Real scalar field scattering in the nearly extremal Schwarzschild–de Sitter space*

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Reasonable approximations are introduced to investigate the real scalar field scattering in the nearly extremal Schwarzschild–de Sitter (SdS) space. The approximations naturally lead to the invertible $x(r)$ and the *global* replacement of the true potential by a Pöshl–Teller one. Meanwhile, the Schrödinger-like wave equation is transformed into a solvable form. Our numerical solutions to the wave equation show that the wave is characteristically similar to the harmonic under the tortoise coordinate x, while the wave piles up near the two horizons and the wavelength tends to its maximum as the potential approaches to the peak under the radial coordinate r.

Keywords: scalar field, Pöshl–Teller potential, near extremal SdS space
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1. Introduction

Black holes are most striking predictions of Einstein's theory of gravitation, which had been absolutely black until the remarkable discovery of Hawking radiation in 1974.[1] Various methods and techniques have been developed to study the various radiation in different black hole spaces.[2−16] Scalar field radiation has also attracted much interest.[17−24]

Brevik and Simonsen[21] studied the massless scalar field in Schwarzschild–de Sitter (SdS) space, where the tangent approximations were introduced to invert $x(r)$ so that the wave equation can be solved. Except for the insufficiency in the fitting, they have provided a new method to solve the equation after all. Afterwards, Tian *et al.*[22] adopted a more precise polynomial approximation to discuss the same topic, and found that the amplitude of the wave arrives at a peak where the potential approaches the maximum. However, this strange behaviour was found by Liu *et al.*[24] to be due to the inappropriate use of the boundary conditions. In the present paper, we introduce some reasonable approximations in the nearly extremal SdS space, which naturally lead to the invertible $x(r)$ and the *global* replacement of the true potential by a Pöshl–Teller one, to investigate the massless scalar field. Furthermore, the global potential replacement indicates that the boundary conditions can be taken in an interval as large as possible.

In the next section we separate the massless scalar field equation and extract the radial equation, then rewrite it in a one dimensional Schrödinger-like form via the definition of tortoise coordinate. According to the properties of the potential near the horizons, we also give the boundary conditions. In Section 3, with the approximations adopted, we turn $x(r)$ into an invertible form and replace the true potential with the Pöshl–Teller potential. In Section 4, the wave equation is solved numerically with the reasonable choices of the parameters and boundary conditions. And Section 5 is devoted to summary and conclusion.

2. Wave equation and boundary conditions

The spherically symmetric SdS space is described by the metric

$$\begin{aligned} \mathrm{d}s^2 = & -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \\ & + r^2\sin^2\theta\mathrm{d}\phi^2, \end{aligned} \tag{1}$$

where

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2} \tag{2}$$

with M denoting the mass of the black hole and a^2 is given in terms of the cosmological constant Λ by $a^2 = 3/\Lambda$. The space is endowed with two horizons:

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the event horizon of the black hole $r_{\rm e}$ and the cosmological horizon $r_{\rm c}$ ($r_{\rm c} > r_{\rm e}$). The function $f(r)$ has zeros at $r_{\rm e}$, $r_{\rm c}$, and $r_{\rm o} = -(r_{\rm e} + r_{\rm c})$, which is of nonphysical significance. In terms of these quantities, $f(r)$ can be expressed as[20]

$$f(r) = \frac{1}{a^2 r}(r - r_{\rm e})(r_{\rm c} - r)(r - r_{\rm o}), \tag{3}$$

and the surface gravity of the horizons can be written in the explicit forms as

$$k_{\rm e} = \frac{(r_{\rm c} - r_{\rm e})(r_{\rm e} - r_0)}{2a^2 r_{\rm e}}, \tag{4}$$

$$k_{\rm c} = \frac{(r_{\rm c} - r_{\rm e})(r_{\rm c} - r_0)}{2a^2 r_{\rm c}} \tag{5}$$

through $k_i = (1/2)|{\rm d}f/{\rm d}r|_{r=r_i}$.

A massless scalar field $\Phi(t, r, \theta, \phi)$ is considered here. With the separable solution

$$\Phi = \frac{1}{\sqrt{4\pi\omega}} \frac{1}{r} R_\omega(r, t) Y_{lm}(\theta, \phi), \tag{6}$$

the scalar field equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu} \right) = 0 \tag{7}$$

can be decomposed as

$$-\frac{1}{f(r)} r^2 \frac{\partial^2}{\partial t^2} \left(\frac{R_\omega}{r} \right) + \frac{\partial}{\partial r} \left[r^2 f(r) \frac{\partial}{\partial r} \left(\frac{R_\omega}{r} \right) \right] - l(l+1) \frac{R_\omega}{r} = 0, \tag{8}$$

$$\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial Y_{lm}}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{lm}}{\partial \phi^2} = -l(l+1) Y_{lm}, \tag{9}$$

where $R_\omega(r, t)$ is the time-dependent radial function and $Y_{lm}(\theta, \phi)$ is the usual spherically harmonic function. It is necessary to eliminate the time variable by the Fourier component $\mathrm{e}^{-\mathrm{i}\,\omega t}$ via

$$R_\omega(r, t) \to \Psi(r) \mathrm{e}^{-\mathrm{i}\,\omega t}. \tag{10}$$

So Eq. (8) can be rewritten as

$$\left[-f(r) \frac{\rm d}{{\rm d}r} \left(f(r) \frac{\rm d}{{\rm d}r} \right) + V(r) \right] \Psi_{\omega l}(r) = \omega^2 \Psi_{\omega l}(r), \tag{11}$$

where the potential $V(r)$ is given by

$$V(r) = f(r) \left[\frac{1}{r} \frac{{\rm d}f(r)}{{\rm d}r} + \frac{l(l+1)}{r^2} \right]. \tag{12}$$

With the tortoise coordinate defined by[25]

$$x = \int \frac{{\rm d}r}{f(r)}, \tag{13}$$

the radial equation (11) can be written in a one-dimensional Schrödinger-like form

$$\left[-\frac{{\rm d}^2}{{\rm d}x^2} + V(r) \right] \Psi_{\omega l}(x) = \omega^2 \Psi_{\omega l}(x). \tag{14}$$

According to Eqs. (3) and (12), we have

$$V(r)_{r \to r_{\rm e}} = V(r)_{r \to r_{\rm c}} \sim 0. \tag{15}$$

Hence Eq. (14) can be reduced to

$$\left[-\frac{{\rm d}^2}{{\rm d}x^2} - \omega^2 \right] \Psi_{\omega l}(x) = 0 \tag{16}$$

near the horizons, and the solutions to Eq. (16) are $\mathrm{e}^{\pm \mathrm{i}\,\omega x}$ or their comprehensive form. Since only the real scalar field is in consideration here, we adopt the real part of the solutions as the boundary condition

$$\Psi_{\omega l}(x) = \cos(\omega x). \tag{17}$$

3. The approximations

Due to the complicated relations between the tortoise coordinate x and the radial coordinate r, Eq. (14) is unsolvable. We should take some techniques to make $x(r)$ invertible. It will be helpful to regard $r_{\rm e}$ and $r_{\rm c}$ as the two fundamental parameters of the SdS space, and to express the black hole mass M and the cosmological constant term a^2 as functions of these variables. The appropriate relations are

$$a^2 = r_{\rm e}^2 + r_{\rm e} r_{\rm c} + r_{\rm c}^2 \tag{18}$$

and

$$2Ma^2 = r_{\rm e} r_{\rm c} (r_{\rm e} + r_{\rm c}). \tag{19}$$

In the case of the nearly extremal SdS black hole, the cosmological horizon $r_{\rm c}$ is very close to the event horizon $r_{\rm e}$, i.e. $(r_{\rm c} - r_{\rm e})/r_{\rm e} << 1$. We can make the following reasonable approximations

$$\begin{aligned} &r_0 \sim -2r_{\rm e}, \quad a^2 \sim 3r_{\rm e}^2, \\ &M \sim \frac{r_{\rm e}}{3}, \quad k_{\rm e} \sim \frac{r_{\rm c} - r_{\rm e}}{2r_{\rm e}^2}. \end{aligned} \tag{20}$$

Furthermore, since r is constrained to run in the region $(r_{\rm e}, r_{\rm c})$, we have $r - r_{\rm o} \sim r_{\rm e} - r_{\rm o} \sim 3r_{\rm e}$ and thus

$$f \sim \frac{(r - r_{\rm e})(r_{\rm c} - r)}{r_{\rm e}^2}. \tag{21}$$

In this limit, we can invert the relation $x(r)$ of Eq. (13) to obtain

$$r = \frac{r_{\rm c} + r_{\rm e}{\rm e}^{-2k_{\rm e}x}}{1 + {\rm e}^{-2k_{\rm e}x}}. \tag{22}$$

Substituting this in the expression for $f(r)$, we find

$$f = \frac{(r_{\rm c} - r_{\rm e})^2}{4r_{\rm e}^2 \cosh^2(k_{\rm e}x)}. \tag{23}$$

As such, and taking the functional form of the potential into account, we see that for the nearly extremal SdS black hole, the wave equation (14) is of the form

$$\left[\frac{{\rm d}^2}{{\rm d}x^2} + \left(\omega^2 - \frac{k_{\rm e}^2 l(l+1)}{\cosh^2(k_{\rm e}x)}\right)\right] \Psi_{\omega l}(x) = 0. \tag{24}$$

The potential in Eq. (24) is the well known Pöshl–Teller potential, which has been widely used to exactly approximate the true potential in many other papers.[25–27]

4. Solutions to the wave equation

To solve the wave equation (24) numerically, we have to specialise the parameters. It was found that the real physical solutions are accepted only if $\Lambda M^2 \leq 1/9$, and $\Lambda M^2 = 1/9$ corresponds to the extremal case with $r_{\rm e} = r_{\rm c} = 3M$. Since the black hole mass is conventionally taken as a unitary constant, we adopt

$$\Lambda = 0.11111, \quad M = 1, \tag{25}$$

which yield $r_{\rm e} = 2.9945$, $r_{\rm c} = 3.0055$ and $M \sim 1$ from above approximations, and this is in accordance with Eq. (20). When solving the equation numerically, we have also taken l and ω as units. To demonstrate the wave properties in the intermediate region and especially near the horizons, the boundary conditions for of Eq. (24) are taken as

$$\begin{aligned} &\Psi_{11}(-3000) = \cos(-3000), \\ &\Psi_{11}(3000) = \cos(3000), \end{aligned} \tag{26}$$

according to Eq. (17), where $x = -3000$ and $x = 3000$ corresponds to $r = 2.9948$ and $r = 3.0052$ respectively. The two values of r are very close to the corresponding horizons. However, it should be mentioned that the above region of x can be extended to an even larger one in numerical computation.

Numerical solutions to the wave equation show that the wave is characteristically similar to the harmonics under the tortoise coordinate (Fig. 1). While the wave piles up near the two horizons and the wavelength tends to its maximum when the potential approaches to the peak under the radial coordinate (Fig. 2). The result is in full agreement with that of Ref. [24] for the Nariai case, where the polynomial approximation was adopted. Therefore, the result obtained here can also serve as a backup to the argument that Pöshl–Teller potential is an exact fitting to the true potential in nearly extremal SdS space.[25]

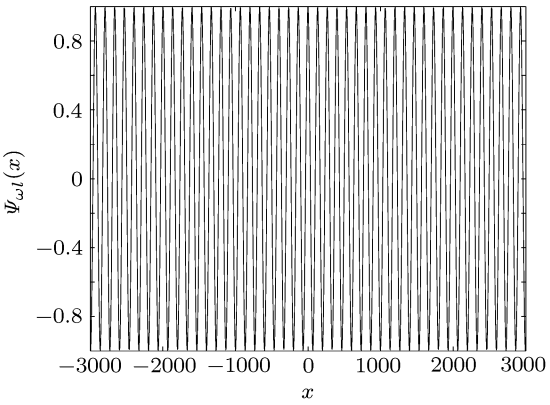


Fig. 1. The variation of the wave amplitude with tortoise coordinate x, $\Lambda = 0.11111$, $M = l = \omega = 1$.

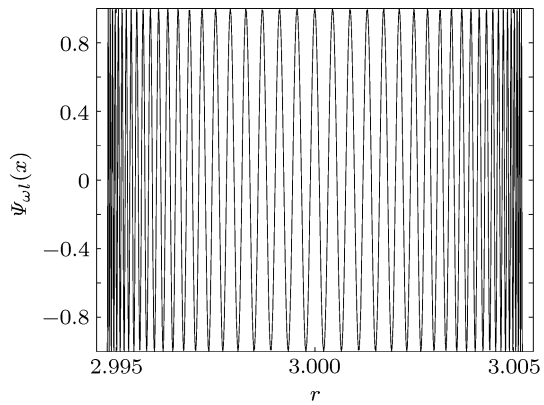


Fig. 2. The variation of the wave amplitude with radial coordinate r, $\Lambda = 0.11111$, $M = l = \omega = 1$.

5. Summary and conclusion

We have calculated the real scalar field scattering in the near extremal SdS space (i.e. the Nariai case in Ref. [24]). In this section, we summarise what have been achieved and give some discussions.

1) Under the approximations adopted, we made $x(r)$ invertible which is the key technique to solve the wave equation. The approximations naturally lead to the replacement of the true potential with a Pöshl–Teller one. This implies that the boundary conditions can be selected in an interval as large as possible to better illustrate the wave properties near horizons. It should be mentioned that the replacement of the potential is a *global* one unlike that under the tangent or the polynomial approximation. Although with the method of polynomial approximation, one can fit the radial coordinate with the tortoise coordinate in high accuracy with suitable choice of the polynomial degree, the maximum fitting interval is limited (for example $x \in [-70, 70]$ in Ref. [24]). Our numerical results show that the wave is characteristically similar to the harmonic under the tortoise coordinate x (Fig. 1), and piles up near the horizons under the radial coordinate r (Fig. 2).

2) The reflection and transmission coefficients can be calculated with the real potential replaced by a square potential (See Ref. [21] for details). However there are some ambiguities in the calculation since we can replace the continuous potential with square potentials of different height. In the present high-frequency case ($\omega = 1$), we can conjecture that the transmission coefficient must approach unity one since the potential barrier is almost fully transparent ($V_{\max} = 7.4074 \times 10^{-7}$).

3) What should also be mentioned is that the method adopted here cannot be extended to the case when the cosmological horizon and the event horizon are widely separated ($\Lambda \ll 1$) which was not considered. With decreasing Λ, the potential barrier will be higher, and this will affect the transmission coefficient and even the amplitude of the wave near the potential peak which needs further investigation. However, we conjecture that the method adopted here could be used in other near extremal black hole spaces, such as near extremal RN or Kerr space which are of interest for their special properties.

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