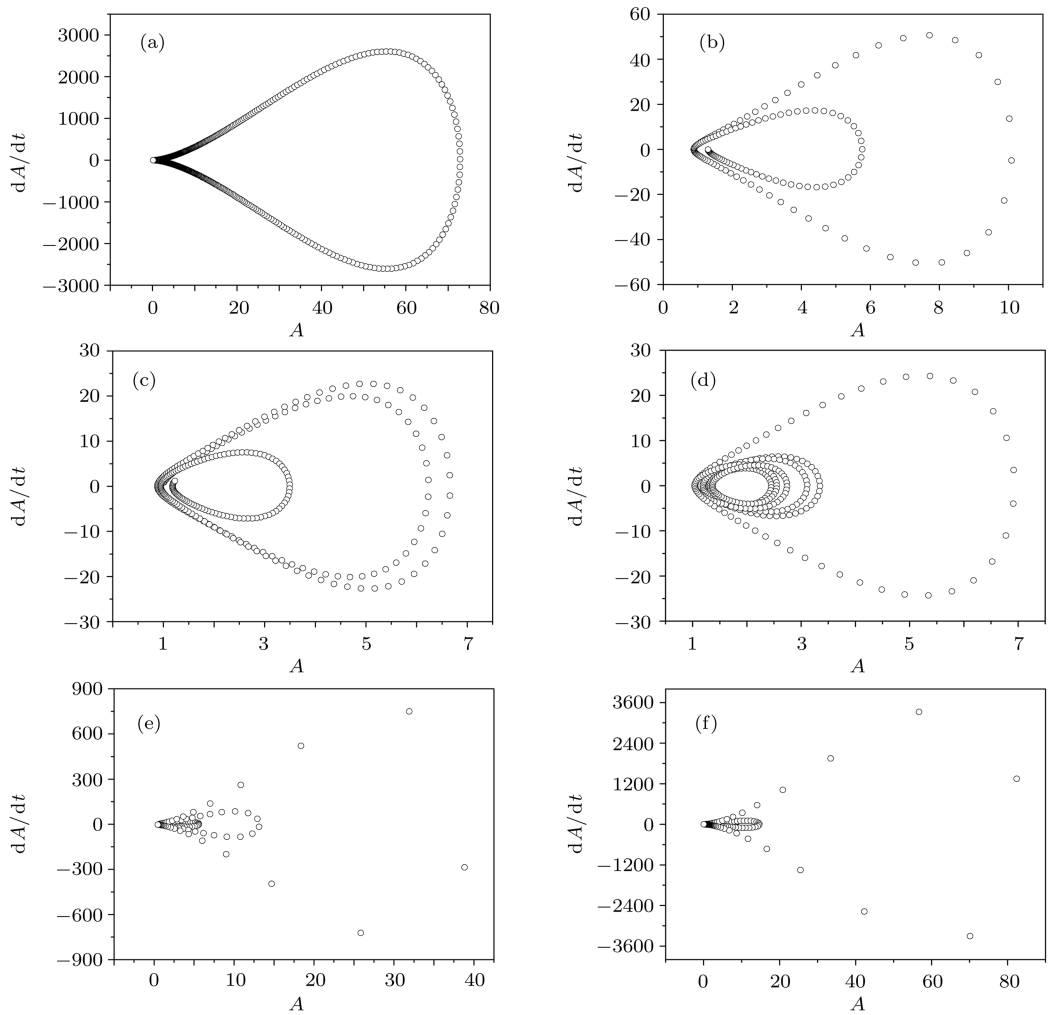


Fig. 3. The attractors with $V_1 = 20.2/3$, $g = -0.4$ and different values of ω and $\alpha\omega_x^2$. (a) $\omega = 5.4$ and $\alpha\omega_x^2 = -0.8$; (b) $\omega = 6$ and $\alpha\omega_x^2 = 0.8$; (c) $\omega = 5.4$ and $\alpha\omega_x^2 = -0.8$; (d) $\omega = 6$ and $\alpha\omega_x^2 = 0.8$; (e) $\omega = 6$ and $\alpha\omega_x^2 = 0.8$; (f) $\omega = 5.4$ and $\alpha\omega_x^2 = -0.8$.

3. Conclusions

In conclusion, for an atomic BEC created initially in a region near the potential centre and driven by the time-periodic harmonic or inverted-harmonic potential, we have demonstrated that the classical Smale-horseshoe chaos certainly exists in the time evolutions of the system. The formally exact solution of the time-dependent GPE, whose amplitude and phase depend on the solutions of the famous Duffing equation with periodic driving and quadratic damping, has been constructed. When the periodic driving is switched off and the number of condensed atoms is conserved, we arrive at the most studied BEC systems with parabolic potentials. Then the exact solutions become the explicit functions of spatiotemporal coordinates and govern the exactly nonstationary states or the exactly stable stationary states.

In recent advancements in applications of BEC, quantum computation with BEC atoms is an interesting subject.[27−29] However, chaos is associated with quantum entanglement[30] and quantum error corrections,[31] which are all key subject in quantum computation; thereby, investigating the chaos and unstable cycle in BEC is very important.

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