The ground-state entanglement in the spin-XX chain with a magnetization current*

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The ground-state entanglement in a transverse spin-1/2 XX chain with a magnetization current is studied. By introducing a magnetization current to the system, a quantum phase transition to current-carrying phase may be presented with the variation of the driving field λ for the magnetic field $h > 1$; and the ground-state entanglement arises simultaneously at the critical point of quantum phase transition. In our model, the introduction of magnetization current may result in more entanglement between any two nearest-neighbour spins.

Keywords: entanglement, current, correlation, concurrence

PACC: 0367, 0560, 7510J, 0570L

1. Introduction

In the past few decades, quantum entanglement has attracted more attentions due to its potential application in quantum-information processing and quantum computation.[1,2] In addition, it has been argued that entanglement may play an essential role in the description and understanding of quantum phase transition[3] presented in many-body condensed-matter systems.[4−8] Therefore, much effort has been made in the past years to investigate the quantum entanglement in various physical systems. Among such systems exactly solvable one-dimensional (1D) quantum spin-1/2 systems (spin chains) with various interactions and anisotropies (e.g., the XY model) have been extensively studied in the context of quantum information science.[9−11]

It has been shown that many 1D integrable quantum many-body systems which are characterized by a macroscopic number of conservation law, will present very anomalous dissipationless transport behaviours.[12] For example, the transport of energy (energy current) is closely related to the first conservation law in these systems.[13] Antal and Rácz suggested a method of studying a nonequilibrium steady state (NESS) of an Ising and isotropic XY chain in transverse field by imposing an energy current,[14] which has been shown to have a measurable effect in a realistic experimental setup,[15] or a magnetization current on the system.[16] They find that the presence of energy and magnetization current changes the large-distance decay of correlations. Furthermore, it has been shown that the introduction of an energy current into the XX chain results in more block entanglement in the system.[17] Motivated by these works, we have studied transverse three-spin Ising and XY models,[18−20] and found that the presence of current was accompanied with sudden changes of quantum entanglement. But we have a poor understanding of the relationship between the magnetization current and bipartite entanglement in an infinite-long spin chain. Here we investigate the nearest-neighbour entanglement in NESS with a magnetization current in a transverse XX chain, and study the changes produced by the presence of the current in the quantum entanglement and in the quantum phase of the system. It is additionally hoped that the connection between quantum entanglement and quantum phase transition will be more clear with the investigation.

The present study investigates the well-known XX model by imposing a magnetization current on the system. The structure of the paper is as follows. Section 2 constructs an effective Hamilton with magnetization current by using the Lagrange multiplier method. Section 3 gives the reduced two-spin density matrix of the nonequilibrium steady state, and reviews a measure of entanglement. Section 4 is devoted to the explicit expression of correlation functions in the

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ground state of the system, and investigates the properties of nearest-neighbour entanglement. Section 5 summarizes the results of this investigation.

2. Magnetization current and effective Hamiltonian

We consider a 1D spin-1/2 XX chain in a transverse field h:

$$H = -J\sum_{\ell=1}^{N}(s_\ell^x s_{\ell+1}^x + s_\ell^y s_{\ell+1}^y + h s_\ell^z), \qquad (1)$$

where s_i^α ($\alpha = x, y, z$) are the well-known 1/2 Pauli operators on the ℓth ($\ell = 1, 2, \ldots, N$) spin of the chain with periodic boundary conditions ($s_{N+1}^\alpha = s_1^\alpha$). Without loss of generality, the coupling strength J is set to $J = 1$ throughout the paper, and h is the transverse field in units of J. With the Jordan–Wigner transformation, the model can be exactly solved in the thermodynamic limit ($N \to \infty$) with energy spectrum

$$\Lambda_\phi = -(\cos\phi + h), \quad (-\pi \le \phi \le \pi). \qquad (2)$$

In the model, the z component of total magnetization $M_z = \sum_\ell s_\ell^z$ is conserved. The conserved quantity of the system is represented by magnetization current operator.[13] The local magnetization current can be defined using a continuity equation for the local magnetization s^z:

$$j_\ell^M = s_\ell^y s_{\ell+1}^x - s_\ell^x s_{\ell+1}^y. \qquad (3)$$

Surprisingly, the expression is in accord with the Dzyaloshinskii–Moriya (DM) interaction which arises from spin–orbit coupling.[21,22] Moreover, the same terms emerge as a part of current corresponding to energy conservation in the transverse XX system.[16]

Easily, one can find that the macroscopic magnetization current $J^M = \sum_\ell j_\ell^M$ commutes with the Hamiltonian (1), i.e., $[J^M, H] = 0$. In other words, the system Hamiltonian and the macroscopic current operator have a series of common eigenstates. Using the Lagrange multiplier method, we can impose a fixed magnetization current J^M to the states of the system,

$$\mathscr{H} = H - \lambda J^M, \qquad (4)$$

where, without loss of generality, $h \ge 0$ and $\lambda \ge 0$ are assumed. The Hamiltonian can be regarded as an effective Hamiltonian with the competition between the J-exchange interaction and λ-DM interaction. Obviously, an eigenstate of the effective Hamiltonian $\mathscr{H}$ must still be a stationary state of H. However, these eigenstates may carry the magnetization current described by J^M, which is just desired in our investigation. In this study, we mainly aim to investigate the ground state of $\mathscr{H}$ which carries a given magnetization current. It is noteworthy that $\mathscr{H}$ is just another equilibrium Hamiltonian mathematically constructed to give nonequilibrium state of H, and the time evolution of the system is still governed by H. Now, the energy spectrum can be solved again in the thermodynamic limit,

$$\Lambda(\lambda) = -(\cos\phi + h + \lambda\sin\phi). \qquad (5)$$

Clearly, the modes with negative energy are occupied in the ground state of $\mathscr{H}$. When $\lambda = 0$, the $\phi = \pm\pi$ gap disappears at $h = 1$ and the ground state reaches its critical point of quantum phase transition as shown by many authors.[23] Figure 1 shows the spectrum of the transverse XX model in the presence of a field (λ) which drives the magnetization current. We can find that the $\phi \to -\phi$ symmetry of the spectrum is broken for $\lambda \ne 0$. The ground-state properties maybe change at some values of ϕ, which depends on the fields h and λ. For the case of $h < 1$, $\Lambda(\lambda)$ always change sign with ϕ increasing for any λ, accordingly, the magnetization current is surely presented. For the case of $h > 1$, with small λ the ground state remains the same of H and no magnetization current flows through the system ($J^M = 0$). When $\lambda > \lambda_c$ ($\lambda_c = \sqrt{h^2 - 1}$) for $h > 1$, the ground state will undergo a quantum phase transition at some ϕ from current-free phase to current-carrying phase. For the sakes of convenience, we define an effective field $\widetilde{h} = h/\sqrt{\lambda^2 + 1}$. When the effective field $\widetilde{h} < 1$ for any values of h, the magnetization, evidently, will be driven to flow (i.e., $J^M > 0$).

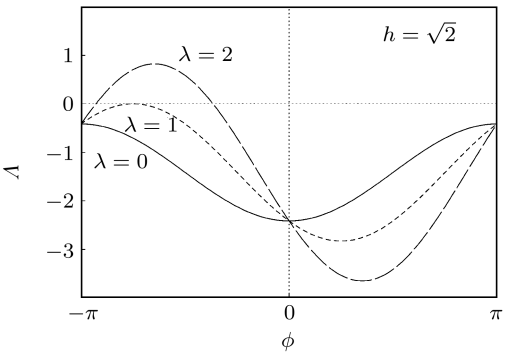


Fig. 1. Spectrum of the transverse XY model in the field λ for $h = \sqrt{2}$.

It should be mentioned that the Hamiltonian $\mathscr{H}$ can be transformed into H using a unitary transformation

$$\mathcal{U} = \exp\left(\mathrm{i}\sum_{\ell=1}^{N}\ell\varphi s_{\ell}^{z}\right). \tag{6}$$

Here, $\varphi = \arctan\lambda$. Then, the NESS and its correlation functions can be derived from the eigenstates of the Hamiltonian H with the unitary transformation.

3. Reduced density matrix and concurrence

To examine properties of pairwise entanglement in the ground state of $\mathscr{H}$, we must calculate two-spin density matrix, which is the joint state of two spins at sites ℓ and $\ell + r$ (r denotes the distance between two sites). The two-spin density matrix can be constructed from the one- and two-site correlation functions, using the operator expansion in terms of tensor products of Pauli matrices.[4] For the two-site density matrix, the operator expansion takes the form

$$\rho_{\ell,\ell+r} = 4\sum_{\alpha,\beta=0}^{3} p_{\alpha\beta}s_{\ell}^{\alpha}\otimes s_{\ell+r}^{\beta}. \tag{7}$$

Here, the coefficients $p_{\alpha\beta}$ are determined by the correlation functions between two spins,

$$p_{\alpha\beta} = G_{r}^{\alpha\beta} = \langle s_{\ell}^{\alpha}\otimes s_{\ell+r}^{\beta}\rangle, \tag{8}$$

here s_i^{α}, s_j^{β} are 1/2 Pauli operators with the convention $s_i^0 = I_i$. So that if the relevant correlation functions are known it is possible to construct the two-spin density matrix completely.

Reflection symmetry about any site in the system means that $\rho_{ij} = \rho_{ji}$. Therefore, we have the symmetric relation $p_{\alpha\beta} = p_{\beta\alpha}$. Due to the global phase-flip symmetry of the system, which implies $[s_{\ell}^{z}s_{\ell+r}^{z}, \rho_{\ell,\ell+r}] = 0$, the coefficients, $p_{01} = p_{10}$, $p_{02} = p_{20}$, $p_{13} = p_{31}$ and $p_{23} = p_{32}$ must be zero. Moreover, the z component of the total magnetization commutes with the effective Hamiltonian, $[M^z, \mathscr{H}] = 0$, which implies that for any two spins there can be no coherent superposition of basis states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. Then, the coefficients of the matrix elements $|\uparrow\uparrow\rangle\langle\downarrow\downarrow|$ and $|\downarrow\downarrow\rangle\langle\uparrow\uparrow|$ in the reduce density matrix must equal to zero, which means $p_{11} = p_{22}$ and $p_{12} = -p_{21}$ (i.e., $G_r^{xx} = G_r^{yy}$, $G_r^{xy} = -G_r^{yx}$).

Therefore, the reduced two-spin density matrix can be expressed as

$$\boldsymbol{\rho}_{\ell,\ell+r} = \begin{pmatrix} u^{+} & 0 & 0 & 0 \\ 0 & w & z & 0 \\ 0 & z^{*} & w & 0 \\ 0 & 0 & 0 & u^{-} \end{pmatrix}, \tag{9}$$

here

$$\begin{aligned} u^{\pm} &= \frac{1}{4} \pm m_z + G_r^{zz}, \\ m_z &= M_z/N = \langle s^z\rangle, \\ w &= \frac{1}{4} - G_r^{zz}, \\ z &= G_r^{xx} + G_r^{yy} + \mathrm{i}G_r^{xy} - \mathrm{i}G_r^{yx}. \end{aligned} \tag{10}$$

To explore entanglement properties of the current-carrying ground state in the transverse spin-XX chain, a well-known way to quantify the entanglement is by means of the concurrence.[24−26] Because the entanglement between a pair of adjacent spins is expected to be dominant compared with a pair of nonnearest-neighbour spins, we focus on the nearest-neighbour concurrence in the following discussions. From the density matrix given in (9), the concurrence quantifying the entanglement between two particles in ρ is readily obtained as[27,28]

$$\mathcal{C} = 2\max\{|z| - \sqrt{u^{+}u^{-}}, 0\}. \tag{11}$$

Therefore, the investigation on concurrence can be performed with correlation functions between two particles. The next section gives these correlation functions to determine the concurrence of ground-state entanglement in NESS.

4. Correlation functions and entanglement

For 1D XY model in a constant transverse field, Barouch and McCoy have given exact and asymptotic solutions of the magnetization and correlation functions.[29,30] In the thermodynamic limit, $N \to \infty$, the magnetization and correlation functions can be expressed as:

$$m_z = -\frac{1}{2\pi}\int_0^{\pi}\frac{h + \cos\phi}{|\Lambda_{\phi}|}\,\mathrm{d}\phi, \tag{12}$$

$$\boldsymbol{G}_r^{xx} = \frac{1}{4}\begin{vmatrix} G_{-1} & G_{-2} & \cdots & G_{-r} \\ G_0 & G_{-1} & \cdots & G_{-r+1} \\ \vdots & & & \vdots \\ G_{r-2} & G_{r-3} & \cdots & G_{-1} \end{vmatrix}, \tag{13}$$

$$\boldsymbol{G}_r^{yy} = \frac{1}{4}\begin{vmatrix} G_1 & G_0 & \cdots & G_{-r+2} \\ G_2 & G_1 & \cdots & G_{-r+3} \\ \vdots & & & \vdots \\ G_r & G_{r-1} & \cdots & G_1 \end{vmatrix}, \tag{14}$$

$$\boldsymbol{G}_r^{zz} = m_z^2 - \frac{1}{4}G_r G_{-r}, \tag{15}$$

where

$$\boldsymbol{G}_k = \frac{1}{\pi}\int_0^\pi \frac{\cos(k\phi)(h+\cos\phi)}{|\Lambda_\phi|}\,\mathrm{d}\phi. \tag{16}$$

As the transformation (6) does not change the z component of the spins, the average magnetization m_z and the correlation function G_r^{zz} in the current-carrying nonequilibrium steady state have their equilibrium form but an effective field $\widetilde{h}$. We can get directly the average magnetization and the nearest-neighbour correlation function in the current-carrying ground state by replacing the field h with $\widetilde{h}$,

$$m_z = \begin{cases} -\dfrac{1}{\pi}\arcsin\dfrac{h}{\sqrt{1+\lambda^2}}, & \widetilde{h} < 1, \\ -\dfrac{1}{2}, & \widetilde{h} \geq 1 \end{cases} \tag{17}$$

and the correlation function in the z direction

$$G_1^{zz} = \begin{cases} m_z^2 - \dfrac{1+\lambda^2-h^2}{\pi^2(1+\lambda^2)}, & \widetilde{h} < 1, \\ m_z^2, & \widetilde{h} \geq 1. \end{cases} \tag{18}$$

Obviously, when $\widetilde{h} \geq 1$, all spin align paralleled with the field $\widetilde{h}$ and, consequently, no magnetization current is presented. Equally important, all other correlation functions vanishe for $\widetilde{h} \geq 1$.

With the help of the unitary transformation (6), the calculation of the correlation functions $G_r^{xx} = G_r^{yy}$ for $\widetilde{h} < 1$ in the ground state of $\mathscr{H}$ can be simplified:

$$\begin{aligned} G_1^{xx} = G_1^{yy} &= \cos(\varphi)\langle\widetilde{\psi}_0|s_\ell^x s_{\ell+1}^x|\widetilde{\psi}_0\rangle \\ &= -\frac{\sqrt{1+\lambda^2-h^2}}{2\pi(1+\lambda^2)}, \end{aligned} \tag{19}$$

where $|\widetilde{\psi}_0\rangle$ is the ground state of H in the effective field $\widetilde{h}$.

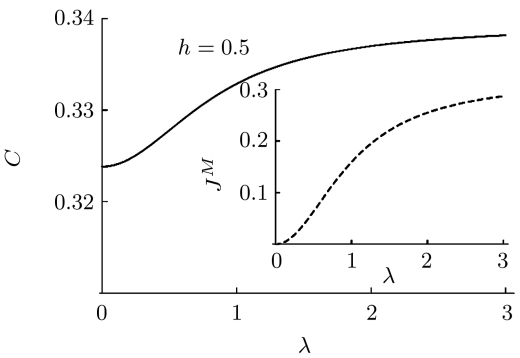
For the spin-XX chain governed by H, the correlation functions $\langle s_i^x s_j^y\rangle$ and $\langle s_i^y s_j^x\rangle$ must be zero. For the NESS governed by $\mathscr{H}$, the correlation functions G_1^{xy} and G_1^{yx} and the magnetization current are both expressed as the product of two fermion operators and can be calculated exactly. They are all zero for $\widetilde{h} \geq 1$ while, for $\widetilde{h} < 1$, one obtains

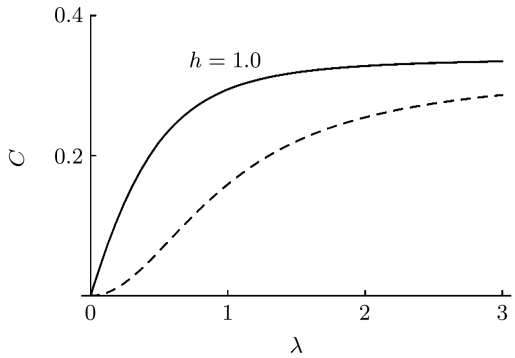
$$G_1^{yx} = -G_1^{xy} = \frac{j^M}{2}, \tag{20}$$

$$\begin{aligned} j^M &= \frac{1}{N}\sum_\ell^N \langle s_\ell^y s_{\ell+1}^x - s_\ell^x s_{\ell+1}^y\rangle \\ &= \frac{\lambda\sqrt{1+\lambda^2-h^2}}{\pi(1+\lambda^2)}. \end{aligned} \tag{21}$$

Obviously, the current increases with the increase of λ, and reaches its maximum at the limit of $\lambda \to \infty$.

Using the solutions obtained above for the thermodynamic limit, we can quantitatively investigate the ground-state entanglement in the transverse XX chain with a magnetization current. The nearest-neighbour concurrence and the magnetization current in NESS are depicted as Fig. 2.





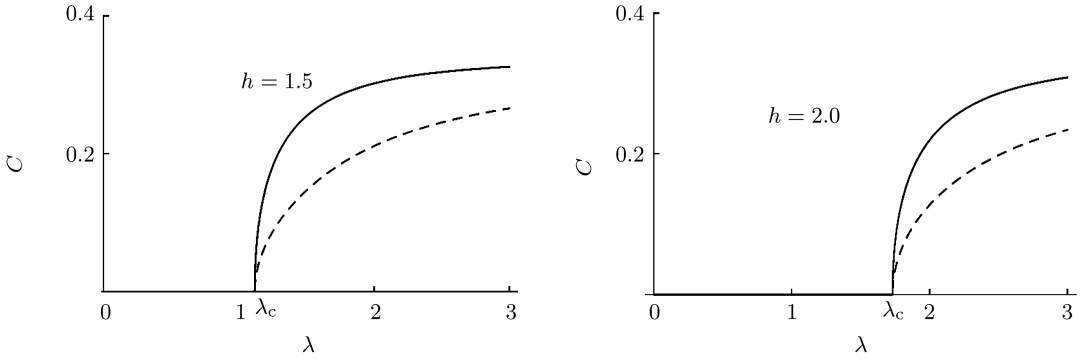


Fig. 2. Concurrence $\mathcal{C}$ of the ground-state entanglement (solid line) and the current of magnetization j^M (dash line) at various current driving λ. The magnetic field is set to $h = 0.5, 1.0, 1.5, 2.0$.

It can be seen that the current of magnetization will be switched on at the critical driving field λ_c ($\lambda_c = \sqrt{h^2 - 1}$). Further increasing λ one entres the current-carrying phase and observes a current growth. Quantum phase transition is realized in this exactly solvable model, which is caused by the competition between J-exchange interaction and λ-DM interaction. As one starts to decrease the magnetic field h, the critical value λ_c approaches zero. From the viewpoint of effective field $\widetilde{h}$, the current will be surely driven by the effective field $\widetilde{h} < 1$. The properties of the nearest-neighbour concurrence seem more complicated. A simultaneity can be observed between the emergence of ground-state entanglement and that of magnetization current at the critical field $\lambda_c \geq 0$ for the magnetic field $h \geq 1$. Moreover, the ground-state entanglement is enhanced significantly with the increase of the current. When $h < 1$, the ground state of $\mathscr{H}$ is initially entangled for $\lambda = 0$, and with increasing λ, the concurrence increases inapparently. The enhancement effect of the current on the ground-state entanglement may be illustrated by the form of effective Hamiltonian $\mathscr{H}$. When the transverse magnetic field h is much less than one, the term of exchange interaction $s^x s^x + s^y s^y$ dominates over the system and, hence, any two neighbour spins in the chain are entangled. The term of current may enhance the exchange between two neighbour spins, then results in more nearest-neighbour entanglement. It is interesting to compare the enhancement effect on entanglement of magnetization current with that of energy current. As noted above, the expression in Eq. (3) is only a part of the expression of energy current. The Hamiltonian when the energy current is introduced can be regarded as an effective Hamiltonian with competition between the two-spin J-exchange interaction, the λ-DM interaction and the three-spin interaction. Therefore, the energy current has a more complex effect on entanglement than the magnetization current.[17] From the above we can come to the conclusion that the introduction of current may lead to a higher level of ground-state entanglement.

5. Discussion and conclusion

It should be noted that this study has examined only the nearest-neighbour entanglement in a spin-XX chain. How about the bipartite entanglement between any two spins behaves when a current is introduced? As noted above, the introduction of current makes a rather drastic change in the long-distance correlations. We believe that the ground-state entanglement between two distant spins maybe behaving anomalously due to the introduction of current. Thus it is a interesting next step to find out the effect of current on the entanglement associated with two distant spins in a spin chain.

In conclusion, in this study we have performed exactly the investigation on the ground-state entanglement of a transverse XX model by introducing a magnetization current. It has been shown that the introduction of a magnetization current can result in more entanglement. For the magnetic field $h > 1$, there is a quantum critical point ($\lambda_c = \sqrt{h^2 - 1}$) of the driving field in the system, and the critical value approaches to zero with the decrease of the magnetic field. When the driving field exceeds its critical point, the ground state of the system enters to the current-carrying phase from the current-free phase and, simultaneously, becomes to entangled between any two nearest-neighbour spins. Moreover, the ground-state entanglement is enhanced significantly with the increase of the current. In contrast, the ground state is initially entangled at $\lambda = 0$ for $h < 1$, and the degree of entanglement grows inapparently with increasing the current.

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