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Gravitational radiation fields in teleparallel equivalent of general relativity and their energies

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We derive two new retarded solutions in the teleparallel theory equivalent to general relativity (TEGR). One of these solutions gives a divergent energy. Therefore, we use the regularized expression of the gravitational energy–momentum tensor, which is a coordinate dependent. A detailed analysis of the loss of the mass of Bondi space–time is carried out using the flux of the gravitational energy–momentum.

Keywords: teleparallel equivalent of general relativity, energy–momentum tensor, Bondi mass, gravitational radiation

PACC: 0420, 0420J, 0430, 0455

1. Introduction

The linearized theory has been developed extensively, however, it seems doubtful whether its results can be fully trusted.[1] The non-linearity of the gravitational field is the most characteristic properties and some of the crucial properties of the field show themselves only through the non-linear terms. Additionally, it is never clear whether solutions derived from the linearization corresponds to exact solutions.[1] In spite of the fact that we have a good deal about exact plane and cylindrical wave solutions, it is doubtful whether such solutions display the characteristics of physically significant waves.[2,3] Einstein's general relativity (GR) theory is a complete one, however, it may not give sensible solutions for situations too far from what is physically reasonable. The simplest field due to a finite source which can be obtained from GR is the spherically symmetric solution, however, Birkhoff's theorem shows that such a solution must be static. For this reason, one cannot truly speak about spherically symmetric waves and thus any description of radiation from a finite system must necessarily involve three coordinate significantly. This makes the mathematical calculations more complicated and thus we have to make use of approximate methods.

Two well established formulas of gravitational energy are the Arnowitt–Deser–Misner (ADM) energy that corresponds to the total energy[4] and the Bondi energy[2,3] that describes the mass of radiating systems in asymptotic flat space–times. The loss of energy of the total mass of a source that radiates gravitational waves is the main feature of the Bondi energy,[2,3] and expressed as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\left(\frac{\mathrm{d}c_0}{\mathrm{d}t}\right)^2, \tag{1}$$

where c_0 is the new function. Equation (1) shows that Bondi energy is related to the loss of the total mass of a source that radiates gravitational waves. The loss of the total mass is given in terms of the minus new function squared, which means that the mass of the source can decrease.

Teleparallel theories are interesting for several reasons: first of all, the GR can be viewed as a particular theory of teleparallelism and, thus, teleparallelism could be considered at the very least as a different view point that can lead to the same results.[5] Second, in this framework, one can define an energy–momentum tensor for the gravitational field which is a true tensor under all general coordinate transformations. This is the reason why teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy–momentum tensor for the gravitational field.[6] The idea was taken over by Pellegrini and Plebański that constructed the general Lagrangian for these theories.[7] The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full

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Poincar group) and, thus, they give an alternative interpretation of GR.[8−27] Therefore, we consider the tetrad theory of gravitation in this work.

It is the aim of the present work to study the variation of the gravitational energy and the corresponding total flux of energy at space-like infinity within the framework of TEGR. To do so we derive the solution given by Bondi[2] in the framework of TEGR and then compute the total mass using the energy–momentum tensor. Also we compute the loss of mass using the flux of the gravitational energy–momentum tensor.[28−33] In Section 2, we briefly review the TEGR theory for gravitational field and its derivation of the field equations. The derivations of energy and total fluxes of the gravitational energy–momentum are given in Section 2. In Section 3, we study the most general tetrad field which contains 16 unknown functions. Two new solutions are given in Section 3. Then, we calculate the total mass of these solutions using the regularized expression in Section 4. Using the flux of the gravitational energy–momentum we also calculate the loss of mass in Section 4. The final section is devoted for the main results.

2. Continuity equation and fluxes of the gravitational and matter fields

In a space–time with absolute parallelism the parallel vector fields $\boldsymbol{e}_a{}^\mu$ defined by the nonsymmetric affine connection

$$\boldsymbol{\Gamma}^\lambda_{\mu\nu} \overset{\text{def.}}{=} \boldsymbol{e}^\lambda_a \boldsymbol{e}^a_{\mu,\nu}, \tag{2}$$

where $e_{a\mu,\ \nu} = \partial_\nu e_{a\mu}$.[1)] The curvature tensor defined by $\boldsymbol{\Gamma}^\lambda_{\mu\nu}$, given by Eq. (2), is identically vanishing. The metric tensor $\boldsymbol{g}_{\mu\nu}$ is defined as

$$\boldsymbol{g}_{\mu\nu} \overset{\text{def.}}{=} \eta_{ab}\ \boldsymbol{e}^a_\mu \boldsymbol{e}^b_\nu, \tag{3}$$

where $\eta_{ab} = (-1,+1,+1,+1)$ is the metric of Minkowski space–time.

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given in Ref. [28]–[33][2)]

$$\begin{aligned}\mathcal{L}_G &= eL_G \\ &= -\frac{e}{16\pi}\left(\frac{\boldsymbol{T}^{abc}\boldsymbol{T}_{abc}}{4} + \frac{\boldsymbol{T}^{abc}\boldsymbol{T}_{bac}}{2} - \boldsymbol{T}^a\boldsymbol{T}_a\right) - L_m \\ &= -\frac{e}{16\pi}\boldsymbol{\Sigma}^{abc}\boldsymbol{T}_{abc} - L_m, \end{aligned} \tag{4}$$

where $e = \det|(e^a_\mu)$. The tensor $\boldsymbol{\Sigma}^{abc}$ is defined as

$$\begin{aligned}\boldsymbol{\Sigma}^{abc} \overset{\text{def.}}{=}\ & \frac{1}{4}\left(\boldsymbol{T}^{abc} + \boldsymbol{T}^{bac} - \boldsymbol{T}^{cab}\right) \\ & + \frac{1}{2}\left(\eta^{ac}\boldsymbol{T}^b - \eta^{ab}\boldsymbol{T}^c\right), \end{aligned} \tag{5}$$

where $\boldsymbol{T}^{abc}$ and $\boldsymbol{T}^a$ are the torsion tensor and the basic vector field defined as

$$\begin{aligned}&\boldsymbol{T}^a_{\mu\nu} \overset{\text{def.}}{=} \boldsymbol{e}^a_\lambda \boldsymbol{T}^\lambda_{\mu\nu} = \partial_\mu \boldsymbol{e}^a_\nu - \partial_\nu \boldsymbol{e}^a_\mu, \\ &\boldsymbol{T}^a_{bc} \overset{\text{def.}}{=} \boldsymbol{e}^\mu_b \boldsymbol{e}^\nu_c \boldsymbol{T}^a_{\mu\nu}, \end{aligned}$$

and

$$\boldsymbol{T}^a \overset{\text{def.}}{=} \boldsymbol{T}^b{}_b{}^a. \tag{6}$$

The quadratic combination $\boldsymbol{\Sigma}^{abc}\boldsymbol{T}_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term.[28−33] L_m represents the Lagrangian density for matter fields.

The gravitational field equations described by L_G are as follows:

$$\begin{aligned}&\boldsymbol{e}_{a\lambda}\boldsymbol{e}_{b\mu}\partial_\nu\left(e\boldsymbol{\Sigma}^{b\lambda\nu}\right) - e\left(\boldsymbol{\Sigma}^{b\nu}_a \boldsymbol{T}_{b\nu\mu}\right. \\ &\left. - \frac{1}{4}e_{a\mu}\boldsymbol{T}_{bcd}\boldsymbol{\Sigma}^{bcd}\right) = \frac{1}{2}\kappa e T_{a\mu}, \end{aligned} \tag{7}$$

where

$$\frac{\delta L_m}{\delta e^{a\mu}} \equiv eT_{a\mu}.$$

It is possible to prove by explicit calculations that the left-hand side of the symmetric field equations (7) is exactly given by[28−33]

$$\frac{e}{2}\left[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\right].$$

Multiplying Eq. (7) by the appropriate inverse tetrad fields leads to the following form:[28−34]

$$\begin{aligned}\partial_\nu\left(-e\boldsymbol{\Sigma}^{a\lambda\nu}\right) =\ & -\frac{e\boldsymbol{e}^{a\mu}}{4}\left(4\boldsymbol{\Sigma}^{b\lambda\nu}T_{b\nu\mu} - \delta^\lambda_\mu \boldsymbol{\Sigma}^{bdc}T_{bcd}\right) \\ & -4\pi e^a_\mu T^{\lambda\mu}. \end{aligned} \tag{8}$$

By restricting the space–time index λ to assume only spatial values then Eq. (8) takes the form[28−33]

$$\begin{aligned}&\partial_0\left(e\boldsymbol{\Sigma}^{a0j}\right) + \partial_k\left(e\boldsymbol{\Sigma}^{akj}\right) \\ &= -\frac{e\boldsymbol{e}^{a\mu}}{4}\left(4\boldsymbol{\Sigma}^{bcj}\boldsymbol{T}_{bc\mu} - \delta^j_\mu \boldsymbol{\Sigma}^{bcd}T_{bcd}\right) \\ &\quad -4\pi e\boldsymbol{e}^a_\mu T^{j\mu}. \end{aligned} \tag{9}$$

[1)] space–time indices $\mu,\ \nu,\cdots$ and SO(3,1) indices a, $b\cdots$ run from 0 to 3. Time and space indices are indicated to $\mu = 0, i$, and $a = (0), (i)$.

[2)] Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

Note that the last two indices of $\boldsymbol{\Sigma}^{abc}$ and $\boldsymbol{T}^{abc}$ are anti-symmetric. Taking the divergence of Eq. (9) with respect to j yields

$$\begin{aligned}&-\partial_0\partial_j\left(-\frac{1}{4\pi}e\boldsymbol{\Sigma}^{a0j}\right)\\&=-\frac{1}{16\pi}\partial_j[ee^{a\mu}\left(4\boldsymbol{\Sigma}^{bcj}\boldsymbol{T}_{bc\mu}-\delta^j_\mu\boldsymbol{\Sigma}^{bcd}\boldsymbol{T}_{bcd}\right)\\&\quad-16\pi(e\boldsymbol{e}^a_\mu\boldsymbol{T}^{j\mu})].\end{aligned}\tag{10}$$

The Hamiltonian formulation of TEGR is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component e_{a0}. Therefore, this quantity will arise as a Lagrange multiplier.[35] The momentum canonically conjugated to $\boldsymbol{e}_{ai}$ is given by $\Pi^{ai}=\delta L/\delta\dot{e}_{ai}$. The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form $L=p\,\dot{q}-H$, in terms of e_{ai},Π^{ai} and the Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form[36]

$$H=e_{a0}C^a+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k,\tag{11}$$

plus a surface term. Here α_{ik} and β_k are Lagrange multipliers that are identified as

$$\alpha_{ik}=\frac{1}{2}(T_{i0k}+T_{k0i})\qquad\text{and}\qquad\beta_k=T_{00k},\tag{12}$$

where C^a, Γ^{ik} and Γ^k are first class constraints.

The constraint C^a is written as $C^a=-\partial_i\Pi^{ai}+h^a$, where h^a is an intricate expression of the field variables. The integral form of the constraint equation $C^a=0$ motivates the definition of the gravitational energy–momentum P^a four-vector[28−33]

$$P^a=-\int_V\mathrm{d}^3x\partial_i\Pi^{ai},\tag{13}$$

where V is an arbitrary volume of the three-dimensional space. In the configuration space we have

$$\Pi^{ai}=-4\kappa\sqrt{-g}\boldsymbol{\Sigma}^{a0i}$$

with

$$\partial_\nu(\sqrt{-g}\boldsymbol{\Sigma}^{a\lambda\nu})=\frac{1}{4\kappa}\sqrt{-g}\boldsymbol{e}^a_\mu(t^{\lambda\mu}+T^{\lambda\mu}),$$

where

$$t^{\lambda\mu}=\kappa\left(4\boldsymbol{\Sigma}^{bc\lambda}\boldsymbol{T}^\mu_{bc}-g^{\lambda\mu}\boldsymbol{\Sigma}^{bcd}\boldsymbol{T}_{bcd}\right).\tag{14}$$

By integrating Eq. (10) over volume V of the three-dimensional space we have[28−33]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-\int_v\mathrm{d}^3x\partial_j\Pi^{aj}\right]=-\phi_g{}^a-\phi_m{}^a,\tag{15}$$

where

$$\begin{aligned}&\phi^a_g\\&=\frac{\kappa}{2}\int_S\mathrm{d}S_j\left[\boldsymbol{e}\boldsymbol{e}^{a\mu}\left(4\boldsymbol{\Sigma}^{bcj}\boldsymbol{T}_{bc\mu}-\delta^j_\mu\boldsymbol{\Sigma}^{bcd}\boldsymbol{T}_{bcd}\right)\right]\end{aligned}\tag{16}$$

is the a component of the gravitational energy–momentum flux and

$$\phi_m{}^a=\int_S\mathrm{d}S_j\left(\boldsymbol{e}\boldsymbol{e}^a_\mu T^{j\mu}\right)\tag{17}$$

is the a component of the matter energy–momentum flux. S represents the spatial boundary of the volume V. Therefore the loss of the gravitational energy $P^{(0)}=E$ is governed by the equation

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\phi_g{}^0-\phi_m{}^0.\tag{18}$$

Maluf and Faria have shown that $\phi_g{}^0$ plays a major role in the Bondi radiating space–time, since it yields the well-known expression for the gravitational energy loss, while $\phi_m{}^0$ is related to the Vaidya space–time.[37]

3. Bondi space–times in TEGR and their associated energies

In this section we are going to use the most general tetrad field which contains 16 unknown functions. We write it in the coordinate $(t,\ r,\ \theta,\ \phi)$ by

$$\left(e^a_\mu\right)=\begin{pmatrix}A_1&A_2&A_3&A_4\\B_1&B_2&B_3&B_4\\F_1&F_2&F_3&F_4\\H_1&H_2&H_3&H_4\end{pmatrix},\tag{19}$$

where $A_i,\ B_i,\ F_i,\ H_i,\ i=1\cdots4$ are unknown functions of r, θ, ϕ and t. Applying Eq. (19) to field equation (7), we obtain a very lengthy and tedious system of partial differential equations. We are going to write a special solutions to these system.

First solution:

The 16 unknown functions in this solution have the form

$$
\begin{aligned}
A_1 &= -e^{\beta}\sqrt{\frac{V}{r}}, \qquad A_2 = -e^{\beta}\sqrt{\frac{r}{V}}, \qquad A_3 = 0, \qquad A_4 = 0, \qquad B_1 = Ure^{\gamma}\cos\theta\cos\phi,\\
B_2 &= -e^{\beta}\sqrt{\frac{r}{V}}\sin\theta\cos\phi, \qquad B_3 = -re^{\gamma}\cos\theta\cos\phi, \qquad B_4 = -re^{-\gamma}\sin\theta\sin\phi,\\
F_1 &= Ure^{\gamma}\cos\theta\sin\phi, \qquad F_2 = -e^{\beta}\sqrt{\frac{r}{V}}\sin\theta\sin\phi, \qquad F_3 = -re^{\gamma}\cos\theta\sin\phi,\\
F_4 &= re^{-\gamma}\sin\theta\cos\phi, \qquad H_1 = -Ure^{\gamma}\sin\theta, \qquad H_2 = -e^{\beta}\sqrt{\frac{r}{V}}\cos\theta,\\
H_3 &= re^{\gamma}\sin\theta, \qquad H_4 = 0.
\end{aligned} \tag{20}
$$

The functions β, γ, V and U appear in Eq. (20) have the following asymptotic behaviours up to $O\left(1/r^2\right)$

$$
\begin{aligned}
\beta &= -\frac{\left(c^2(u,\theta)+d^2(u,\theta)\right)}{4r^2}+O\left(\frac{1}{r^4}\right), \qquad \gamma = \frac{c(u,\theta)}{r}+O\left(\frac{1}{r^3}\right),\\
U &= -\frac{l(u,\theta)}{r^2}+O\left(\frac{1}{r^3}\right), \qquad V = r-2M(u,\theta)-\frac{l_1(u,\theta)}{r},
\end{aligned} \tag{21}
$$

where time t is defined as $t = u + r$ which is the retarded time. The quantities $l(u,\theta)$ and $l_1(u,\theta)$ are defined as

$$
\begin{aligned}
l(u,\theta) &= \frac{\partial c(u,\theta)}{\partial\theta}+2c(u,\theta)\cot\theta,\\
l_1(u,\theta) &= \frac{\partial d(u,\theta)}{\partial\theta}+d(u,\theta)\cos\theta-\left(\frac{\partial c(u,\theta)}{\partial\theta}\right)^2-4c(u,\theta)\left(\frac{\partial c(u,\theta)}{\partial\theta}\right)\cot\theta\\
&\quad -\frac{1}{2}c^2(u,\theta)\left(1+8\cot^2\theta\right),
\end{aligned} \tag{22}
$$

with $M(u,\theta)$, $c(u,\theta)$ and $d(u,\theta)$ are the mass aspect, the news function and the dipole aspect respectively. Solution given by Eq. (20) with Eqs. (21) and (22) satisfies the field equations (7) up to $O\left(1/r^2\right)$. The metric associated with solution given by Eq. (20) has the form

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(\frac{V}{r}e^{2\beta}-U^2r^2e^{2\gamma}\right)\mathrm{d}u^2-2e^{2\beta}\mathrm{d}u\mathrm{d}r\\
&\quad -2Ur^2e^{2\gamma}\mathrm{d}u\mathrm{d}\theta+r^2\left(e^{2\gamma}\mathrm{d}\theta^2+e^{2\gamma}\sin^2\theta\mathrm{d}\phi^2\right),
\end{aligned} \tag{23}
$$

which is the Bondi space–time.[11]

Second solution:

The 16 unknown functions in this case have the following forms:

$$
\begin{aligned}
A_1 &= \sqrt{\frac{Ve^{2\,\beta}}{r}+r^2e^{2\,\gamma}U^2\cosh(2\delta)+r^2e^{-2\,\gamma}W^2\cosh(2\delta)+2\,r^2WU\sinh(2\delta)},\\
A_2 &= -B_2 = \frac{-e^{2\,\beta}}{\sqrt{\frac{Ve^{2\,\beta}}{r}+r^2e^{2\,\gamma}U^2\cosh(2\delta)+r^2e^{-2\,\gamma}W^2\cosh(2\delta)+2\,r^2WU\sinh(2\delta)}},\\
A_3 &= -B_3 = \frac{-r^2\left(e^{2\,\gamma}U\cosh(2\delta)+W\sinh(2\delta)\right)}{\sqrt{\frac{Ve^{2\,\beta}}{r}+r^2e^{2\,\gamma}U^2\cosh(2\delta)+r^2e^{-2\,\gamma}W^2\cosh(2\delta)+2\,r^2WU\sinh(2\delta)}},\\
A_4 &= -B_4 = \frac{-r^2\sin\theta\left(e^{-2\,\gamma}W\cosh(2\delta)+U\sinh(2\delta)\right)}{\sqrt{\frac{Ve^{2\,\beta}}{r}+r^2e^{2\,\gamma}U^2\cosh(2\delta)+r^2e^{-2\,\gamma}W^2\cosh(2\delta)+2\,r^2WU\sinh(2\delta)}},\\
B_1 &= F_1 = F_2 = H_1 = H_2 = H_3 = 0, \quad F_3 = re^{\gamma}\sqrt{-\cosh(2\delta)},\\
F_4 &= -\frac{r\sinh(2\delta)\sin\theta}{e^{\gamma}\sqrt{-\cosh(2\delta)}}, \quad H_4 = \sqrt{-r^2e^{-2\,\gamma}\cosh(2\delta)\sin^2\theta+\frac{r^2\sinh^2(2\delta)\sin^2\theta}{e^{2\,\gamma}\cosh(2\delta)}}.
\end{aligned} \tag{24}
$$

The functions β, γ, V, U, W and δ appearing in Eq. (24) have the following asymptotic behaviours up to $O\left(1/r^2\right)$:

$$\begin{aligned}\beta &= -\frac{\left(c^2(u,\theta)+d^2(u,\theta)\right)}{4r^2}+O\left(\frac{1}{r^4}\right), \qquad \gamma=\frac{c(u,\theta)}{r}+O\left(\frac{1}{r^3}\right), \quad \delta=\frac{d\,(\theta,u)}{r}+O\left(\frac{1}{r^3}\right)\\ U &= -\frac{l(u,\theta)}{r^2}+O\left(\frac{1}{r^3}\right), \qquad V=r-2M(u,\theta)-\frac{l_1(u,\theta)}{r}, \quad W=-\frac{l_2(u,\theta)}{r^2}+O\left(\frac{1}{r^3}\right).\end{aligned} \tag{25}$$

The quantities $l(u,\theta)$ and $l_1(u,\theta)$ are defined in Eq. (22) and $l_2(u,\theta)$ has the following form:

$$l_2(u,\theta) = \frac{\partial d(u,\theta)}{\partial\theta}+2d(u,\theta)\cot\theta. \tag{26}$$

The metric associated with solution given by Eq. (24) has the following form:

$$\begin{aligned}\mathrm{d}s^2 = &\left(\frac{V}{r}e^{2\beta}+r^2U^2e^{2\gamma}\cosh(2\delta)+r^2W^2e^{-2\gamma}\cosh(2\delta)+2r^2UW\sinh(2\delta)\right)\mathrm{d}u^2-2e^{2\beta}\mathrm{d}u\mathrm{d}r\\ &-2r^2\left(Ue^{2\gamma}\cosh(2\delta)+W\sinh(2\delta)\right)\mathrm{d}u\mathrm{d}\theta-2r^2\sin\theta\left(We^{-2\gamma}\cosh(2\delta)+U\sinh(2\delta)\right)\mathrm{d}u\mathrm{d}\phi\\ &+r^2\left(e^{2\gamma}\cosh(2\delta)\mathrm{d}\theta^2+e^{-2\gamma}\cosh(2\delta)\sin^2\theta\,\mathrm{d}\phi^2+2\sinh(2\delta)\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi\right),\end{aligned} \tag{27}$$

which is the Bondi space–time.[38] Solutions (20) and (24) satisfy field equations (7) up to $O\left(1/r^3\right)$.

Calculate the energy associated with solution (20) by calculating the torsion tensor and basic vector[1)], we finally have the expression up to $O\left(1/r^0\right)$, i.e.,

$$E\cong\frac{1}{2}\int_0^\pi \sin\theta\,\{M(u,\theta)-2r\}\,\mathrm{d}\theta, \tag{28}$$

which depends on the radial coordinate. Therefore, in this case we are going to use the regularized expression for the gravitational energy–momentum.[28–33]

4. Regularized expression for gravitational energy–momentum and localization of energy

For a space–time in which the physical quantities, i.e., $M(u,\theta)$ etc. equal zero, it is usual to consider a set of tetrad fields such that the torsion $T^\lambda_{\mu\nu}=0$ in any coordinate system. However, in general an arbitrary set of tetrad fields does not satisfy $T^\lambda_{\mu\nu}=0$. It might be argued, therefore, that the expression for the gravitational energy–momentum (13) is restricted to a particular class of tetrad fields, namely, to the class of frames such that $T^\lambda_{\mu\nu}=0$, if E^μ_a represents the space–time of the tetrad field in which the physical quantities equal zero.[34] To explain this, we calculate the space–time in which the physical quantities equal zero for the tetrad field of Eq. (19) using (20) which is given by

$$(E^\mu_a)=\begin{pmatrix}-1 & -1 & 0 & 0\\ 0 & -\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ 0 & -\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ 0 & -\cos\theta & -r\sin\theta & 0\end{pmatrix}. \tag{29}$$

1) The details of components of those quantities are given in the appendix.

Expression (29) yields the following non-vanishing torsion components:

$$T^2_{12} = \frac{2}{r} = T^3_{13}. \tag{30}$$

We will denote $T^a_{\mu\nu}(E) = \partial_\mu E^a_\nu - \partial_\nu E^a_\mu$ and $\mathit{\Pi}^{aj}(E)$ as the expression of $\mathit{\Pi}^{aj}$ constructed out of the tetrad E^a_μ. The regularized form of the gravitational energy–momentum P^a is defined as[34,36]

$$P^a = -\int_V \mathrm{d}^3x \partial_k \left[\mathit{\Pi}^{ak}(e) - \mathit{\Pi}^{ak}(E)\right]. \tag{31}$$

The reference space–time is determined by tetrad fields E^a_μ, obtained from e^a_μ by requiring the vanishing of the physical parameters like mass, angular-momentum, etc. Equation (31) can have the following form:[34,36]

$$P^a = -\oint_{S\to\infty} \mathrm{d}S_k \left[\mathit{\Pi}^{ak}(e) - \mathit{\Pi}^{ak}(E)\right], \tag{32}$$

where the surface S is established at space-like infinity. Equation (32) transforms as a vector under the global SO(3,1) group.[28–33]

We use Eq. (32) to calculate the gravitational energy of Eq. (20). We need to calculate the quantity

$$\boldsymbol{\Sigma}^{(0)01} = e^{(0)}_0 \boldsymbol{\Sigma}^{001} + e^{(0)}_a \boldsymbol{\Sigma}^{a01}.$$

Evaluate the above equation, then we will find

$$\mathit{\Pi}^{(0)1}(e) \cong \frac{1}{4}\sin\theta \left\{M(u,\theta) - 4r\right\}. \tag{33}$$

The expression of $\mathit{\Pi}^{(0)1}(E)$ is obtained just by making $M(u,\theta) = 0$ in Eq. (33), thus it is given by

$$\mathit{\Pi}^{(0)1}(E) \cong -r\sin\theta. \tag{34}$$

Substituting Eqs. (33) and (34) to Eq. (32) we finally obtain the gravitational energy of the tetrad field of Eq. (19) using solution given by Eq. (20) in the following form:

$$P^{(0)} \cong \frac{1}{2}\int_0^\pi \sin\theta\, M(u,\theta)\, \mathrm{d}\theta, \tag{35}$$

which is the Bondi energy.[2,3]

The flux of the gravitational energy is determined by putting $a = (0)$ in Eq. (16) and by making $j = 1$, i.e., by integrating over a surface S of constant radius r, and requiring $r \to \infty$. Equation (16) in such a case has the following form:

$$\phi_g{}^{(0)} = \frac{\kappa}{2}\int_S \mathrm{d}S_1 e \left(4e^{(0)\mu}\boldsymbol{\Sigma}^{bc1}T_{bc\mu} - e^{(0)1}\boldsymbol{\Sigma}^{bcd}T_{bcd}\right). \tag{36}$$

Applying Eq. (36) to the tetrad field (19) and using Eq. (20), we finally have

$$\phi^{(0)}_g = \frac{1}{2}\int_0^\pi \sin\theta \left\{c_{,0}(u,\theta)\right\}^2 \mathrm{d}\theta, \tag{37}$$

which is the value of the loss of the gravitational mass of the Bondi space–time.[2,3] In that case Eq. (1) is satisfied for the solution given by Eq. (20).

Using the regularized form of the energy given by Eq. (32) to calculate the gravitational energy of Eq (24), we finally have

$$\mathit{\Pi}^{(0)1}(e) \cong \frac{1}{4}\sin\theta \left\{M(u,\theta) - r\right\}. \tag{38}$$

The expression of $\mathit{\Pi}^{(0)1}(E)$ is obtained just by making $M(u,\theta) = 0$, in Eq. (38), and it is given by

$$\mathit{\Pi}^{(0)1}(E) \cong -r\sin\theta. \tag{39}$$

Thus the gravitational energy of Eq. (24) is given by

$$P^{(0)} \cong \frac{1}{2}\int_0^\pi \sin\theta\, M(u,\theta)\, \mathrm{d}\theta, \tag{40}$$

which is the Bondi energy.[2,3]

Repeat the same calculations done for the first solution we finally, obtain the flux of the gravitational energy of the second solution in the following form:

$$\phi_g{}^{(0)} = \frac{1}{2}\int_0^\pi \sin\theta \left(\left\{c_{,0}(u,\theta)\right\}^2 + \left\{d_{,0}(u,\theta)\right\}^2\right) \mathrm{d}\theta, \tag{41}$$

which is the value of the loss of mass of the Bondi space–time.[2,3,38] The flux given by Eq. (41) will coincide with that of Eq. (37) when the function $d(u,\theta) = 0$ which means that the function δ appears in solution (24) must vanish.

5. Main results and discussion

The main results of the present paper are follows:

• We have obtained two new special solutions which satisfy field equations (7). These solutions reproduce the Bondi space–time.[2,3,28–33]

• We have calculated the energy associated with the first solution and found that it depends on radial coordinate. This is due to the fact that if the physical quantities, i.e., $M(u,\theta)$ are set to be equal to zero the components of the torsion tensor do not vanish as they should be! Therefore, we have used the regularized expression to recalculate the energy associated with the two solutions. We have shown that the formula of

energy associated with these solutions are coincident with that given by Bondi.[2,3]

• We have used the flux of the energy–momentum of gravitational and matter fields.[37] The flux of matter field has no effect since the solutions are vacuum ones to the field equations. The non-vanishing value of the flux of the gravitational energy–momentum indicates the emission of the gravitational waves.

• Maluf and Faria[37] have applied the flux of the gravitational energy–momentum to a tetrad which is different from that we have obtained here. Our results are in coincident with those obtained before.[37]

• Zhang[38] has discussed within the Reimannian geometry the properties of the metric (27). Here we have discussed the gravitational radiation of this metric within the teleparallel geometry by using the flux of the gravitational energy–momentum. Also we have shown that our results are in coincident with those obtained before.[38]

• A further work that can be done for the solutions obtained in this work is to study their spatial momentum and angular-momentum. This will be our future work.

Appendix

Here in this appendix we write explicitly the calculations used in Eqs. (28) and (38). For Eq. (28), the non-vanishing components of torsion are

$$
\begin{aligned}
&T^0{}_{01} = -\frac{T^1{}_{01}}{2} \cong -2\,\frac{M_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),\ T^0{}_{02} \cong \frac{c_{,\theta}(u,\theta)\sin\theta + 2\,c(u,\theta)\cos\theta - M_{,\theta}(u,\theta)\sin\theta}{r\sin\theta} + O\left(\frac{1}{r^2}\right),\\
&T^0{}_{12} = \frac{-T^1{}_{12}}{2} \cong 2\,\frac{M_{,\theta}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),\quad T^1{}_{02} = T^3{}_{23} \cong -\frac{c_{,\theta}(u,\theta)\sin\theta + 2\,c(u,\theta)\cos\theta}{r\sin\theta} + O\left(\frac{1}{r^2}\right),\\
&T^2{}_{12} = T^3{}_{13} \cong \frac{2}{r} + O\left(\frac{1}{r^2}\right),\quad T^3{}_{03} = -\frac{c_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),
\end{aligned}
\tag{42}
$$

and the non-vanishing components of the basic vector are

$$
T^0 \cong -2\,\frac{2 + M_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),\quad T^1 = \frac{4 + M_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),
\tag{43}
$$

where

$$
M_{,i} = \frac{\partial M}{\partial x^i}.
$$

For Eq. (38) the non-vanishing components of the torsion tensor are

$$
\begin{aligned}
&T^2{}_{20} = -T^3{}_{30} \cong -\frac{c_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right),\quad T^2{}_{23} \cong 2\,\frac{\{d(u,\theta)\cos\theta + d_{,\theta}(u,\theta)\sin\theta\}}{r} + O\left(\frac{1}{r^2}\right),\\
&T^2{}_{30} \cong -\frac{2d_{,0}(u,\theta)\,\sin\theta}{r} + O\left(\frac{1}{r^2}\right),\quad T^3{}_{23} \cong \frac{r\cos\theta - c_{,\theta}(u,\theta)\sin\theta}{r\sin\theta} + O\left(\frac{1}{r^2}\right),
\end{aligned}
\tag{44}
$$

and the non-vanishing component of the basic vector is

$$
T^0 \cong 2\,\frac{1 + M_{,0}(u,\theta)}{r} + O\left(\frac{1}{r^2}\right).
\tag{45}
$$

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