Teleparallel equivalent theory of (1+1)-dimensional gravity

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A theory of (1+1)-dimensional gravity is constructed on the basis of the teleparallel equivalent of general relativity. The fundamental field variables are the tetrad fields $e_i{}^\mu$ and the gravity is attributed to the torsion. A dilatonic spherically symmetric exact solution of the gravitational field equations characterized by two parameters M and Q is derived. The energy associated with this solution is calculated using the two-dimensional gravitational energy–momentum formula.

Keywords: teleparallel equivalent theory of (1+1)-dimensional gravity, two-dimensional spherically symmetric dilatonic black hole, energy

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1. Introduction

Since the work of Hawking,[1−6] the classical and the quantum analyses of singularities in gravity have led to many discussions[7−16] on some fundamental physical problems that are expected to arise in the theory of quantum gravity. Technical difficulties have precluded this problem especially the perturbative non-renormalizability of quantum gravity from being solved.

Witten *et al.*[17,18] have discovered a $(1+1)$-dimensional model of gravity with non-trivial dynamics which contains many features of the four-dimensional gravitational theory, in particular, the formation and the evaporation of black holes and naked singularities. Now one has a potentially useful model in examining some of the interesting physics underlying the problems posed by Hawking's work. With this motivation, a lot of effort has been made to understand the $(1+1)$-dimensional dilaton gravity.[19]

Semiclassical physics of black holes, combining issues of spacetime, thermodynamics and quantum theory, is one of the most interesting areas in physics. In our real four-dimensional world high mathematical complexities obscure the analysis of interplay among these aspects. This explains why $(1+1)$-dimensional black hole physics has became so popular in the last decade.[20] In addition $(1+1)$-dimensional gravitational models are much simpler because of a lesser degree of freedom than the higher-dimensional counterparts in studying black holes and cosmology. In fact, enormous studies related to the intriguing gravitational issues have been conducted in this lower-dimensional regime (see Ref. [21] and references therein). The $(1+1)$-dimensional gravitational theory has attracted considerable attention recently.[22−29] In particular, the extension of the spacetime geometry by including torsion has encompassed a class of interesting models with rich mathematical and physical contents (see Ref. [30] and references therein).

The $(1+1)$-dimensional gravity models retain many properties of their higher-dimensional counterparts but are considerably simpler and thus can be used as toy models for tackling the conceptual problems of quantum gravity while at the same time avoiding the technical difficulties that arise from the non-linear dynamics of gravity theories in higher dimensions. The most prominent one of them, the Schwarzschild black hole, is not merely a toy model but of considerable interest for general relativity in four (or higher)-dimensions.[31]

At present, teleparallel theory seems to be popular again. There is a trend of analysing the basic solutions of general relativity with teleparallel

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theory and comparing the results. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory[32−44] or metric-affine gravity.[45] Physics relevant to geometry may be related to the teleparallel description of gravity.[46−50] Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques (see Ref. [51] and references therein). A teleparallel approach is used for positive-gravitational-energy proof.[52] A relation between spinor Lagrangian and teleparallel theory was established.[53] In metric-affine generalization of teleparallelism, Obukhov[30] have shown that there is an inconsistency in the coupling of spinors. Mielke[54] demonstrated the consistency of the coupling of the Dirac fields to the TEGR. However, Andrade and Pereira[55−57] have shown that this demonstration is not correct. They[55−57] have also studied the general teleparallel gravity model within the framework of the metric affine gravity theory. Nester *et al.*[58] have considered the quasilocal centre of mass (COM) in tetrad-teleparallel gravity. They have used the covariant Hamiltonian formalism, in which quasilocal quantities are given by the Hamiltonian boundary term, along with the covariant asymptotic Hamiltonian boundary expressions. Consideration of the COM not only gave the most restrictive asymptotic conditions on the variables but also presented strong constraints on the acceptable expressions.[58−60] Mielke *et al.*[61] have demonstrated the complete integrability of a two-dimensional teleparallel theory of gravity in vacuum.

For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy–momentum are related to the geometrical description of the gravitational field rather than an intrinsic drawback of the theory.[62,63] Møller[64] has shown that the problem of energy–momentum complex has no solution in the framework of gravitational field theories based on Riemannian space–time. In a series of papers,[64−67] he obtained a general expression for a satisfactory energy–momentum complex in the teleparallel space-time. Xu and Jing[68] derived the field equation with a cosmological term and studied the energy of the general four-dimensional stationary axisymmetric space-time in the context of the Hamiltonian formulation of the TEGR.

It is well known that TEGR[46−63] provides an alternative description of Einstein's general relativity. In this theory the gravitational field is described by the tetrad field $e^a{}_\mu$. In fact the first attempt to construct a theory of the gravitational field in terms of a set of four linearly independent vector fields in the Weitzenböck geometry was made by Einstein.[69−71]

A well-given mathematically consistent expression for the gravitational energy has been developed.[63] It arises in the realm of the Hamiltonian formulation of the TEGR[72] and satisfies several crucial requirements for any acceptable definition of gravitational energy. The gravitational energy–momentum P^a [63,73] obtained in the framework of the TEGR has been investigated in the context of several distinct configurations of the gravitational filed. For asymptotically flat space–time P^0 yields the ADM energy (short for Richard Arnowitt, Stanley Deser, and Charles Misner).[74] In the context of tetrad theories of gravity, asymptotically flat space–time may be characterized by the asymptotic boundary condition

$$e_{a\mu} \cong \eta_{a\mu} + \frac{1}{2} h_{a\mu}(1/r), \tag{1}$$

and by the condition $\partial_\mu e^a{}_\mu = \mathrm{O}(1/r^2)$ in the asymptotic limit $r \to \infty$, with $\eta_{ab} = (-1, +1, +1, +1)$ being the metric of Minkowski space–time. An important property of tetrad fields that satisfy Eq. (3) is that in the flat space–time limit one has $e^a{}_\mu(t, x, y, z) = \delta^a{}_\mu$, and therefore the torsion tensor $\boldsymbol{T}^a{}_{\mu\nu} = 0$. Maluf *et al.*[75] have extended the definition P^a for the gravitational energy–momentum[63,72] to any arbitrary tetrad fields, i.e., for the tetrad fields that satisfy $\boldsymbol{T}^a{}_{\mu\nu} \neq 0$ for the flat space–time. The redefinition is the only possible consistent extension of P^a, valid for the tetrad fields that do not satisfy Eq. (3).

It would be significant to develop various theories of gravity also for $(1+1)$-dimensional case, which will bring toy models to examining basic concepts in theories of gravity. The purpose of this paper is to develop

a teleparallel equivalent of general relativity (TEGR) (1 + 1)-dimensional gravity. In Section 2, we give a brief review of tetrad fields. The theory of the TEGR in two-dimension is given in Section 3. In Section 4, two exact solutions are derived. In section 5, the calculations of energy for the two solutions are given. Final section is devoted to discussion and conclusion.

2. Dreibeins, covariant derivative and teleparallelism

The two-dimensional space–time M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}\mathrm{d}x^\mu \bigotimes \mathrm{d}x^\nu$ $(\mu,\nu=0,1)$ related to the fields $\boldsymbol{e}^k = \mathrm{e}^k{}_\mu \mathrm{d}x^\mu$, $(k=0,1)$ through the relation $g_{\mu\nu} = \eta_{ij}e^i{}_\mu e^j{}_\nu$ with $\eta_{ij} = (-1,1)$. Here $\{x^\mu;\mu=0,1\}$ is a local coordinate of the space–time. The fields $\boldsymbol{e}_k = e^\mu{}_k\partial/\partial x^\mu$, which are the dual to $\boldsymbol{e}^k$, are the dreibein fields. The field strength of $e^k{}_\mu$ is given by[1)]

$$T^k{}_{\mu\nu} = \Gamma^k{}_{[\mu\nu]}, \qquad \text{with} \qquad \Gamma^k{}_{\mu\nu} = \partial_\mu e^k{}_\nu. \tag{2}$$

We define the covariant derivative of the Lorentzian vector field V^k as

$$\nabla_l V^k = e^\mu{}_l \partial_\mu V^k. \tag{3}$$

For the world vector fields $\boldsymbol{V} = v^\mu\partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma^\mu{}_{\nu\lambda}$ is given by

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\lambda\nu}V^\lambda. \tag{4}$$

The requirement

$$\nabla_l V^k = e^\nu{}_l e^k{}_\mu \nabla_\nu V^\mu, \quad \text{for} \quad V^\mu = e^\mu{}_k V^k \quad \text{leads to} \quad \Gamma^\mu{}_{\lambda\nu} = e^\mu{}_k \partial_\nu e^k{}_\lambda, \tag{5}$$

and hence we have the relations

$$T^k{}_{\mu\nu} = e^k{}_\lambda T^\lambda{}_{\mu\nu}, \quad R^\mu{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\tau\lambda}\Gamma^\tau{}_{\nu\rho} - \Gamma^\mu{}_{\tau\rho}\Gamma^\tau{}_{\nu\lambda} \equiv 0, \tag{6}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda}g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda}g_{\mu\rho} \equiv 0, \quad \text{where} \quad T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}. \tag{7}$$

The components $T^\lambda{}_{\mu\nu}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion and of the curvature tensor, respectively. Equation (5) implies the teleparallelism and it together with (6) means that M is the Weitzenböck space–time. Also from Eq. (6) we obtain

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\} + \boldsymbol{K}^\lambda{}_{\mu\nu}, \quad \text{with} \quad \left\{{}^{\lambda}_{\mu\nu}\right\} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}\right), \tag{8}$$

and the $\boldsymbol{K}^\lambda{}_{\mu\nu}$ denotes for the contorsion tensor

$$\boldsymbol{K}^\lambda{}_{\mu\nu} = -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu\right) = -K_\mu{}^\lambda{}_\nu. \tag{9}$$

In what follows, the field components $e^k{}_\mu$ and $e^\mu{}_k$ are used to convert Latin and Greek indices, similar to the case of V^μ and V^k. Also, rasing and lowering the indices $k,l,m\ldots$ are accomplished with the aid of $\eta_{ij} = (\eta^{ij})^{-1}$ and η^{ij} respectively.

3. Lagrangian density and gravitation field equation

In this paper we follow the notions given in Ref. [63]. The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[2),[62]]

1) We will denote the symmetric part by ( ), for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$.

2) Throughout this paper we use the relativistic units, $c = G = 1$.

$$\mathcal{L}_G = ee^{-2\phi}L_G = -ee^{-2\phi}\left(\frac{\boldsymbol{T}^{abc}\boldsymbol{T}_{abc}}{4} + \frac{\boldsymbol{T}^{abc}\boldsymbol{T}_{bac}}{2} - \boldsymbol{T}^a T_a\right) - L_m = -\frac{ee^{2\phi}}{16\pi}\boldsymbol{\Sigma}^{abc}\boldsymbol{T}_{abc} - L_m, \tag{10}$$

where $e = \det(e^a{}_\mu)$ and ϕ is the dilaton. The tensor $\boldsymbol{\Sigma}^{abc}$ is defined by

$$\boldsymbol{\Sigma}^{abc} \stackrel{\text{def.}}{=} \frac{1}{4}\left(\boldsymbol{T}^{abc} + \boldsymbol{T}^{bac} - \boldsymbol{T}^{cab}\right) + \frac{1}{2}\left(\eta^{ac}\boldsymbol{T}^b - \eta^{ab}\boldsymbol{T}^c\right). \tag{11}$$

$\boldsymbol{T}^{abc}$ and $\boldsymbol{T}^a$ are the torsion tensor and the basic vector field defined by

$$\boldsymbol{T}^a{}_{\mu\nu} \stackrel{\text{def.}}{=} e^a{}_\lambda T^\lambda{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu, \qquad \boldsymbol{T}^a{}_{bc} \stackrel{\text{def.}}{=} e_b{}^\mu e_c{}^\nu T^a{}_{\mu\nu}, \qquad \boldsymbol{T}^a \stackrel{\text{def.}}{=} T^b{}_b{}^a. \tag{12}$$

The quadratic combination $\boldsymbol{\Sigma}^{abc}\boldsymbol{T}_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term.[62] L_m represents the Lagrangian density for matter fields.

The electromagnetic Lagrangian density $L_{e.m.}$ is[76,77]

$$\mathcal{L}_{e.m.} = e\ e^{-2\phi}\ L_{e.m.} = \frac{1}{4}e\ e^{-2\phi}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{13}$$

with $F_{\mu\nu}$ being the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$ and is defined by[1)]

$$F_{\mu\nu} \stackrel{\text{def.}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Finally the dilaton Lagrangian density $L_{\rm D}$ is[78]

$$\mathcal{L}_{\rm D} = ee^{-2\phi}\ L_{\rm D} = 4e\ e^{-2\phi}\left(\bigtriangledown\phi\right)^2. \tag{14}$$

The gravitational, electromagnetic, dilaton and the cosmological constant $\Lambda = -2\lambda^2$ field equations for the system described by $L_G - L_{e.m.} + L_{\rm D} + 4e^{-2\phi}\lambda^2$ are as follows:

$$\begin{aligned}
&e_{a\lambda}e_{b\mu}\partial_\nu\left(e\boldsymbol{\Sigma}^{b\lambda\nu}\right) - e\left(\boldsymbol{\Sigma}^{b\nu}{}_a\boldsymbol{T}_{b\nu\mu} - \frac{1}{4}e_{a\mu}\boldsymbol{T}_{bcd}\boldsymbol{\Sigma}^{bcd}\right) - e\ e_{a\mu}\ \lambda^2 = -\frac{1}{2}ee^{2\phi}T_{a\mu},\\
&\nabla_\mu\left(e^{-2\phi}F^{\mu\nu}\right) = 0,\\
&4\lambda^2 + 2\nabla^2\phi - 4\left(\nabla\phi\right)^2 + \frac{1}{4}F^2 = 0,
\end{aligned} \tag{15}$$

where

$$e^{2\phi}\boldsymbol{T}_{\mu\nu} = 2\left\{\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\left(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - \nabla^2\phi\right) - \frac{1}{4}\left(g_{\nu\sigma}F_{\mu\rho}F^{\sigma\rho} - \frac{1}{4}g_{\mu\nu}F^2\right)\right\},$$

with $\boldsymbol{T}_{\mu\nu}$ being the energy–momentum tensor. It is possible to prove by explicit calculations that the left–hand side of the symmetric field equation (15) is exactly given by[62]

$$\frac{e}{2}\left[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e) - 2\ e_{a\mu}\ \lambda^2\right].$$

4. 2-dimensional spherically symmetric dilatonic black hole solution

The tetrad space having two unknown functions of radial coordinate with spherical symmetry, can be written as

$$(\boldsymbol{e}_a{}^\mu) = \begin{pmatrix} A & 0 \\ 0 & B(r) \end{pmatrix}. \tag{16}$$

Applying Eq. (16) to the field equation (15) we obtain simple algebraic equations that have the forms

$$e^{2\phi}T_{00} = \frac{2\lambda^2}{A^2(r)},$$

1) Heaviside-Lorentz rationalized units will be used throughout this paper.

$$e^{2\phi}T_{11} = \frac{2\lambda^2}{B^2(r)}. \tag{17}$$

From the above equations, we obtain

$$\begin{aligned} A(r) &= \frac{1}{\sqrt{1-\dfrac{M}{\lambda}\mathrm{e}^{-2r\lambda}+\dfrac{Q^2}{8\lambda^2}\mathrm{e}^{-4r\lambda}}}, \\ B(r) &= \sqrt{1-\frac{M}{\lambda}\mathrm{e}^{-2r\lambda}+\frac{Q^2}{8\lambda^2}\mathrm{e}^{-4r\lambda}}, \\ F_{\mathrm{tr}} &= Q\mathrm{e}^{-2r\lambda}, \qquad\qquad \phi = -r\lambda. \end{aligned} \tag{18}$$

Solution given by Eq. (18) satisfies Eq. (15). The tetrad (16) takes the form

$$(e_i{}^{\mu}) = \begin{pmatrix} \dfrac{1}{\sqrt{1-\dfrac{M}{\lambda}\mathrm{e}^{-2r\lambda}+\dfrac{Q^2}{8\lambda^2}\mathrm{e}^{-4r\lambda}}} & 0 \\ 0 & \sqrt{1-\dfrac{M}{\lambda}\mathrm{e}^{-2r\lambda}+\dfrac{Q^2}{8\lambda^2}\mathrm{e}^{-4r\lambda}} \end{pmatrix}, \tag{19}$$

with the associated Riemannian metric

$$\mathrm{d}s^2 = -\eta_1\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\eta_1}, \qquad \text{where} \qquad \eta_1 = \left[1-\frac{M}{\lambda}\mathrm{e}^{-2r\lambda}+\frac{Q^2}{8\lambda^2}\mathrm{e}^{-4r\lambda}\right], \tag{20}$$

which is a static 2D dilatonic spherically symmetric black hole solution.[23,24] In this coordinate system the metric approaches the flat space–time when the radial coordinate, i.e., $r \to \infty$. There is no singularity as $r \to 0$ and when $r \to \frac{1}{2\lambda}\left[\frac{2M+\sqrt{4M^2-2Q^2}}{4\lambda}\right]$, Eq. (20) will suffer a singularity.

Now we discuss the singularity of the above space–time by calculating the scalar concomitans of the torsion and the curvature tensors. Using Eq. (18) we have the non-vanishing components of the Riemann Christoffel curvature, Ricci tensor, torsion tensor and the basic vector in the following forms:

$$\begin{aligned} R_{0101}(\{\}) &= R_{1010}(\{\}) = \mathrm{e}^{-2r\lambda}\left(2M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right), \\ R_{00}(\{\}) &= \frac{\mathrm{e}^{-2r\lambda}\left(2M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)\left(8\lambda^2 - 8M\lambda\mathrm{e}^{-2r\lambda} + Q^2\mathrm{e}^{-4r\lambda}\right)}{8\lambda^2}, \\ R_{11}(\{\}) &= \frac{8\lambda^2\mathrm{e}^{-2r\lambda}\left(2M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)}{8M\lambda\mathrm{e}^{-2r\lambda} - 8\lambda^2 - Q^2\mathrm{e}^{-4r\lambda}}, \\ T_{001} &= \frac{\mathrm{e}^{-2r\lambda}\left(4M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)}{4\lambda}, \\ T_1 &= \frac{2\lambda\mathrm{e}^{-2r\lambda}\left(4M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)}{8M\lambda\mathrm{e}^{-2r\lambda} - 8\lambda^2 - Q^2\mathrm{e}^{-4r\lambda}}. \end{aligned} \tag{21}$$

The scalars of the Riemann Christoffel curvature tensor, Ricci tensor, Ricci scalar, the torsion tensor and the basic vector in the following forms:

$$\begin{aligned} R^{\mu\nu\rho\sigma}(\{\})R_{\mu\nu\rho\sigma}(\{\}) &= 2R^{\mu\nu}(\{\})R_{\mu\nu}(\{\}) = 4\mathrm{e}^{-4r\lambda}\left(M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)^2, \\ R(\{\}) &= -2\mathrm{e}^{-2r\lambda}\left(2M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right), \\ T^{\mu\nu\rho}T_{\mu\nu\rho} &= 2T^{\mu}T_{\mu} = \frac{\mathrm{e}^{-4r\lambda}\left(4M\lambda - Q^2\mathrm{e}^{-2r\lambda}\right)^2}{Q^2\mathrm{e}^{-4r\lambda} - 8M\lambda\mathrm{e}^{-2r\lambda} + 8\lambda^2}. \end{aligned} \tag{22}$$

We can conclude from Eq. (22) that as $r \to 0$ solution (18) does not suffer any singularity which can be confirmed from Eq. (22) as $r \to 0$ we have

$$\begin{aligned} &R^{\mu\nu\rho\sigma}(\{\})R_{\mu\nu\rho\sigma}(\{\}) = 2R^{\mu\nu}(\{\})R_{\mu\nu}(\{\}) \to 4(2M\lambda - Q^2)^2, \qquad R \to -2(2M\lambda - Q^2), \\ &T^{\mu\nu\rho}T_{\mu\nu\rho} = 2T^{\mu}T_{\mu} \to \frac{\left(4M\lambda - Q^2\right)^2}{Q^2 - 8M\lambda + 8\lambda^2}. \end{aligned} \tag{23}$$

When the radial coordinate $r \to \infty$ we can see from Eq. (22) that solution (18) will be the flat space–time. From Eq. (22) we see that the scalars of the Riemann Christoffel curvature tensor, Ricci tensor, Ricci scalar, the torsion tensor and the basic vector are vanishing, conforming with the fact that solution (18) when $r \to \infty$ is a flat space–time. When $r \to \frac{1}{2\lambda} \ln \left[\frac{2M+\sqrt{4M^2-2Q^2}}{4\lambda}\right]$ we see that the torsion tensor and the basic vector suffer this singularity.

5. Energy associated with spherically symmetric solutions

Now we are going to calculate the energy content of the tetrad field (19). Before this we give a brief review of the derivation of the gravitational energy-momentum in a four-dimensional background.

Multiplying the symmetric part of Eq. (15) with the appropriate inverse tetrad field yields[63,75]

$$\partial_\nu \left(-e\boldsymbol{\Sigma}^{a\lambda\nu}\right) = -\frac{ee^{a\mu}}{4}\left(4\boldsymbol{\Sigma}^{b\lambda\nu}T_{b\nu\mu} - \delta^\lambda{}_\mu \boldsymbol{\Sigma}^{bdc}T_{bcd}\right) - 4\pi e^a{}_\mu T^{\lambda\mu}. \tag{24}$$

In order to restrict the spacetime index λ, we assume only spatial values. Then Eq. (24) takes the form[63]

$$\partial_0 \left(e\boldsymbol{\Sigma}^{a0j}\right) + \partial_k \left(e\boldsymbol{\Sigma}^{akj}\right) = -\frac{ee^{a\mu}}{4}\left(4\boldsymbol{\Sigma}^{bcj}T_{bc\mu} - \delta^j{}_\mu \boldsymbol{\Sigma}^{bcd}T_{bcd}\right) - 4\pi e e^a{}_\mu T^{j\mu}. \tag{25}$$

Note that the last two indices of Σ^{abc} and T^{abc} are anti-symmetric. Taking the divergence of Eq. (25) with respect to j yields

$$-\partial_0\partial_j\left(-\frac{1}{4\pi}e\boldsymbol{\Sigma}^{a0j}\right) = -\frac{1}{16\pi}\partial_j\left[ee^{a\mu}\left(4\boldsymbol{\Sigma}^{bcj}T_{bc\mu} - \delta^j{}_\mu\boldsymbol{\Sigma}^{bcd}T_{bcd}\right) - 16\pi(ee^a{}_\mu T^{j\mu})\right]. \tag{26}$$

In the Hamiltonian formulation of the TEGR,[40−42,72] the momentum canonically conjugated to the tetrad components e_{aj} is given by

$$\Pi^{aj} = -\frac{1}{4\pi}e\boldsymbol{\Sigma}^{a0j},$$

and the energy contained within a volume V of the three-dimensional space-like hypersurface is defined by[40−42,68]

$$P^0 = E = -\int_V \mathrm{d}^3x\partial_j\Pi^{0j} = \frac{1}{8\pi}\int_V \mathrm{d}^3x\partial_j(eT^j). \tag{27}$$

If no condition is imposed on the tetrad field, P^a transforms as a vector under the global SO(3,1) group. It describes the gravitational energy-momentum with respect to observers adapted to $e^a{}_\mu$. We assume that the spacetime is asymptotically flat. Therefore, the energy of the gravitational energy-momentum for a physical system for a two-dimensional gravitational background is given as

$$P^{(0)} = E = \frac{1}{2}\int \mathrm{d}^1x\partial_j(eT^j). \tag{28}$$

In order to evaluate the energy distribution of the exterior two-dimensional black hole of Eq. (19) we use Eqs. (28) and (12) to evaluate the nonzero components of T^j. The non-vanishing components of T^j are as follows:

$$T^1 = -\frac{\mathrm{e}^{-2\lambda r}\sqrt{8\lambda^2 - 8M\lambda\mathrm{e}^{-2\lambda r} + Q^2\mathrm{e}^{-4\lambda r}}\left(4M\lambda - Q^2\mathrm{e}^{-2\lambda r}\right)}{8\lambda^2}, \qquad e = 1, \tag{29}$$

using Eq. (29) in Eq. (28) we have

$$\begin{aligned} P^{(0)} = E_{\mathrm{ext}}(R) &= \frac{1}{2}\int_R^\infty \mathrm{d}r\frac{\partial}{\partial r}(T^j) = \frac{-\sqrt{2}}{32}\frac{\mathrm{e}^{-2\lambda r}\sqrt{8\lambda^2 - 8M\lambda\mathrm{e}^{-2\lambda r} + Q^2\mathrm{e}^{-4\lambda r}}\left(4M\lambda - Q^2\mathrm{e}^{-2\lambda r}\right)}{\lambda^2} \\ &\cong \frac{m}{2R^2} \quad \text{up to} \quad O(1/R^2). \end{aligned} \tag{30}$$

Therefore, the energy associated with a two-dimensional Schwarzschild black hole in a sphere of radius R will be

$$E(R) \cong m\left(1 - \frac{1}{2R^2}\right). \tag{31}$$

As we see from Eq. (31) that the energy of two-dimensional Schwarzschild black hole is distributed to its interior as well as its exterior up to $O(1/R^2)$.

6. Main results and discussion

In this paper we have constructed a two-dimensional TEGR theory. The field variable in this theory is the dreibein field $e_i{}^\mu$ and the gravity is attributed to the torsion. The gravitational Lagrangian of the TEGR is given by Eq. (10).

A static spherically symmetric black hole solution is obtained, which represents the 2D dilatonic black hole. To study the physics of this solution we have calculated the scalars accompanied by this solutions like the scalar of Riemann-Christoffel, the Ricci tensor, the torsion tensor and the basic vector. Mielke *et al.*[35] have obtained exact wave, charged black hole and dilaton solutions. They[61] have proved in the vacuum, the integrability of general two-dimensional Poincaré gauge theory. The solutions given in Ref. [61] are characterized by the non-vanishing of the curvature tensor as well as the torsion tensor.

We first, obtain the exact expression of the gravitational energy-momentum in two-dimensional gravitational background given by Eq. (28). We have found the explicit expression for the energy distribution associated with a two-dimensional dilatonic black hole. It is shown that the energy is distributed in its interior as well as its exterior.

As seen clearly from Eq. (30), the exterior energy begins from $O(1/R^2)$. Therefore, the exterior energy of $O(1/R)$ does not appear in the two-dimensional theory! This is in discrepancy with the 4-dimensional gravity theory. This may be due to the fact that Eq. (28) depends mainly on the basic vector T^a and from its definition as given by expression (12) its construction will behave as the solution given in Eq. (18) like $O(1/R^2)$ and $O(1/R^3)$. This case does not happen to the 4-dimensional gravity theory because the construction of Eq. (28) will behave like $O(1/R)$.[79]

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