Impulsive control of stochastic system under the sense of stochastic asymptotical stability*

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This paper studies the stochastic asymptotical stability of stochastic impulsive differential equations, and establishes a comparison theory to ensure the trivial solution's stochastic asymptotical stability. From the comparison theory, it can find out whether the stochastic impulsive differential system is stable just by studying the stability of a deterministic comparison system. As a general application of this theory, it controls the chaos of stochastic Lü system using impulsive control method, and numerical simulations are employed to verify the feasibility of this method.

Keywords: stochastic impulsive system, stochastic asymptotical stability, comparison theory, impulsive control

PACC: 0545, 0547

1. Introduction

Control and synchronization of chaos have become important research subjects in the studies of the nonlinear dynamical system during the past decades, among which, the impulsive method is important. The impulsive method allows the stabilization and synchronization of chaotic systems to use only a small control impulse, furthermore, the impulsive control technique is also suitable to deal with systems which can not endure continuous disturbance. Using this method we find that the response system receives the information from the driven system only in discrete times and the amount of conveyed information is decreased. This is suitable in practice because of the reduced control cost.[1] In the works we have found that Yang and other scholars[2−15] studied the stability problem of the following deterministic impulsive differential system:

$$\begin{cases} \dot{X} = a(t, X), & t \neq \tau_i, \\ \Delta X = U(i, X) = X(\tau_i^+) - X(\tau_i), & t = \tau_i, \\ X(t_0) = X_0, & i = 1, 2, \ldots, \end{cases} \tag{1}$$

and a comparison theory is proposed for system (1), the comparison theory is employed to design the impulsive control and synchronization method for different chaotic systems. On the other hand, for the following stochastic impulsive system:

$$\begin{cases} \mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}w(t), & t \neq \tau_i, \\ \Delta X = U(i, X) = X(\tau_i^+) - X(\tau_i^-), & t = \tau_i, \\ X(t_0^+) = X_0, & i = 1, 2, \ldots. \end{cases} \tag{2}$$

Wu *et al.*[16] investigated the p-moment stability, and a judging theory of this problem was constructed. Yang *et al.*[17−21] studied the p-moment stability of stochastic impulsive differential equations with delays and its application to impulsive control or impulsive synchronization. Li *et al.*[22−24] tried to give a comparison theory about the asymptotical p-moment stability in system (2), and applied the theory to chaotic control and synchronization in the Chua system. To the best of our knowledge, there are few results about the stochastic asymptotical stability for system (2), therefore, it is necessary to consider this problem. In this paper, we aim at gaining an insight into the trivial solution's stochastic asymptotical stability for system (2) and attempting to study its application to chaotic control. As an example, we use the theory to judge whether stochastic Lü system with impulsive control signals is stable, from which we can judge whether this system can be controlled with the impulsive control method.

2. Basic definitions and lemma

To make the present work self-contained, we address some basic definitions and results in this sec-

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tion, which can be found in Refs. [22]–[26]. Let R^n be n-dimensional Euclidean space, $\|\cdot\|$ be the norm of element $x \in R^n$ and $\|x\| \triangleq \sqrt{\sum_{i=1}^{n} x_i^2}$, $R^{n\times m}$ be an $n \times m$ matrix, $R_+ = \{x : x \in R, x \geq 0\}$, $S_\rho = \{X : |X| \leq \rho, X \in R^n\}$, and $R_{t_0} = \{t : t \in R_+, t \geq t_0\}$. Let $C^{1,2}(R_{t_0} \times R^n, R_+)$ denote the family of all non-negative functions from $R_{t_0} \times R^n$ to R_+ which are once continuously differentiable with respect to the first variable and twice with respect to the second variable. In system (2), $f(\cdot,\cdot) \in C\,[R_{t_0} \times R^n, R^n]$, $g(\cdot,\cdot) \in C\,[R_{t_0} \times R^n, R^{n\times m}]$, and $w(t)$ is a normal m-dimensional Wiener process, denote

$$\Gamma = \{\tau_i : i = 1, 2, \ldots, \quad t_0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_i < \tau_{i+1} < \ldots\},$$

and $\lim_{i\to\infty} \tau_i = +\infty$, where τ_i means the moment when impulsive happens. We also assume that $f(t,0) = 0$, $g(t,0) = 0$ and $U(t,0) = 0$, which guarantee the existence of trivial solution in system (2). In the present paper, we always assume that there is only one stochastic process $X(t) = X(t, t_0, X_0)$ satisfying system (2), whose left side is continuous: $X(\tau_i^-) = \lim_{t\to\tau_i-0} X(t) = X(\tau_i)$, and the right side is limitable : $X(\tau_i^+) = \lim_{t\to\tau_i+0} X(t)$. Let

$$\tau_{t_0}(\rho) = \begin{cases} \inf\{t : t \geq t_0, X(t) \notin S_\rho\}, & t \in T, T \neq \emptyset, \\ \infty, & X(t) \in S_\rho \text{ for all } t < \infty, \end{cases}$$

and $t \wedge \tau_{t_0}(\rho) = \min(t, \tau_{t_0}(\rho))$, $\widetilde{x}(t) = x(t \wedge \tau_{t_0}(\rho))$.

Definition 1. The trivial solution of system (2) is said to be

(i) stochastically stable if for any $\varepsilon > 0$, $\varepsilon' > 0$, $t_0 > 0$, there exists $\delta = \delta(t_0, \varepsilon, \varepsilon') > 0$ (δ is continuous at t_0 for every ε and ε') such that $\|X_0\| < \delta$ implies

$$P_{X_0 t_0}\{\|X\,(t)\| < \varepsilon', \text{ for all } t \geq t_0\} \geq 1 - \varepsilon; \quad (3)$$

(ii) stochastically asymptotically stable if the trivial solution of system (2) is stochastically stable and for any $\varepsilon > 0$, $\varepsilon' > 0$, $t_0 > 0$, there exist $\delta = \delta(t_0) > 0$ and $T = T(t_0, \varepsilon, \varepsilon')$ such that $\|X_0\| < \delta$ implies

$$P_{X_0 t_0}\{\|X\,(t)\| < \varepsilon', \text{for all } t \geq T + t_0\} \geq 1 - \varepsilon. \quad (4)$$

Definition 2. For each $V(t,X) \in C^{1,2}(R_{t_0} \times R^n, R_+)$, define an operator L from $R_{t_0} \times R^n$ to R as

$$LV(t,X) = V_t(t,X) + V_X(t,X)f(t,X) + \frac{1}{2}\text{trace}(g^{\mathrm{T}}(t,X)V_{XX}\, g(t,X)),$$

where

$$V_t(t,X) = \partial V(t,X)/\partial t,\ V_{XX}(t,X) = [\partial^2 V(t,X)/\partial x_i \partial x_j]_{n\times n},$$

and

$$V_X(t,X) = [\partial V(t,X)/\partial x_1, \partial V(t,X)/\partial x_2, \ldots, \partial V(t,X)/\partial x_n].$$

Definition 3. We say that a continuous function $\phi : [0, r] \to R_+$ belongs to class $\mathcal{K}$, if $\phi(0) = 0$ and ϕ is strictly increasing on $[0, r]$.

Definition 4. Let $V(t,X) \in C^{1,2}(R_{t_0} \times R^n, R_+)$, if the following inequalities

$$\begin{cases} V_t(t,X) \leq G(t, V(t,X)), \\ LV(t,X) \leq G(t, V(t,X)), \\ V(\tau_k, X + U(k,X)) \leq \psi_k(V(\tau_k, X)) \end{cases} \quad (5)$$

hold, where $G : R_+ \times R \to R$, which is continuous on every $(\tau_{k-1}, \tau_k]$ for $X \in R$, $k = 0, 1, 2, \ldots$, $\lim_{(t,y)\to(\tau_k^+, x)} G(t,y) = G(\tau_k^+, x)$ exists, $\psi_k : R_+ \to R_+$ is nondecreasing and concvex, then the following system

$$\begin{cases} \omega = G(t, \omega), & t \neq \tau_k, \\ \omega(\tau_k^+) = \psi_k(\omega(\tau_k)), & k = 0, 1, 2, \ldots, \\ \omega(t_0^+) = \omega_0 \geq 0 \end{cases} \quad (6)$$

is called the comparison system of system (2).

3. Stochastic asymptotical stability of stochastic impulsive differential equation

As we know, the most important thing in chaotic control and synchronization is the stability problem of

the control system or the synchronization error system. In this paper, we try to propose a comparison theory about the stochastic asymptotical stability of system (2), which is a nice tool in the study of stability.[22]

Theorem 1. For system (2) and its comparison system (8), if there exists $V(t,X) \in C^{1,2}(R_{t_0} \times R^n, R_+)$, such that

(H_1) $\beta(\|X(t)\|) \leq V(t,X) \leq \alpha(\|X(t)\|)$, where $\alpha(\cdot)$, $\beta(\cdot) \in \mathcal{K}$ and $\beta(\cdot)$ is convex,

(H_2) $LV(t,X) \leq H(t,V(t,X))$ holds for $t \neq \tau_k$,

(H_3) there exists $\rho_0 > 0$ such that $X \in S_{\rho_0}$ implies that $X + U(k,X) \in S_{\rho_0}$ for all k and $V(t, X + U(k,X)) \leq \psi_k(V(t,X))$ holds for $t = \tau_k$, $X \in S_{\rho_0}$, then the trivial solution's stability properties of comparison system (8) imply the trivial solution's corresponding stochastic stability properties of system (2).

Proof: The proof of this theory can be found in Refs. [23] and [24].

Theorem 2. Let $H(t,\omega) = \dot{\lambda}(t)\omega$, $\lambda \in C^1(R_+, R_+)$, $\dot{\lambda}(t) \geq 0$, $\psi_k(\omega) = d_k\omega$, $d_k \geq 0$ for all k, then the origin of system (3) is

(i) stable if

$$\lambda(\tau_{k+1}) + \ln(d_k) \leq \lambda(\tau_k)$$

holds for all k;

(ii) asymptotically stable if

$\lambda(\tau_{k+1}) + \ln(\gamma d_k) \leq \lambda(\tau_k)$, where $\gamma > 1$ holds for all k.

Proof: The proof of this theorem can be found in Ref. [3].

4. Impulsive control of Lü system with parameter white-noise excitation

As an application of theorem 1, we use it to judge whether the stochastic Lü system can be controlled with the impulsive method.

For Lü system with parameter white-noise excitation

$$\begin{cases} \dot{x} = -ax + ay + \beta x \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, \\ \dot{y} = cy - xz + \beta y \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, \\ \dot{z} = xy - bz + \beta z \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, \end{cases} \tag{7}$$

where a, b, c, and β are positive parameters, and w_t is normal Wiener process as mentioned above. When these parameters are chosen as $a = 36.0$, $b = 3.0$, $c = 28.0$, $\beta = 0.01$, Lü system is chaotic. The time history are provided in Fig. 1.

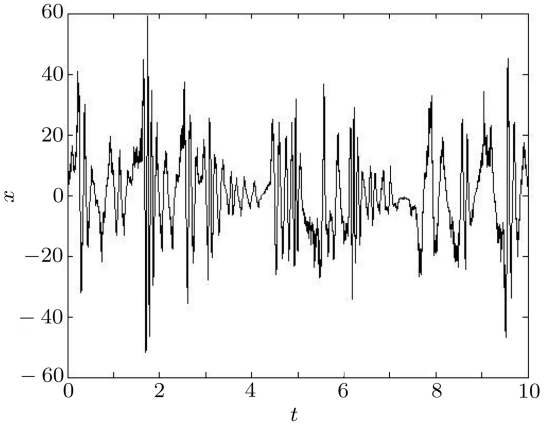


Fig. 1. Time history for system (7).

A kind of impulsive control system of (7) can be provided as:

$$\begin{cases} \dot{x} = -ax + ay + \beta x \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, \\ \dot{y} = cy - xz + \beta y \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, & t \neq \tau_i, \\ \dot{z} = xy - bz + \beta z \dfrac{\mathrm{d}w_t}{\mathrm{d}t}, \\ \dot{x}(\tau_i^+) = (1+k_1)x(\tau_i), \\ \dot{y}(\tau_i^+) = (1+k_2)y(\tau_i), & i = 1, 2, \cdots, \\ \dot{z}(\tau_i^+) = (1+k_3)z(\tau_i), \end{cases} \tag{8}$$

where k_1, k_2 and k_3 are control parameters, $\tau_{i+1} - \tau_i = d$ for all $i = 0, 1, 2, \ldots$. In the following, we will find out whether system (8) satisfies the conditions of theory 1. Let $V(t,X) = x^2 + y^2 + z^2$, it is easy to find that, (H_1) of theorem 1 holds.

$$\begin{aligned} L(V(t,X)) = & -2ax^2 + 2axy \\ & + 2cy^2 - 2bz^2 + \beta^2(x^2 + y^2 + z^2) \\ & \leq (\beta^2 - a)x^2 + (\beta^2 + 2c + a)y^2 \\ & + (\beta^2 - 2b)z^2. \end{aligned}$$

Let $\lambda_1(t) = \max\{\left|\beta^2 - a\right|, \left|\beta^2 + 2c + a\right|, \left|\beta^2 - 2b\right|\}$, we can find

$L(V(t,X(t))) \leq \lambda_1(t)V(t,X(t))$, let $H(t, V(t,X)) = \lambda_1(t)V(t,X(t))$, condition (H_2) of theorem 1 holds for all $t \in R_{t_0} \backslash \Gamma$. And for $i = 0, 1, 2, \ldots$, we find

$$\begin{aligned} V(\tau_i, X(\tau_i^+)) &= (1+k_1)^2 x^2(\tau_i) + (1+k_2)^2 y^2(\tau_i) \\ &\quad + (1+k_3)^2 z^2(\tau_i) \\ &\leq (1+k)^2 V(\tau_i, X(\tau_i)) \end{aligned}$$

holds, where $(1+k)^2 = \max\{(1+k_1)^2, (1+k_2)^2, (1+k_3)^2\}$, $-2 < k < 0$.

Let $\psi_i(V(t,X)) = (1+k)^2 V(\tau_i, X(\tau_i))$, then (H_3) of theorem 1 holds. So we have the following comparison system of system (8)

$$\begin{cases} \dot{X} = \lambda_1(t)X, & t \neq \tau_i, \\ X(\tau_i^+) = (1+k)^2 X(\tau_i), & i = 1, 2, \ldots, \\ X(t_0^+) = X_0 \geq 0, \end{cases} \tag{9}$$

where $X = (x, y, z)^{\mathrm{T}}$, so the trivial solution's stability properties of comparison system (9) imply the trivial solution's corresponding stochastic stability properties of system (8) . From theorem 2, if $\lambda_1(t)d + \ln(\gamma(1+k)^2) \leq 0$ i.e. $0 \leq d \leq -(\ln\gamma + 2\ln(1+k))/\lambda_1(t)$ satisfies, system (9) is asymptotically stable, from which, system (8) is stochastically asymptotically stable. Let $a = 36$, $b = 3$, $c = 28$, $\beta = 0.01$, $\gamma = 1$, then $\lambda_1(t) = 92.0001$, Fig. 2 shows the stable region of system (8). The whole region under the curve is the stable region.

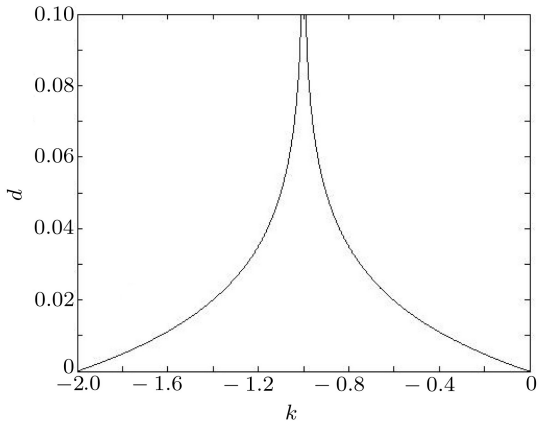


Fig. 2. The estimate of boundaries of stable region in Lü system.

Numerical experimental results are shown in Fig. 3 with $k_1 = -1.2$, $k_2 = k_3 = -1.3$ and $d = 0.03$, which are under the curve in Fig. 2. Figure 3 shows the evolution of $x(t)$, $y(t)$ and $z(t)$. We can see that the system approaches the origin as time goes by.

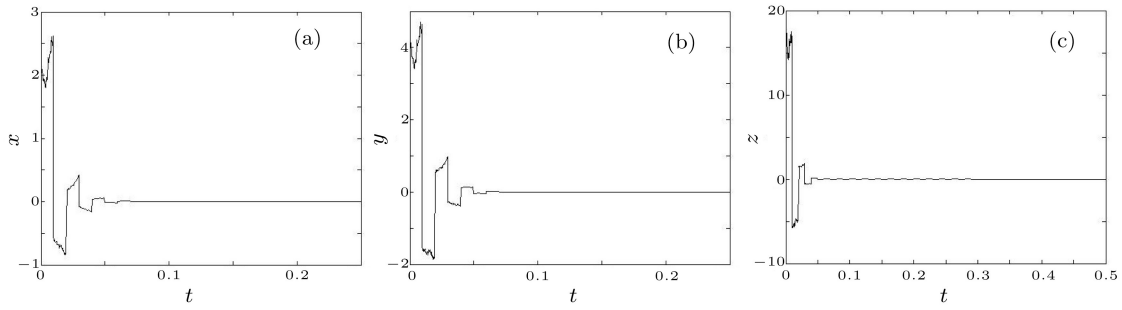


Fig. 3. Time history of system (8) under the stable state.

However, when parameters are chosen as $k_1 = -1.5$, $k_2 = k_3 = -1.6$ and $d = 0.03$, which are outside the stable region in Fig. 2, the system under control would not approach the origin as time goes by, as can be found in Fig. 4.

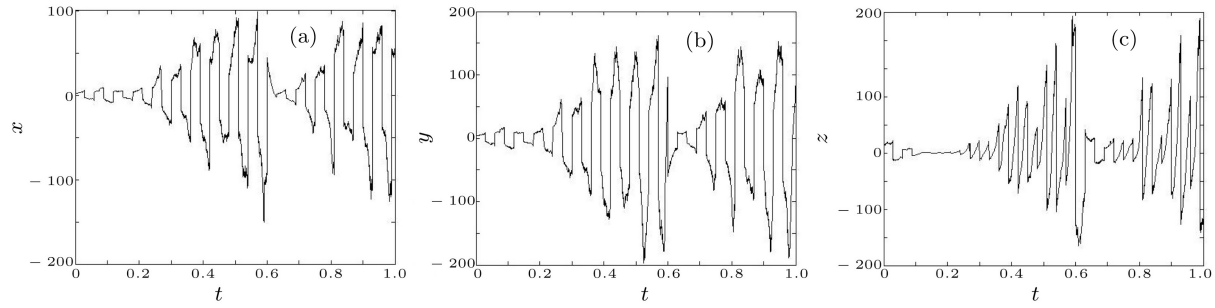


Fig. 4. Time history of system (8) under the unstable state.

5. Conclusion

In this paper, we study the stochastic asymptotical stability of a nonlinear system with both noise perturbations and impulsive effects, and a comparison theory is derived to study the stochastic asymptotical stability. The chaotic Lü system with white-noise excitation is taken as an example for the proposed control scheme and numerical simulation is shown to be consistent with the theoretical analysis. The main contribution of this work is that the most important stability in stochastic theory, stochastic asymptotical stability, is derived for the first time in stochastic impulsive differential systems. Furthermore, it is a nice tool for its general applicability, as can be showed in the Lü system with both white-noise perturbations and impulsive effects.

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