Relative entropy of entanglement of two-qubit 'X' states*

Huang Jie-Hui(黄接辉)†, Liu Nian-Hua(刘念华), Liu Jiang-Tao(刘江涛), Yu Tian-Bao(于天宝), and He Xian(何 弦)

Department of Physics, Nanchang University, Nanchang 330031, *China*

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Two closest single-qubit states could be diagonalised by the same unitary matrix, which helps to find the relative entropy of entanglement of a two-qubit 'X' state. We formulate two binary equations for the relative entropy of entanglement and the corresponding closest separable state of a given two-qubit 'X' state. This approach can be applied to get the relative entropy of entanglement of many widely-discussed two-qubit states, such as pure states, Werner states, and so on.

Keywords: relative entropy of entanglement, two-qubit state
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1. Introduction

For a mixed entangled state, different entanglement measures[1−7] may present different results for its degree of entanglement. Among all entanglement measures, the entanglement of formation and concurrence for two-qubit systems,[5] which are connected to each other through a one-to-one correspondence relation, are wildly-discussed owing to their computability for a two-qubit state. Although he generalisation of entanglement of formation or concurrence to high dimensions or multipartite systems is not easy,[8−10] the relevant research is attracting more and more interests.[11,12] On the contrary, another entanglement measure, the relative entropy of entanglement (REE), which characterizes the minimal 'distance' of a given entangled state deviated from the set of all separable states,[3] i.e.,

$$E_{\mathrm{R}}(\boldsymbol{\rho}) \equiv \min_{\sigma\in\mathcal{D}} S(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \min_{\boldsymbol{\sigma}\in\mathcal{D}} \mathrm{Tr}(\boldsymbol{\rho}\ln\boldsymbol{\rho} - \boldsymbol{\rho}\ln\boldsymbol{\sigma}), \quad (1)$$

where the target state $\boldsymbol{\sigma}$ being called the closest separable state (CSS), is independent of the number of subsystems and their dimensionality. Although many important properties of the REE are expected or even verified, how to calculate the REE of a general entangled state remains an open question. Even for the simplest case of two-qubit states, the analytical form of REE is known only for some special states with high symmetry.[3,4,13−17]

Recently, a convergent iterative algorithm is proposed to numerically calculate the REE of a bipartite quantum state, which helps to find the best separable approximation for a given entangled state, and also provides the corresponding optimal entanglement-witness measurement.[18,19] Opposite to the original problem of finding the CSS of a given entangled state, there is still an open fundamental problem to find an entangled state for a given CSS, which has investigated very recently.[20−23] It is shown that the REE and CSS of many special two-qubit states with high symmetry can be obtained analytically. In this study, we investigate the REE and CSS of a two-qubit 'X' state, which is currently, to our knowledge, the most general form we can analytically solve for a two-qubit state. As a special case of the results presented here, the REE and CSS of a simpler two-qubit 'X' state, with six non-zero elements in the 4×4 density matrix, is investigated in Ref. [17], and the two methods present the same results in this special case.

2. The closest state of a single-qubit state

Before we present the details of finding the REE and CSS of a two-qubit 'X' state, we introduce an interesting theorem on the closest state of a single-qubit state first, which will be used in the following context as an important precondition. Given a single-qubit

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†Corresponding author. E-mail: hjhxyx@yahoo.com.cn
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state represented by a 2×2 density matrix,

$$\boldsymbol{\rho} = \begin{pmatrix} a & c\mathrm{e}^{\mathrm{i}\,\phi} \\ c\mathrm{e}^{-\mathrm{i}\,\phi} & b \end{pmatrix}, \tag{2}$$

with the four real numbers a, b, c, and ϕ satisfying $ab \geq c^2$ and $\phi \in [0, 2\pi)$, what is the closest state, denoted by $\boldsymbol{\sigma}$ here, so that the 'distance' measured by $S(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \mathrm{Tr}[\boldsymbol{\rho}(\ln \boldsymbol{\rho} - \ln \boldsymbol{\sigma})]$, is minimized? Since the density matrix in Eq. (2) can be transformed to a real 2×2 density matrix through a 'local' transformation U,

$$\boldsymbol{\rho}' = \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \boldsymbol{U}\boldsymbol{\rho}\boldsymbol{U}^{\dagger} \tag{3a}$$

with

$$\boldsymbol{U} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\phi} \end{pmatrix}, \tag{3b}$$

and any local transformation does not change the entanglement among the subsystems, we assume all the elements in density matrix in Eq. (2) are real numbers, and the phase term is equal to zero, i.e. $\phi = 0$.

Now we suppose that the two eigenvalues of the target state $\boldsymbol{\sigma}$ are fixed as η_1 and η_2. What is the closest state to the state $\boldsymbol{\rho}'$?

A single-qubit state with fixed eigenvalues η_1 and η_2 could be expressed by a diagonal matrix,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}, \tag{4a}$$

and a two-dimensional unitary matrix,

$$\boldsymbol{O} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{4b}$$

through the following relation,

$$\boldsymbol{\sigma} = \boldsymbol{O}\boldsymbol{\Lambda}\boldsymbol{O}^{\dagger}. \tag{4c}$$

Here we can confine the angle θ in Eq. (4b) in the range $[0, \pi)$, because the matrix $\boldsymbol{\sigma}$ in Eq. (4c) periodically depends on the angle θ, and the corresponding period is π. The 'distance' between the single-qubit state (3) and the state (4) is

$$S(\boldsymbol{\rho}'||\boldsymbol{\sigma}) = \mathrm{Tr}(\boldsymbol{\rho}' \ln \boldsymbol{\rho}') - [(a\cos^2\theta + b\sin^2\theta - c\sin(2\theta))\ln\eta_1 + (a\sin^2\theta + b\cos^2\theta + c\sin(2\theta))\ln\eta_2]. \tag{5}$$

The extreme value of $S(\boldsymbol{\rho}'||\boldsymbol{\sigma})$ requires $\partial S/\partial\theta = 0$, which leads to

$$(b - a)\sin(2\theta) = 2c\cos(2\theta). \tag{6}$$

In the range $[0, \pi)$, there are two angles, denoted as θ_1 and θ_2 temporarily, satisfying the above equation. The two angles, θ_1 and θ_2, which have a $\pi/2$ angle difference to each other, determine the maximal and minimal values of the 'distance' between the single-qubit state (3) and the restricted state (4) through Eq. (5). The standard way to judge whether it is a maximal 'distance' or a minimal 'distance' is through the second order derivative of the 'distance' with respect to the angle θ, i.e., $\partial^2 S/\partial\theta^2$. The condition $\partial^2 S/\partial\theta^2 > 0$ corresponds to a minimal 'distance', and the condition $\partial^2 S/\partial\theta^2 < 0$ implies a maximal 'distance' (The special case with $\partial S/\partial\theta = 0$ and $\partial^2 S/\partial\theta^2 = 0$, corresponding to a trivial case $a - b = c = 0$ in the density matrix in Eq. (3a), is easy to treat and will be excluded in the current consideration).

In fact, the density matrix in Eq. (3a) is also diagonalised by the unitary matrix $\boldsymbol{O}$ under the condition (6), with two eigenvalues,

$$\begin{aligned} \lambda_1 &= a\cos^2\theta + b\sin^2\theta - c\sin(2\theta), \\ \lambda_2 &= a\sin^2\theta + b\cos^2\theta + c\sin(2\theta). \end{aligned} \tag{7}$$

That is to say, the two density matrices $\boldsymbol{\rho}'$ and $\boldsymbol{\sigma}$ are diagonalised by the same unitary matrix when the "distance' between them is minimised or maximised, and the corresponding 'distance' is now,

$$S(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \mathrm{Tr}(\boldsymbol{\rho}\ln\boldsymbol{\rho}) - (\lambda_1\ln\eta_1 + \lambda_2\ln\eta_2). \tag{8}$$

So we have another way to distinguish the minimal 'distance' from the maximal 'distance'. Under the condition (6), if the four eigenvalues of ρ and σ, which can be diagonalised by the same unitary matrix $\boldsymbol{O}$, satisfy $(\lambda_1 - \lambda_2)(\eta_1 - \eta_2) > 0$, the corresponding 'distance' is the minimal one; and the other case $(\lambda_1 - \lambda_2)(\eta_1 - \eta_2) < 0$ answers for the maximal 'distance'. This criteria is equivalent to the previous one via $\partial^2 S/\partial\theta^2$. Since the relation (6) is independent on the eigenvalues η_1 and η_2, the above discussions are also valid when another restriction is imposed on the target density matrix $\boldsymbol{\sigma}$, rather than fixing its two eigenvalues. For example, we can assume that the two

elements σ_{11} and σ_{22} are fixed, or the elements σ_{12} and the sum $\sigma_{11}+\sigma_{22}$ are fixed, which does not change the conclusion that two closest single-qubit state should be able to be diagonalised by the same unitary matrix. At present, we do not know whether this conclusion is valid or not for high dimensional systems or multipartite systems, but it is very useful, at least for our present aim of finding the REE and CSS of a two-qubit 'X' state.

3. The REE and CSS of a two-qubit 'X' state

In this section, we prove that the REE of an entangled two-qubit 'X' state is solvable through two binary equations, and the corresponding CSS can also be obtained in this way. The issue on the REE and CSS of a separable state is trivial and easy, which is excluded in our present consideration. In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, an entangled two-qubit 'X' state has the following form,

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{9}$$

Without the loss of generality, we assume $|\rho_{14}| > \sqrt{\rho_{22}\cdot\rho_{33}} \geq |\rho_{23}|$ (the condition $|\rho_{14}| > \sqrt{\rho_{22}\cdot\rho_{33}}$ is to ensure a non-vanishing entanglement[24,25]) and all elements in the above density matrix are real numbers. An entangled two-qubit 'X' state with complex coherence terms $\rho_{14}(=\rho_{41}^*)$ and $\rho_{23}(=\rho_{32}^*)$, or with $|\rho_{14}| < |\rho_{23}|$, can be transformed into such a form like Eq. (9) through local operations. Among the four eigenvalues of the density matrix in Eq. (9), the first and the last ones, i.e. λ_1 and λ_4, are entirely determined by the four corner elements, ρ_{11}, ρ_{14}, ρ_{41}, and ρ_{44}, and independent on the rest four non-zero elements. Similarly, the two eigenvalues, λ_2 and λ_3, are related to the four inner elements ρ_{22}, ρ_{23}, ρ_{32}, and ρ_{33}, and have nothing to do with the four corner elements. In other words, the two-qubit 'X' state (9) could be considered as spanned in a Hilbert space composed of two independent two-dimensional Hilbert spaces. Such density matrix can be diagonalised by a 4×4 unitary matrix $\boldsymbol{U}$ with the form of,

$$\boldsymbol{U} = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\beta & \sin\beta & 0 \\ 0 & -\sin\beta & \cos\beta & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix}, \tag{10a}$$

through the following relation,

$$\boldsymbol{\rho} = U\mathrm{Diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}U^\dagger. \tag{10b}$$

For the sake of simplicity, we assume $\lambda_1 \geq \lambda_4$ and $\lambda_2 \geq \lambda_3$, and these two conditions can always be satisfied through the adjustment of the angles α and β in the transformation (10a).

To find the closest state of such a two-qubit 'X' state, we only need to search the states in the same Hilbert space, which implies the target CSS of the state (9), σ, has a similar form, with eight zero-elements in the density matrix. Furthermore, a CSS should be an edge state in the set of all separable states,[13–17,20–23] which means the partial transpose of the CSS σ, i.e. σ^Γ, is rank deficient, or say, the partial transpose σ^Γ has at least one zero-eigenvalue.[24,25] Since we have assumed $|\rho_{14}| > |\rho_{23}|$ for the entangled two-qubit 'X' state (9), the only possibility is to set $|\sigma_{14}| = \sqrt{\sigma_{22}\cdot\sigma_{33}}$ in the CSS. Thus the CSS of the two-qubit 'X' state (9) should have the following form,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 & \pm\sqrt{xy} \\ 0 & x & \sigma_{23} & 0 \\ 0 & \sigma_{32} & y & 0 \\ \pm\sqrt{xy} & 0 & 0 & \sigma_{44} \end{pmatrix}. \tag{11}$$

The 'distance' between the two states (9) and (11) is now in fact the sum of the 'distance' between two 'unnormalised' single-qubit states,

$$\bar{\boldsymbol{\rho}} = \begin{pmatrix} \rho_{11} & \rho_{14} \\ \rho_{41} & \rho_{44} \end{pmatrix} \quad \text{and} \quad \bar{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & \pm\sqrt{xy} \\ \pm\sqrt{xy} & \sigma_{44} \end{pmatrix}, \tag{12}$$

and the 'distance' between the other two 'unnormalised' single-qubit states,

$$\bar{\bar{\boldsymbol{\rho}}} = \begin{pmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{pmatrix} \quad \text{and} \quad \bar{\bar{\boldsymbol{\sigma}}} = \begin{pmatrix} x & \sigma_{23} \\ \sigma_{32} & y \end{pmatrix}, \tag{13}$$

i.e., $S(\boldsymbol{\rho}||\boldsymbol{\sigma}) = S(\bar{\boldsymbol{\rho}}||\bar{\boldsymbol{\sigma}}) + S(\bar{\bar{\boldsymbol{\rho}}}||\bar{\bar{\boldsymbol{\sigma}}})$, where the four 'unnormalised' single-qubit states, $\bar{\boldsymbol{\rho}}$, $\bar{\bar{\boldsymbol{\rho}}}$, $\bar{\boldsymbol{\sigma}}$, and $\bar{\bar{\boldsymbol{\sigma}}}$, are constructed by the elements in the density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ in Eqs. (9) and (11). Thus the problem of minimising the 'distance' between two two-qubit 'X' states now turn to minimise the 'distance' of two single-qubit states, which has been discussed in Section 2.

According to the theorem presented in the previous section, two closest or farthest single-qubit states should be diagonalised by the same matrix, so the elements in the two matrices $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ should satisfy,

$$\sigma_{11} = \frac{1}{2}(1-x-y) \mp \sqrt{xy}\mathrm{ctg}(2\alpha), \tag{14a}$$

$$\sigma_{44} = \frac{1}{2}(1-x-y) \pm \sqrt{xy}\mathrm{ctg}(2\alpha), \tag{14b}$$

$$\sigma_{23} = \sigma_{32} = \frac{1}{2}(y-x)\mathrm{tg}(2\beta), \quad (\cos(2\beta) \neq 0), \tag{14c}$$

where we have used the result in Eq. (6) and the normalisation condition $\sigma_{11} + \sigma_{44} + x + y = 1$. It is easy to check that the original two-qubit 'X' state (9) and its CSS (11) with elements shown in Eq. (14) are diagonalised by the matrix (10a) simultaneously. Thus, given an entangled two-qubit 'X' state (9) with $\rho_{22} \neq \rho_{33}$, a necessary condition of its closest separable state (11) is that they should be diagonalised by the same unitary matrix. Just as we have mentioned above, to ensure the 'distance' is minimised rather than maximised, the eigenvalues of the state (9) and its CSS (11), diagonalised by the same unitary matrix (10a), should satisfy $(\lambda_1-\lambda_4)(\eta_1-\eta_4) > 0$ and $(\lambda_2-\lambda_3)(\eta_2-\eta_3) > 0$.

The density matrix (11) under the condition (14) has eigenvalues,

$$\eta_1 = \frac{1}{2}(1-x-y) + \frac{\sqrt{xy}}{|\sin(2\alpha)|}, \tag{15a}$$

$$\eta_2 = \frac{1}{\cos(2\beta)}(x\cos^2\beta - y\sin^2\beta), \tag{15b}$$

$$\eta_3 = \frac{1}{\cos(2\beta)}(y\cos^2\beta - x\sin^2\beta), \tag{15c}$$

$$\eta_4 = \frac{1}{2}(1-x-y) - \frac{\sqrt{xy}}{|\sin(2\alpha)|}. \tag{15c}$$

The two real numbers x and y should satisfy $x > y$ for the case $\cos(2\beta) > 0$ and $x < y$ for the case $\cos(2\beta) < 0$, so that the other condition $(\lambda_2-\lambda_3)(\eta_2-\eta_3) > 0$ for minimising the 'distance' between the two "unnormalised' single-qubit states $\bar{\bar{\boldsymbol{\rho}}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ is satisfied. Other cases with $\sin(2\alpha) = 0$ or $\cos(2\beta) = 0$ will be discussed in the following as exceptions.

Since the state (11) and the original two-qubit 'X' state (9) with $\rho_{22} \neq \rho_{33}$ could be diagonalised by the same unitary matrix, their 'distance' measured by relative entropy is directly determined by their eigenvalues,

$$S(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \sum_{i=1}^{4}(\lambda_i \ln\lambda_i - \lambda_i \ln\eta_i), \tag{16}$$

where λ_i and η_i ($i = 1, 2, 3, 4$) are the eigenvalues of the original two-qubit 'X' state (9) and its CSS (11) diagonalised by the same unitary matrix (10a), and every η_i is a function of the variables x and y (see Eqs. (15)). The following task is to find the global CSS of the state (9) with respect to the two variables x and y. As a concave function of x and y, which has been verified through our numerical program, the above 'distance' has only one minimal point in the allowed range of x and y. This minimal point is located at the position where the derivatives of $S(\boldsymbol{\rho}||\boldsymbol{\sigma})$ with respect to the two variables, x and y, reach zero, i.e. $\partial S/\partial x = 0$ and $\partial S/\partial y = 0$, which means,

$$\frac{|\sin(2\alpha)| - \sqrt{\dfrac{y}{x}}}{|\sin(2\alpha)|(1-x-y) + 2\sqrt{xy}}\lambda_1 + \frac{|\sin(2\alpha)| + \sqrt{\dfrac{y}{x}}}{|\sin(2\alpha)|(1-x-y) - 2\sqrt{xy}}\lambda_4 + \frac{\cos^2\beta}{y\sin^2\beta - x\cos^2\beta}\lambda_2 + \frac{\sin^2\beta}{y\cos^2\beta - x\sin^2\beta}\lambda_3 = 0, \tag{17a}$$

$$\frac{|\sin(2\alpha)| - \sqrt{\dfrac{x}{y}}}{|\sin(2\alpha)|(1-x-y) + 2\sqrt{xy}}\lambda_1 + \frac{|\sin(2\alpha)| + \sqrt{\dfrac{x}{y}}}{|\sin(2\alpha)|(1-x-y) - 2\sqrt{xy}}\lambda_4 + \frac{\sin^2\beta}{x\cos^2\beta - y\sin^2\beta}\lambda_2 + \frac{\cos^2\beta}{x\sin^2\beta - y\cos^2\beta}\lambda_3 = 0. \tag{17b}$$

The diagonalisation of the original state (9) brings out all parameters in the above two equations, such as α, β, and $\lambda_i (i = 1, 2, 3, 4)$, whose solution in the range $0 \leq x, y \leq 1$ and $x + y \leq 1$ gives birth to the target CSS. Here we remind that the global 'distance' of a given entangled state deviated from the set of all separable states, measured by relative entropy, has no finite upper bound, which implies that the only solution in the range $0 \leq x, y \leq 1$ and $x + y \leq 1$ of the above two equations answers for the minimal value of the 'distance' (16). That is to say, besides the condition $(\lambda_1 - \lambda_4)(\eta_1 - \eta_4) > 0$, which has been taken into account in the derivation of the equations (17), the other necessary condition for minimising the 'distance' (16), i.e. $(\lambda_2 - \lambda_3)(\eta_2 - \eta_3) > 0$, is satisfied automatically by the solution of the above two equations, which is also in agree with our numerical tests. Thus the CSS of a general two-qubit 'X' state (9) could be found through the following two steps: (i) diagonalising the density matrix in decreasing order, and finding out the corresponding unitary matrix $\boldsymbol{U}$ in Eqs. (10); (ii) solving the two binary equations (17) and substituting the result into the Eqs. (11) and (14), then we derive the target CSS. The corresponding REE could be calculated through Eq. (16), with λ_i and η_i ($i = 1$, 2, 3, 4) being the eigenvalues of the original two-qubit state and its CSS, which are diagonalised by the same unitary matrix.

The above discussion does not always keep valid for all two-qubit 'X' states. Just as is displayed in Eqs. (14), the final equality is valid only when $\cos(2\beta) \neq 0$, which implies an extra restriction on the density matrix (9), $\rho_{22} \neq \rho_{33}$. Now we discuss the two exceptions associated with $\rho_{22} = \rho_{33}$. The first case is to assume the two elements ρ_{22} and ρ_{33} in Eq. (9), which equal to each other now, to be non-zero numbers. The corresponding CSS in form of Eqs. (11) and (14) should be revised as,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 & \pm x \\ 0 & x & \sigma_{23} & 0 \\ 0 & \sigma_{32} & x & 0 \\ \pm x & 0 & 0 & \sigma_{44} \end{pmatrix} \tag{18a}$$

with

$$\begin{aligned} \sigma_{11} &= \frac{1}{2} - x \mp x\mathrm{ctg}(2\alpha), \\ \sigma_{44} &= \frac{1}{2} - x \pm x\mathrm{ctg}(2\alpha), \\ \sigma_{23} &= \sigma_{32} = \mp\frac{\lambda_2 - \lambda_3}{\lambda_2 + \lambda_3} x, \end{aligned} \tag{18b}$$

and only one variable x is left. Just like the case discussed above, here the choice of '+' or '–' in the equations (18), which is determined by the value of $\sin(2\alpha)$ and $\cos(2\beta)$, is to satisfy the condition $(\lambda_1 - \lambda_4)(\eta_1 - \eta_4) > 0$ and $(\lambda_2 - \lambda_3)(\eta_2 - \eta_3) > 0$ (λ_i and η_i ($i = 1, 2, 3, 4$) are the eigenvalues of the density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$). Following the same method used above, we can obtain the equation for variable x,

$$\begin{aligned} &\frac{\lambda_1}{x + \dfrac{|\sin(2\alpha)|}{2(1 - |\sin(2\alpha)|)}} + \frac{\lambda_4}{x - \dfrac{|\sin(2\alpha)|}{2(1 + |\sin(2\alpha)|)}} \\ &+ \frac{\lambda_2 + \lambda_3}{x} = 0, \end{aligned} \tag{19}$$

whose solution gives birth to the CSS of the two-qubit 'X' state (9) under the restriction $\rho_{22} = \rho_{33}$ ($\neq 0$). The corresponding REE can also be calculated through Eq. (16).

The other exception with zero-elements ρ_{22} and ρ_{33} in the density matrix (9), which also implies $\rho_{23} = \rho_{32} = 0$, is much easier to treat. Here we directly present its CSS,

$$\sigma = \mathrm{diag}\left(\rho_{11}, 0, 0, \rho_{44}\right), \tag{20}$$

and its REE,

$$E_{\mathrm{R}}(\rho) = \lambda_1 \ln \lambda_1 + \lambda_4 \ln \lambda_4 - \rho_{11} \ln \rho_{11} - \rho_{44} \ln \rho_{44}, \tag{21}$$

where λ_1 and λ_4 are the two non-zero eigenvalues of the density matrix (9) under the restriction $\rho_{22} = \rho_{33} = 0$.

Thus we have finished the discussions of the CSS and REE of all possible types of two-qubit 'X' state. By solving a two binary equations (17) or a single variable equation (19), we can find out the CSS and REE of a two-qubit 'X' state. The validity of this approach has already been tested by us through three examples of a rank-3 state, a Werner state[26] and a pure state, and all results agree well with the previous ones in Refs. [4] and [27], where some of the results are achieved by a numerical method and no analytical proof is presented there.

4. Conclusion

To summarise, quantum relative entropy can be used to measure the 'distance' (or 'closeness') between two quantum states, which is independent of the number of subsystems and their dimensionality. We proved that two closest single-qubit states, with one of them having a restriction, should be able to be

diagonalised by the same matrix, which is a very useful theorem. If it can be extended to high dimensional and multipartite systems, such as the case of a two-qubit 'X' state with $\rho_{22} \neq \rho_{33}$, discussed in Section 3, the issue on finding the relative entropy of entanglement (REE) and the closest separable state (CSS) of a quantum state can be much simplified. By applying this theorem into the case of two-qubit 'X' states, we formulate two binary equations for the REE and CSS of a given two-qubit 'X' state. The validity of this method is tested through three examples, including a rank-3 state, a two-qubit Werner state and two-qubit pure states. The final results well agree with the previous ones discussed in the literature. In fact, all the examples of entangled two-qubit state with analytical REE and CSS, discussed in Refs. [3,4,13–17], can be easily explained by using our formula. As another important property expected for the REE, the REE of a bipartite state should be bounded by the entanglement of formation of the same state, which is satisfied by the examples mentioned here.

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