The 0-1 test algorithm for chaos and its applications*

Sun Ke-Hui(孙克辉)[a)†], Liu Xuan(刘 璇)[a)], and Zhu Cong-Xu(朱从旭)[b)]

[a)] *School of Physics Science and Technology, Central South University, Changsha* 410083, *China*
[b)] *School of Information Science and Engineering, Central South University, Changsha* 410083, *China*

(Received 4 March 2010; revised manuscript received 21 May 2010)

To determine whether a given deterministic nonlinear dynamic system is chaotic or periodic, a novel test approach named zero-one (0-1) test has been proposed recently. In this approach, the regular and chaotic motions can be decided by calculating the parameter K approaching asymptotically to zero or one. In this study, we focus on the 0-1 test algorithm and illustrate the selection of parameters of this algorithm by numerical experiments. To validate the reliability and the universality of this algorithm, it is applied to typical nonlinear dynamic systems, including fractional-order dynamic system.

Keywords: chaos, 0-1 test, fractional-order system, Lyapunov exponent
PACC: 0545

1. Introduction

A long-standing fundamental issue in the theory of chaotic dynamics analysis is the determination of whether a nonlinear dynamic system is regular, chaotic or random. Many testing methods have been proposed, such as calculating the maximal Lyapunov exponent, observing the phase trajectory directly, calculating the power spectrum and so on. The usual testing method is the calculation of the maximal Lyapunov exponent. A positive maximal Lyapunov exponent indicates chaos. This approach has been widely used for integer-order dynamical systems whose equations are known. If the equations are not known or one wishes to examine experimental data or a fractional-order dynamical system,[1−10] one has to employ the embedding theory of Takens and phase space reconstruction[11] to calculate the maximal Lyapunov exponent.

Recently, a new test approach for distinguishing regular from chaotic dynamics in deterministic dynamical systems, named '0-1 test', was proposed by Gottwald and Melbourne.[12] The input is the time series of a relevant variable and the output is zero or one. This test has two main advantages over computing the maximal Lyapunov exponent. (i) This test applies directly to time series data, so that phase space reconstruction is not required. Moreover, the form and nature of the underlying dynamical system is irrelevant; and the test applies equally well to continuous time systems, discrete time systems, experimental data,[13] maps, integer-order systems, and fractional-order systems. (ii) It is a binary test (in principle, the test only yields 0 or 1), so a numerically computed value, say 0.01, yields a definite conclusion, whereas such a value for the maximal Lyapunov exponent would not.

In order to verify the availability and reliability of the '0-1 test', we focus especially on the selection of the parameters and the application to some typical chaotic systems in this study. This paper is organised as follows. In Section 2, we briefly review the '0-1 test' algorithm and the relevant details of its implementation. In Section 3, the '0-1 test' is applied to three typical dynamical systems and compared with the method of computing the maximal Lyapunov exponent. In Section 4, we discuss the results and present our conclusions.

2. Principle of the 0-1 test algorithm

2.1. The 0-1 test algorithm

A set of discrete data $\phi(n)$ sampled at times $n = 1, 2, 3, \ldots$, $\phi(n)$ represents a one-dimensional observable data set. Choose a constant $c \in R^+$ at random and define[14]

$$p(n) = \sum_{j=1}^{n} \phi(j)\cos(\theta(j)), \quad n = 1, 2, 3, \ldots, \quad (1)$$

*Project supported by the National Natural Science Foundation of of China (Grant No. 60672041).
†Corresponding author. E-mail: kehui@csu.edu.cn
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$$s(n) = \sum_{j=1}^{n} \phi(j) \sin(\theta(j)), n = 1, 2, 3, \ldots, \quad (2)$$

where

$$\theta(j) = jc + \sum_{i=1}^{j} \phi(i), j = 1, 2, 3, \ldots, n. \quad (3)$$

Then, on the basis of function $p(n)$ (or alternatively of the function $s(n)$), define the mean square displacement as

$$M(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2, \quad n = 1, 2, 3, \ldots. \quad (4)$$

If the behaviour of $p(n)$ ($s(n)$ behaves in the same way) is Brownian, then $M(n)$ grows linearly in time. If the behaviour of $p(n)$ is bounded, then $M(n)$ is bounded. The asymptotic growth rate is defined by

$$K = \lim_{n\to\infty} \frac{\log M(n)}{\log n}. \quad (5)$$

Just as the same for continuous time system, K=0 means that the system is regular, and K=1 means that the system is chaotic.

Suppose that we have a finite amount of data $\phi(n)$, $1 \le n \le N$, then we define

$$M(n) = \frac{1}{N-n} \sum_{j=1}^{N-n} [p(j+n) - p(j)]^2. \quad (6)$$

If $1 \ll n \ll N$, it follows that $M(n)$ scales with n in a way similar to the case described by Eq. (4). Therefore, by plotting $\log(M(n) + 1)$ against $\log n$ for $1 \le n \le N/10$, the test still consists of checking whether K is close to 0 or close to 1.

Note that, another interesting feature of the '0–1 test' is that the inspection of the dynamics of the (p, s)-trajectories provides a simple visual test of whether the underlying dynamics is chaotic or nonchaotic.[15] Namely, bounded trajectories in the (p, s)-plane imply regular dynamics, whereas Brownian-like (unbounded) trajectories imply chaotic dynamics.

2.2. The parameters selection of the 0-1 test algorithm

Three parameters need to be chosen in the 0-1 test algorithm, they are c, N, and n.

For a short data set, there is an inconvenient resonance phenomenon, where a given choice of c may resonate with frequencies in the underlying dynamics. For example, if the signal is 2π-periodic and we choose c as an integer, then a simple argument using the Fourier series for $\phi(n)$ shows that typically $p(n)$ will grow linearly yielding K=2. A near-integer choice of c results in K=0 eventually, but the convergence is very slow. For a small iteration number, the computed value of K is likely to be closer to 1 than to 0. If the 'bad' choices of c are dense, the situation resulted from a quasiperiodic dynamic case is even worse compared with a chaotic dynamics. Nevertheless, there is zero probability of making a bad choice in theory. In practice, numerical experimentation shows that even for short data series, the bad choices of c are rare but do occur from time to time. Our resolution of this problem is to choose several values of c at random, and compute K for each choice of c. Then we take the median value of K (We do not take the average of K, because if c does fail, it will fail quite badly).

To illustrate the selection of parameter of the 0-1 test algorithm, Hénon map was taken as an example, which is defined by

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n, \\ y_{n+1} = bx_n, \end{cases} \quad (7)$$

here, a and b are the system parameters. When $1.07 \le a \le 1.4$, $b = 0.3$, the system is chaotic. We compare the effects of taking m different values of c randomly within the range from 1 to 11 in Fig. 1. Here, m=1, 10, 100 and 1000. It is clear that taking one choice of c is not effective and that taking 10 values of c is a dramatic improvement, though some periodic windows are still poorly defined. This is remedied by going to 100 values of c. When m increasing to 1000 does not make a significant further improvement but reduce efficiency. Taking all of the above into account, we choose m=100.

The resonance phenomenon can be clearly shown in Fig. 2, especially in Fig. 2(a), where it is a quasiperiodic dynamics. That is why we must choose several values of c at random, computing K for each choice of c, and then take the median value of K.

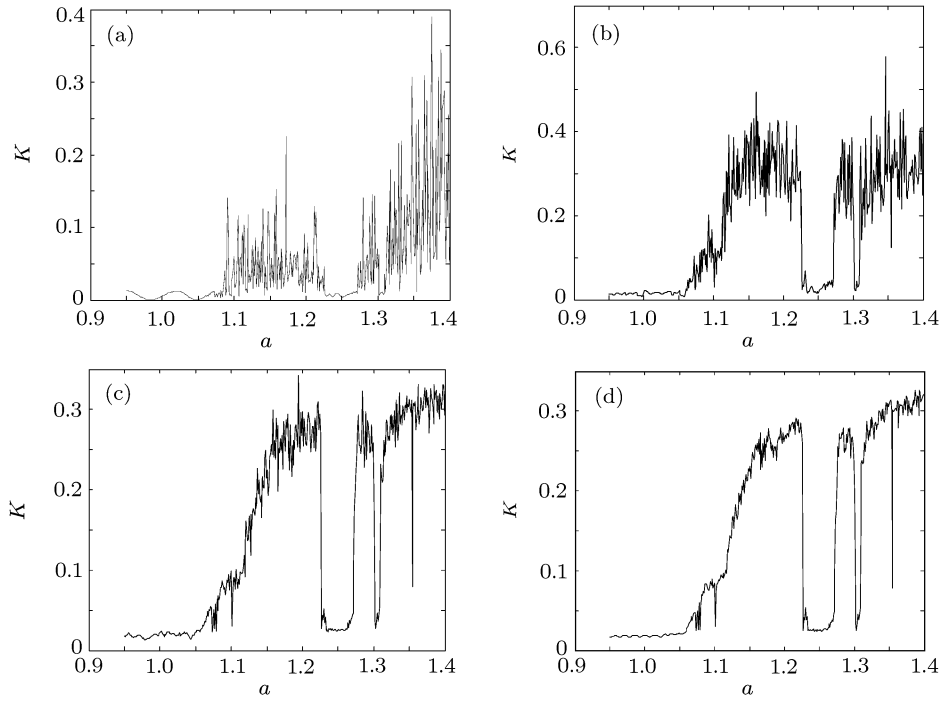


Fig. 1. Curves of K versus a for the Hénon map. The median value of K is taken from m values of c where (a) m=1, (b) m=10, (c) m=100, (d) m=1000.

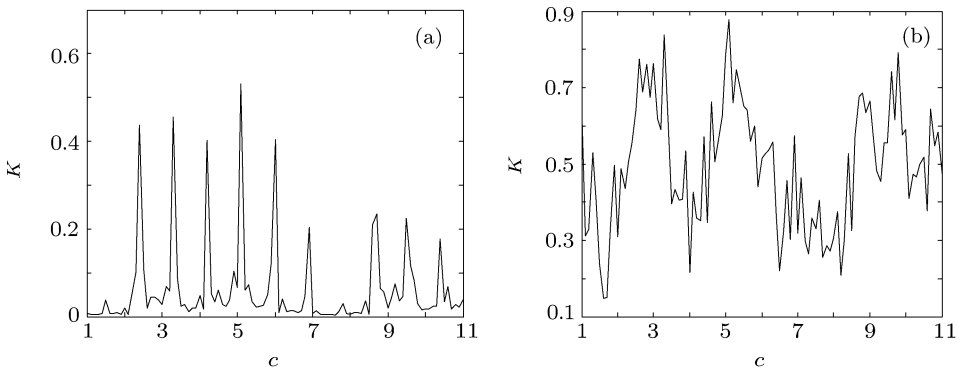


Fig. 2. Curves of K versus c for the Hénon map. (a) a=1.24, (b) a=1.4.

For the selection of N, it should tend to infinity in theory, but this is not feasible in practice. As the algorithm is achieved with the loop, and there are nested loops. The greater the sequence length selected, the longer the loop will increase, and the execution time of the algorithm will significantly increase. So the efficiency of the implementation of the algorithm will decrease significantly. Taking all of above into account, an appropriate value of the sequence length N should be selected long enough.

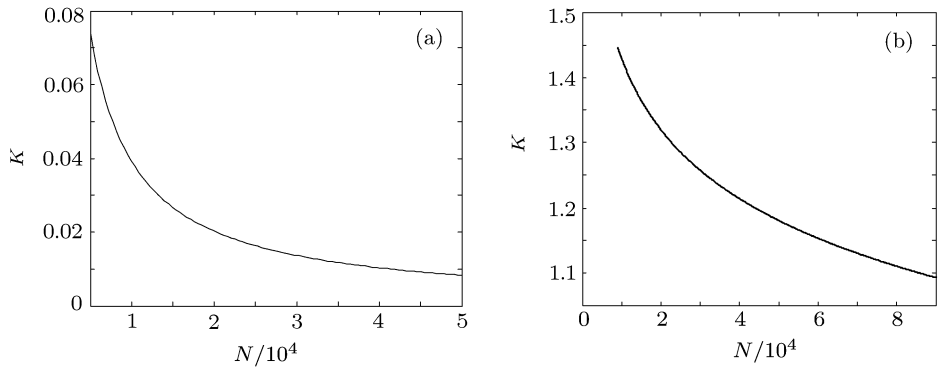


Fig. 3. Curves of K versus N for the Hénon map. (a) a=1.24, (b) a=1.4.

The impact of parameter N on the value of K is shown in Fig. 3. When the other parameters are determined, both in periodic or chaotic system, K decreases with the increasing of N. The greater the N is, the closer to ideal value of K.

For the selection of n, there is no special requirement. Based on the principle of $1 \leq n \leq N/10$, any choice of n is sufficient. The impact of parameter n on the value of K is shown in Fig. 4. When the other parameters are determined, the value of K shows an oscillating increase with an increase of n when the dynamics is periodic, but when the dynamic system is chaotic, the impact is not significant. Thus an appropriate value of n should be selected as small as possible.

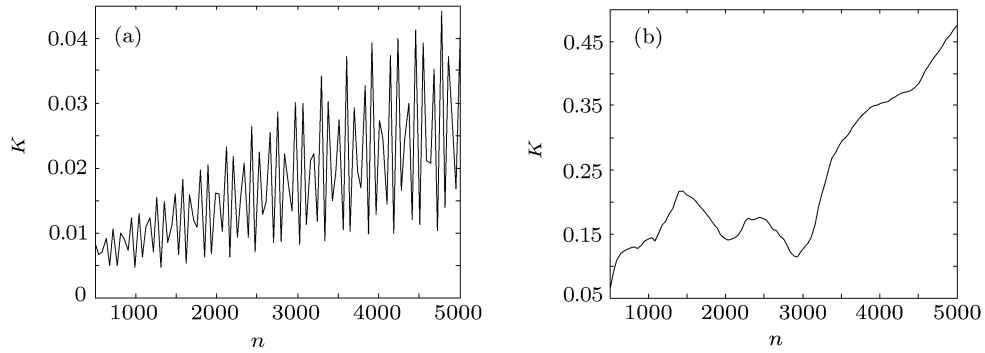


Fig. 4. Curves of K versus n for the Hénon map. (a) a=1.24, (b) a=1.4.

The test is a binary test. The output is 0 or 1, with no grey areas in principle. However, there are grey areas in practice for any test. The suitable parameter can only reduce the grey areas, but can not avoid it.

3. Applications of the 0-1 test algorithm

To demonstrate the reliability and universality of the test, we apply the test to Hénon map, integer-order simplified Lorenz system and its fractional-order counterpart.

3.1. 0-1 test for Hénon map

Hénon map is a typical two-dimensional discrete chaotic system, which is described by Eq. (7). Set b=0.3, x_0=0.5, y_0=0.5, and let a vary from 0.95 to 1.40. The step size for a is 0.001. The plot of the largest Lyapunov exponent (LE) versus a is shown in Fig. 5. It shows that system (7) is chaotic over most of the range $a \in$(1.07, 1.40) with some periodic windows. Next, we carry out the 0-1 test with a=1.24 and a=1.4 respectively, and we obtain K=0.0104 at a=1.24, and K=0.6173 at a=1.4. The state space plots are shown in Fig. 6 and the translation components (p, s) are shown in Fig. 7.

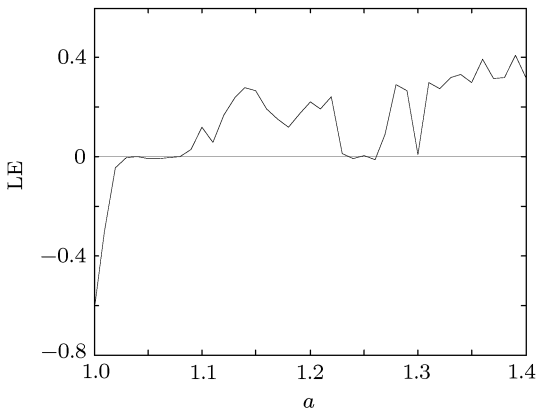


Fig. 5. The largest Lyapunov exponent versus a for the Hénon map.

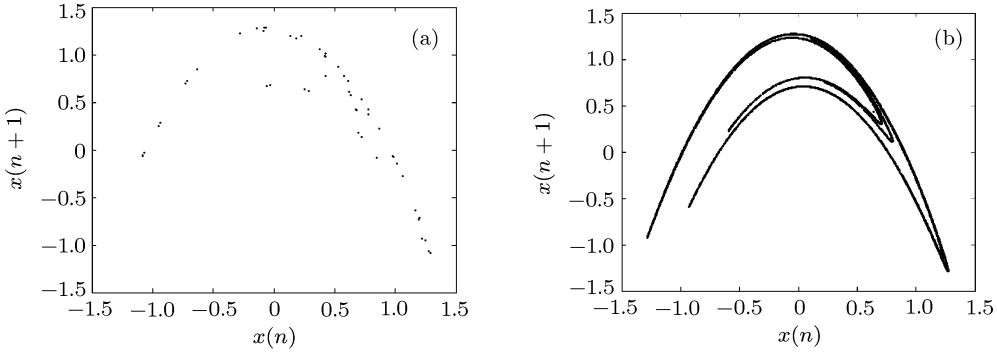


Fig. 6. State space plots for the Hénon map. (a) a=1.24, (b) a=1.4.

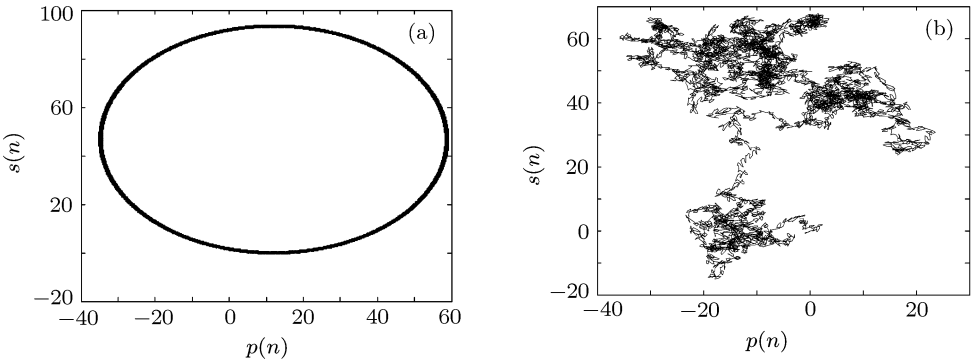


Fig. 7. Dynamics of the translation components (p, s) of the Hénon map. (a) a=1.24, (b) a=1.4.

3.2. 0-1 test for the integer-order simplified Lorenz system

The integer-order simplified Lorenz system is described by[16]

$$\begin{cases} \dot{x} = 10(y - x), \\ \dot{y} = (24 - 4a)x - xz + ay, \\ \dot{z} = xy - 8z/3, \end{cases} \tag{8}$$

where a is the system parameter. When $a \in (-1.59, 7.75)$, it is chaotic with some periodic windows.[16] The largest Lyapunov exponent versus a is shown in Fig. 8. Now, we carry out the 0-1 test with a=7.2 and $a = -1$ respectively, and we obtain K=0.0736 at a=7.2, and K=0.7630 at $a = -1$. The state space plots are shown in Fig. 9 and the translation components (p, s) are shown in Fig. 10.

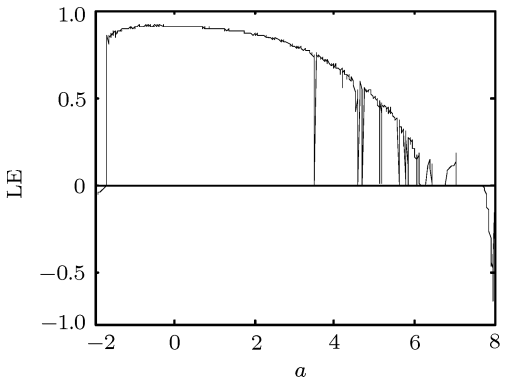


Fig. 8. The largest Lyapunov exponent versus a for the integer-order simplified Lorenz system.

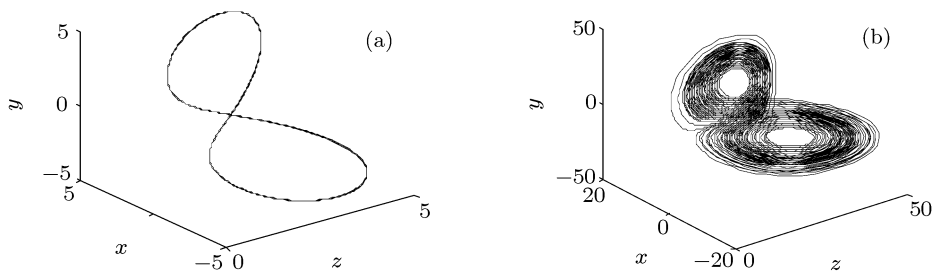


Fig. 9. State space plots for the integer-order simplified Lorenz system. (a) $a = 7.2$, (b) $a = -1$.

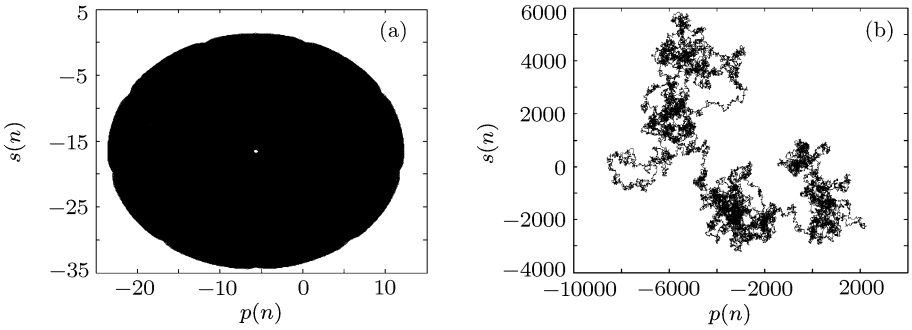


Fig. 10. Dynamics of the translation components (p, s) of the integer-order simplified Lorenz system. (a) $a = 7.2$, (b) $a = -1$.

3.3. 0-1 test for the fractional-order simplified Lorenz system

Consider the fractional-order simplified Lorenz system given by[17]

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}} = 10(y-x), \\ \dfrac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\beta}} = (24-4a)x - xz + ay, \\ \dfrac{\mathrm{d}^{\gamma}z}{\mathrm{d}t^{\gamma}} = xy - \frac{8}{3}z, \end{cases} \tag{9}$$

where α, β, γ are the fractional-orders. In this paper, we take $0.95 < \alpha = \beta = \gamma = q \leq 1.15$. Let a=5 and q vary from 0.95 to 1.15. The plot of the largest Lyapunov exponent versus q is shown in Fig. 11. It shows that system (9) is chaotic over most of the range $q \in (0.95, 1.15)$ but with some periodic windows. We carry out the 0-1 test with q=1.12 and q=0.98 respectively, and we obtain K=0.0807 at q=1.12, and K=0.9348 at q=0.98. The state space plots are shown in Fig. 12 and the translation components (p, s) are shown in Fig. 13.

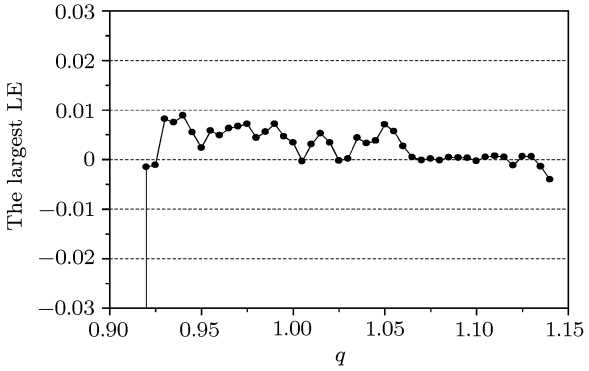


Fig. 11. The largest Lyapunov exponent versus q for the fractional-order simplified Lorenz system.

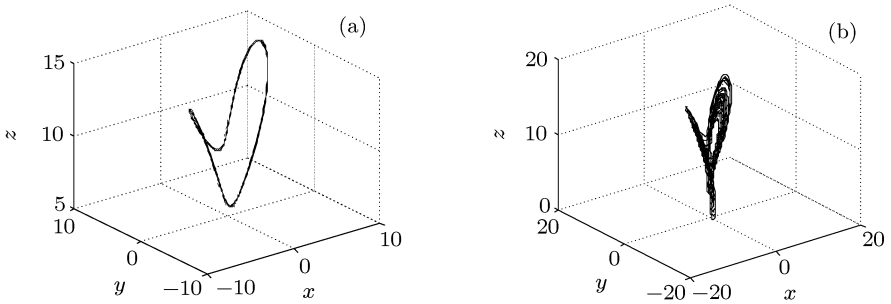


Fig. 12. State space plots of the fractional-order simplified Lorenz system with different q. (a) q=1.12, (b) q=0.98.

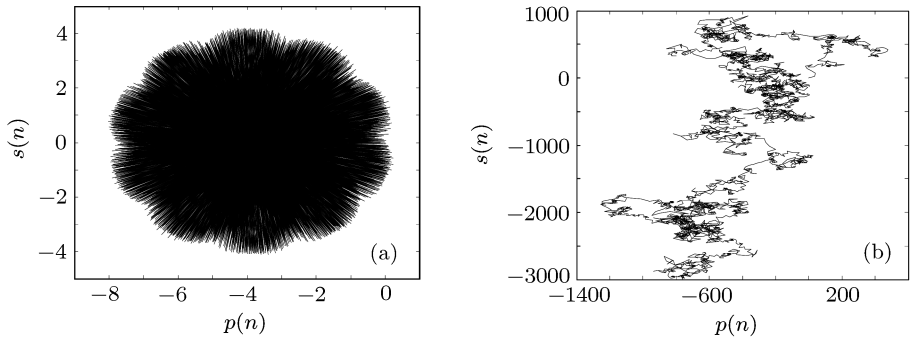


Fig. 13. Dynamics of the translation components (p, s) of the fractional-order simplified Lorenz system. (a) q=1.12, (b) q=0.98.

3.4. Discussions

For the discrete system, Fig. 1, Fig. 6 and Fig. 7 show the application of the test to Hénon map. The periodic window is obvious because the test is focused on discrete data, although K does not reach 1 when the system is chaotic. This is due to insufficient length of the time series. $K \leq 0.1$ yields the definite conclusion that the dynamics is not chaotic.

For the periodic windows of a continuous system (either an integer-order or a fractional-order), the state space plots are limit cycle and the translation components (p, s) depict bounded trajectories. The influence of the parameters on the value of K is the same. In both cases, the value of K approaches 0, clearly indicating the presence of regular dynamics (periodic or quasi-periodic).

For the chaotic regions of a continuous system (either an integer-order or a fractional-order), the state space plots are a strange attractor and the translation components (p, s) depict Brownian-like (unbounded) trajectories. The influence of the parameters on the value of K is also the same. In both cases, the value of K approaches 1, clearly indicating the presence of chaotic dynamics.

Although the 0-1 test has many advantages over calculating the largest Lyapunov exponent, and it avoids certain well-documented drawbacks of traditional tests,[18,19] it will not replace traditional methods completely.

4. Conclusions

We have investigated a simple, convenient test for chaos and tested its efficiency with an analysis of different examples. The computation cost is low, both in terms of programming efforts and in terms of actual computation time. This distinction is clear by means of the diagnostic variable K which has values either close to 0 or close to 1. With the sufficient length of the time series, $K \leq 0.1$ indicates that the dynamics is not chaotic, and $K > 0.1$ indicate that the dynamics is chaotic. The advantage of this method is that it applies directly to the time series data. The most powerful aspect of this method is that it is independent of the nature of the data under consideration. In particular the equations of the underlying dynamical system do not need to be known, and there is no practical restriction on the dimension of the underlying data. Since the test is a binary test, it can only distinguish between nonchaotic and chaotic dynamics, and there are grey areas in practice. So we should manage to avoid it in the practice.

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