Nonlinear feedback synchronisation control between fractional-order and integer-order chaotic systems

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This paper focuses on the synchronisation between fractional-order and integer-order chaotic systems. Based on Lyapunov stability theory and numerical differentiation, a nonlinear feedback controller is obtained to achieve the synchronisation between fractional-order and integer-order chaotic systems. Numerical simulation results are presented to illustrate the effectiveness of this method.

Keywords: chaos synchronisation, fractional-order chaotic system, nonlinear feedback control, numerical differentiation

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1. Introduction

Chaos is a ubiquitous and extremely complex nonlinear phenomenon, and has been intensively studied in the last three decades.[1] Since Pecora and Carroll[2] first proposed the drive-response method to achieve synchronisation of chaotic systems in 1990, a number of synchronisation approaches have been developed such as linear and nonlinear feedback control,[3−6] Ott–Grebgi–Yorke method,[7] sliding mode control,[8] adaptive control[9,10] and linear matrix inequality (LMI) technique.[11] During the last two decades, scholars have been devoted to studying chaos control and synchronisation problems. The main chaotic systems are the following: Lorenz chaotic system,[12] Chen and Lü chaotic system[13] and Liu chaotic system.[14] The systems referred to in these references are integer-order chaotic systems. In Ref. [15], the author studied the fractional-order chaotic system. As is known, fractional calculus has 300 years of history in mathematics, however, the research of the fractional-order chaotic system is just beginning. Synchronisation of chaotic systems has long been based on two fractional-order systems,[16−18] or two integer-order systems.[19−21] However, synchronisation between the integer-order chaotic system and the fractional-order chaotic system is thoroughly a new domain. In this paper, based on Lyapunov stability theory and numerical differentiation, a nonlinear feedback controller is obtained to achieve the synchronisation between the fractional-order chaotic system and the integer-order chaotic system. Numerical simulation results are presented to illustrate the effectiveness of this method.

2. Fractional derivative and its approximate calculation

There are several definitions for fractional derivative, but the following definition is often used in many papers:

$$D_*^\alpha y(x) = J^{n-\alpha} y^{(n)}(x), \quad \alpha > 0, \tag{1}$$

where $n-1 < \alpha < n$ (n is an integer and α is a fraction); $y^{(n)}(x)$ is the general n-order derivative; J^β ($\beta = n-\alpha$) is the β-order Riemann–Liouville integral operator, which is expressed as

$$J^\beta z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) \mathrm{d}(t), \beta > 0. \tag{2}$$

The operator D_*^α is generally called 'α-order Caputo differential operator'.[22]

Riemann–Liouville fractional derivative for a function $f(t)$ is expressed as

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-\tau)^{\beta-1} f(\tau) \mathrm{d}(\tau), \quad (\beta = n-\alpha). \tag{3}$$

Laplace transform for the formula (3) is written as

$$L\left[\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right] = s^\alpha L[f(t)] - \sum_{k=0}^{n-1} s^k \left[\frac{\mathrm{d}^{\alpha-1-k} f(t)}{\mathrm{d}t^{\alpha-1-k}}\right]_{t=0}. \tag{4}$$

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If the initial value of function $f(t)$ is zero, formula (4) can be expressed as

$$L\left[\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right] = s^\alpha L[f(t)]. \tag{5}$$

The purpose of Laplace transform for formula (3) is to transform the time domain into the frequency domain so that numerical calculation can be performed.[23] In Ref. [24], in the deduction of the expansion of $1/s^\alpha$ $(0.1 < \alpha < 0.9)$, 2 dB and 3 dB are set to be the approximation errors. In this paper, α and the approximation error are set to be 0.9 and 2 dB, respectively. Therefore,

$$\frac{1}{s^{0.9}} \approx \frac{2.2675(s+1.292)(s+215.4)}{(s+0.01292)(s+2.154)(s+359.4)}. \tag{6}$$

3. The synchronisation between fractional-order unified chaotic system and the Chen chaotic system

For the fractional-order unified chaotic system, we have

$$\begin{aligned}
\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} &= (25\delta + 10)(y - x),\\
\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} &= (28 - 35\delta)x - xz + (29\delta - 1)y,\\
\frac{\mathrm{d}^\alpha z}{\mathrm{d}t^\alpha} &= xy - \frac{1}{3}(8 + \delta)z,
\end{aligned} \tag{7}$$

where x, y, and z are state variables; δ is a parameter with $\delta \in [0, 1]$. Choose $\alpha = 0.9$, when $\delta \in [0, 0.8)$, the system (7) is a generalised fractional-order Lorenz chaotic system; when $\delta = 0.8$, the system (7) is a generalised fractional-order Lü chaotic system; when $\delta \in (0.8, 1]$, the system (7) is a generalised fractional-order Chen chaotic system.

For $\alpha = 0.9$ and $\delta = 0.5$, the generalised fractional-order Lorenz chaotic attractor is shown in Fig. 1(a).

For $\alpha = 0.9$ and $\delta = 0.8$, the generalised fractional-order Lü chaotic attractor is shown in Fig. 1(b).

For $\alpha = 0.9$ and $\delta = 0.9$, the generalised fractional-order Chen chaotic attractor is shown in Fig. 1(c).

For $\alpha = 1$ and $\delta = 1$, the Chen chaotic attractor is shown in Fig. 1(d).

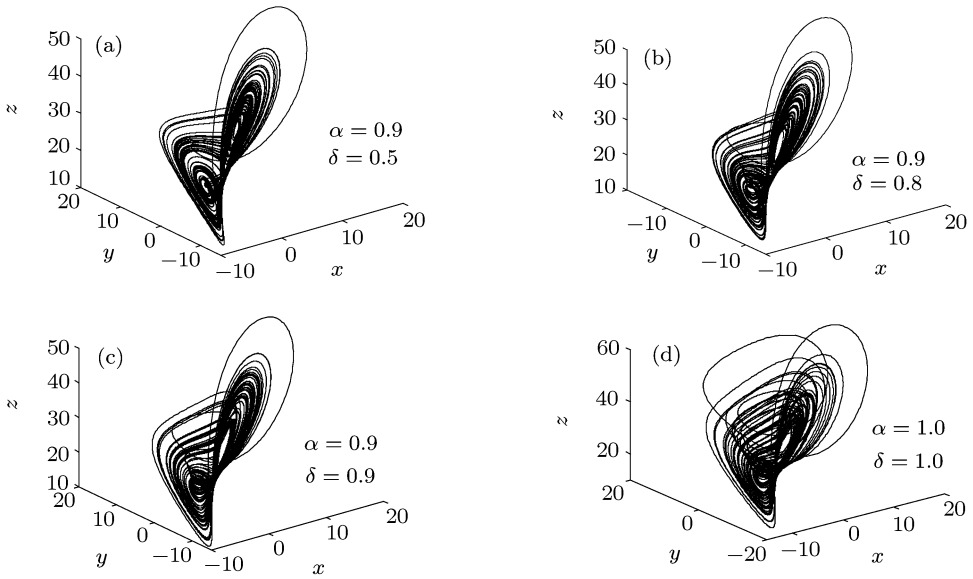


Fig. 1. Chaotic attractors.

In this paper, the α-order unified chaotic system is used as a driving system, which is described as

$$\begin{aligned}
\frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} &= (25\delta + 10)(y_1 - x_1),\\
\frac{\mathrm{d}^\alpha y_1}{\mathrm{d}t^\alpha} &= (28 - 35\delta)x_1 - x_1 z_1 + (29\delta - 1)y_1,\\
\frac{\mathrm{d}^\alpha z_1}{\mathrm{d}t^\alpha} &= x_1 y_1 - \frac{1}{3}(8 + \delta)z_1,
\end{aligned} \tag{8}$$

where x_1, y_1, and z_1 are state variables, δ is a parameter with $\delta \in [0, 1]$.

The response system is the Chen system under control, which is described as

$$\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + u_1,\\
\dot{y}_2 &= (c - a)x_2 - x_2 z_2 + c y_2 + u_2,\\
\dot{z}_2 &= x_2 y_2 - b z_2 + u_3,
\end{aligned} \tag{9}$$

where x_2, y_2, and z_2 are state variables, $\boldsymbol{u}(t) = (u_1, u_2, u_3)^{\mathrm{T}}$ is a controller, a, b, and c are parameters, with $a = 35$, $b = 3$, and $c = 28$ in this paper. Let $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, and $e_3 = z_2 - z_1$, where $\boldsymbol{e}(t) = (e_1, e_2, e_3)^{\mathrm{T}}$ is synchronisation error vector. Our object is to design a controller $\boldsymbol{u}(t) = (u_1, u_2, u_3)^{\mathrm{T}}$ so that the response system (9) asymptotically approaches the driving system (8) and finally the synchronisation will be accomplished, and the limit of the synchronisation error vector $\boldsymbol{e}(t) = (e_1, e_2, e_3)^{\mathrm{T}}$ approaches zero,

$$\lim_{t\to\infty} \boldsymbol{e}(t) = 0. \tag{10}$$

The synchronisation error system between the driving system (8) and the response system (9) can be expressed as

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_2 - \dot{x}_1,\\
\dot{e}_2 &= \dot{y}_2 - \dot{y}_1,\\
\dot{e}_3 &= \dot{z}_2 - \dot{z}_1,
\end{aligned} \tag{11}$$

where $\dot{x}_2$, $\dot{y}_2$, and $\dot{z}_2$ are obtained from the response system (9), while no exact expressions of $\dot{x}_1$, $\dot{y}_1$, and $\dot{z}_1$ can be obtained from the driving system (8). Therefore, numerical differentiation method is used to obtain $\dot{x}_1$, $\dot{y}_1$, and $\dot{z}_1$.

According to the definition of derivative, derivative is approximately expressed using difference quotient as

$$f'(a) \approx \frac{f(a + h) - f(a)}{h}, \tag{12}$$

$$f'(a) \approx \frac{f(a) - f(a - h)}{h}, \tag{13}$$

where $h(> 0)$ is a small increment. Formulae (12) and (13) are called the pre-difference formula and the post-difference formula, respectively. The post-difference formula is used in this paper.

The Lyapunov function for the synchronisation error system (11) is chosen as

$$V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) > 0. \tag{14}$$

Then, taking the derivation of Eq. (14) with respect to time results, we obtain

$$\begin{aligned}
\dot{V}(t) &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3\\
&= e_1(\dot{x}_2 - \dot{x}_1) + e_2(\dot{y}_2 - \dot{y}_1) + e_3(\dot{z}_2 - \dot{z}_1)\\
&= e_1[a(y_2 - x_2) - \dot{x}_1 + u_1] + e_2[(c - a)x_2 - x_2 z_2 + c y_2 - \dot{y}_1 + u_2] + e_3[x_2 y_2 - b z_2 - \dot{z}_1 + u_3]\\
&= e_1[a(y_1 - x_1) + a e_2 - a e_1 - \dot{x}_1 + u_1] + e_2[(c - a)x_1 + (c - a)e_1 - e_1 e_3 - x_1 e_3 - x_2 z_1\\
&\quad + c e_2 + c y_1 - \dot{y}_1 + u_2] + e_3[x_2 y_1 + x_1 e_2 + e_1 e_2 - b z_1 - b e_3 - \dot{z}_1 + u_3].
\end{aligned} \tag{15}$$

For the above object, the following nonlinear feedback controllers are designed

$$\begin{aligned}
&u_1 = \dot{x}_1 - a(y_1 - x_1) - k_1 e_1, \quad u_2 = \dot{y}_1 - (c - a)x_1 - c(e_1 + e_2) + x_2 z_1 - c y_1 - k_2 e_2,\\
&u_3 = \dot{z}_1 - x_2 y_1 + b z_1 - k_3 e_3.
\end{aligned} \tag{16}$$

Substituting Eqs. (16) into the formula (15), $\dot{V}(t)$ can be expressed as

$$\dot{V}(t) = -(a + k_1)e_1^2 - k_2 e_2^2 - (b + k_3)e_3^2, \tag{17}$$

where k_1, k_2, and k_3 are control parameters. Since $a = 35$ and $b = 3$ are greater than zero, if we choose k_1, k_2, and k_3 also greater than zero, then $\dot{V}(t) < 0$, and then $\lim_{t\to\infty} \boldsymbol{e}(t) = 0$. Finally the synchronisation will be accomplished.

4. Simulation results

In what follows, through calculating system error e (the error of the synchronisation error system), mean value $E(e)$ (the average of synchronisation errors e_1, e_2, and e_3) and system variance $D(e)$ (the average of the variance of synchronisation errors e_1, e_2, and e_3), numerical simulation results show the effectiveness of this method for synchronisation between the fractional-order unified chaotic system and the Chen chaotic system, where

$$e = \sqrt{e_1^2 + e_2^2 + e_3^2}, \quad E(e) = \frac{e_1 + e_2 + e_3}{3} \quad \text{and} \quad D(e) = \frac{(e_1 - E(e))^2 + (e_2 - E(e))^2 + (e_3 - E(e))^2}{3}.$$

4.1. Case 1 synchronisation between generalised fractional-order Lorenz chaotic system and the Chen chaotic system

In this subsection, synchronisation between the driving system (8) and the response system (9) is demonstrated. The step size in the simulation is chosen as 0.001 s. We choose $\alpha = 0.9$ and $\delta = 0.5$ in the driving system (8). The values of initial conditions for the driving and response systems are $(x_1(0), y_1(0), z_1(0)) = (1, 2, 3)$ and $(x_2(0), y_2(0), z_2(0)) = (-10, 10, 15)$, respectively; and the control parameters are chosen to be $k_1 = k_2 = k_3 = 1000$. Numerical simulation results are shown in Fig. 2.

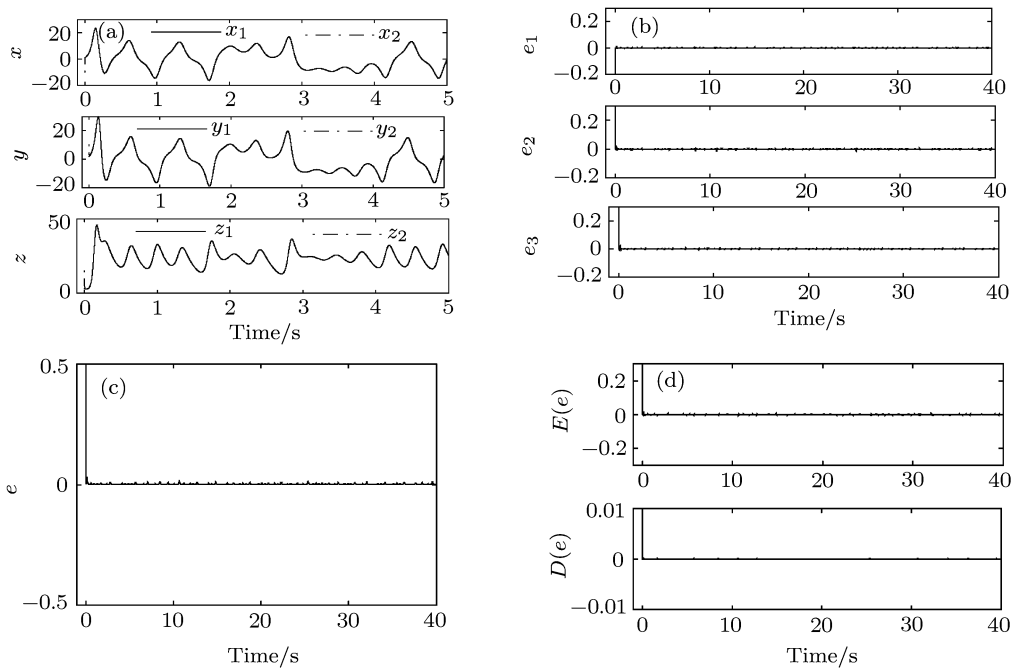


Fig. 2. (a) State trajectories of the driving system (8) and the response system (9). (b) Variations of synchronisation errors e_1, e_2, and e_3 with time t. (c) Variation of system error e with time t. (d) Variations of $E(e)$ and $D(e)$ with time t.

4.2. Case 2 synchronisation between generalised fractional-order Lü chaotic system and the Chen chaotic system

In this subsection, synchronisation between the driving system (8) and the response system (9) is demonstrated. The step size in the simulation is chosen as 0.001 s. We choose $\alpha = 0.9$ and $\delta = 0.8$ in the driving system (8). The values of initial conditions for the driving and response systems are $(x_1(0), y_1(0), z_1(0)) = (1, 2, 3)$ and $(x_2(0), y_2(0), z_2(0)) = (-10, 10, 15)$, respectively; and the control parameters are chosen to be $k_1 = k_2 = k_3 = 1000$. Numerical simulation results are shown in Fig. 3.

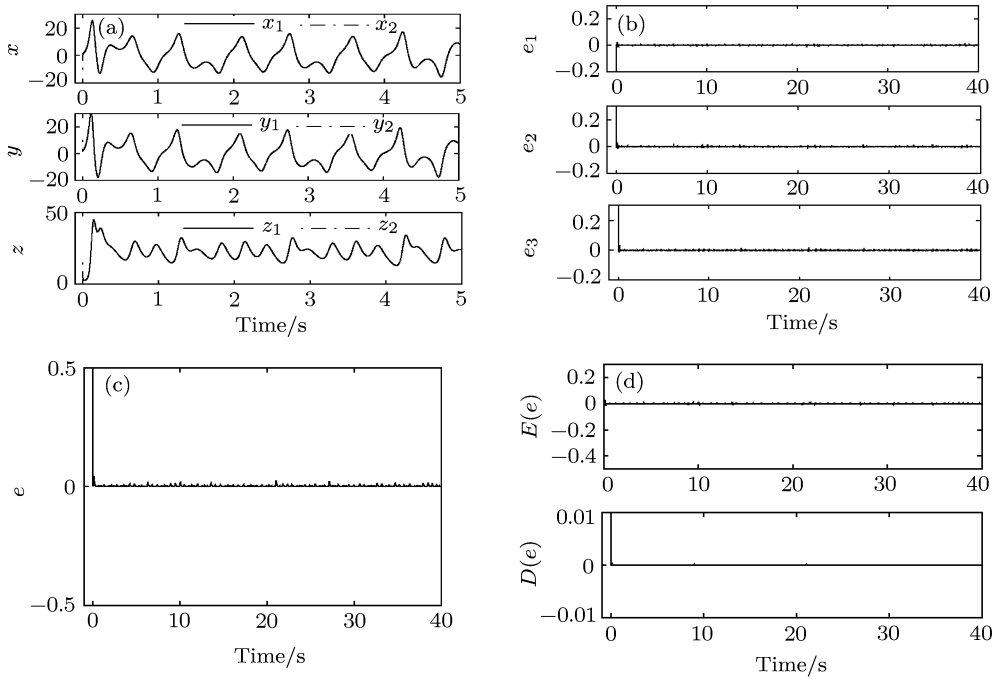


Fig. 3. (a) State trajectories of the driving system (8) and the response system (9). (b) Variations of synchronisation errors e_1, e_2, and e_3 with time t. (c) Variation of system error e with time t. (d) Variations of $E(e)$ and $D(e)$ with time t.

4.3. Case 3 synchronisation between generalised fractional-order Chen chaotic system and the Chen chaotic system

In this subsection, synchronisation between the driving system (8) and the response system (9) is demonstrated. The step size in the simulation is chosen as 0.001 s. We choose $\alpha = 0.9$ and $\delta = 0.9$ in the driving system (8). The values of initial conditions for the driving and response systems are $(x_1(0), y_1(0), z_1(0)) = (1, 2, 3)$ and $(x_2(0), y_2(0), z_2(0)) = (-10, 10, 15)$, respectively; and the control parameters are chosen to be $k_1 = k_2 = k_3 = 1000$. Numerical simulation results are shown in Fig. 4.

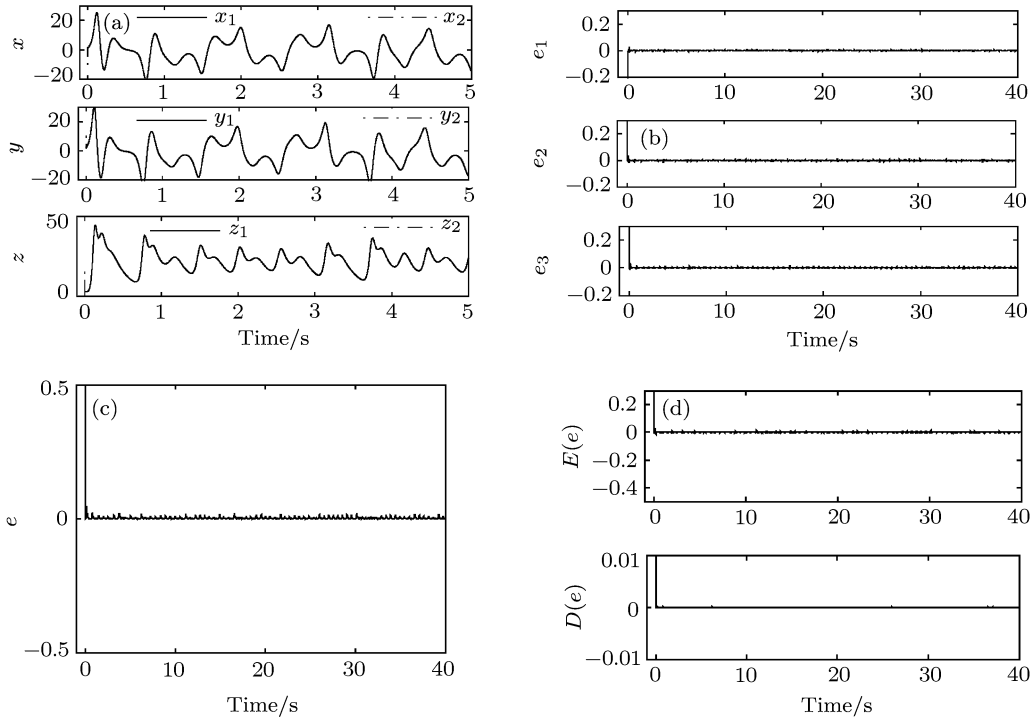


Fig. 4. (a) State trajectories of the driving system (8) and the response system (9). (b) Variations of synchronisation errors e_1, e_2, and e_3 with time t. (c) Variation of system error e with time t. (d) Variations of $E(e)$ and $D(e)$ with time t.

5. Conclusions

First, Lyapunov stability theory and numerical differentiation have been used to obtain the nonlinear feedback controller $u(t)$. Then, the synchronisation between the fractional-order unified chaotic system and the integer-order chaotic system can be achieved via nonlinear feedback control method. Present numerical simulation results reveal that synchronisation errors are not equal to zero, and that system error e, as well as mean value $E(e)$ and system variance $D(e)$ have extremely small values. This is because approximation error in formula (6) is 2 dB and difference quotient is used to express derivative approximately in this paper. Finally, even if the model of the driving system is unknown, as long as the state variables and parameters of the driving system are known, synchronisation between fractional-order chaotic system and the integer-order chaotic system can be accomplished by using numerical calculation and nonlinear feedback control method.

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