Nonlinear waves in a fluid-filled thin viscoelastic tube*

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In the present paper the propagation property of nonlinear waves in a thin viscoelastic tube filled with incompressible inviscid fluid is studied. The tube is considered to be made of an incompressible isotropic viscoelastic material described by Kelvin–Voigt model. Using the mass conservation and the momentum theorem of the fluid and radial dynamic equilibrium of an element of the tube wall, a set of nonlinear partial differential equations governing the propagation of nonlinear pressure wave in the solid–liquid coupled system is obtained. In the long-wave approximation the nonlinear far-field equations can be derived employing the reductive perturbation technique (RPT). Selecting the exponent α of the perturbation parameter in Gardner–Morikawa transformation according to the order of viscous coefficient η, three kinds of evolution equations with soliton solution, i.e. Korteweg–de Vries (KdV)–Burgers, KdV and Burgers equations are deduced. By means of the method of traveling-wave solution and numerical calculation, the propagation properties of solitary waves corresponding with these evolution equations are analysed in detail. Finally, as a example of practical application, the propagation of pressure pulses in large blood vessels is discussed.

Keywords: fluid-filled tube, solitary wave, KdV–Burgers equation, Kelvin–Vogit model

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1. Introduction

The study of nonlinear waves has great significance for solving complex problems in engineering and science. In particular, the establishment of the soliton theory is a great achievement in modern nonlinear sciences. This theory has been applied widely to various fields, including hydrodynamics,[1] plasma and fibre communications. In present years, there has seen some progress in studying nonlinear waves in solid structures. In this paper we study the propagation of nonlinear waves of the solid–liquid coupled system in a fluid-filled pressurised tube.

The propagation of waves in fluid-filled tubes has been an ongoing area of study for at least 200 years. Thomas Young 1808 first derived a formula of propagation speed of pressure pulse in a fluid-filled elastic tube. This is known as the Moens–Korteweg formula, because it was popularised and modified by Korteweg and Moens (1878). In last century the investigation of the pressure waves in fluid-filled elastic tubes continued to progress rapidly. Up to the 1980's much of the work still utilised linear theory.[2]

From the 1980's, with nonlinear science development, began the investigation of nonlinear waves, especially the solitary waves, in a fluid-filled elastic tube. One of the original studies for examining solitary waves in fluid-filled elastic tubes was completed by Hashizume.[3] Particularly, it is worth mentioning that Demiray and his collaborators[4,5] carried out a series of studies and published many valuable research papers related to nonlinear waves in fluid-filled tubes.

In this paper, the following basic assumptions are used in deriving governing equations:

(i) The axial stress and displacement of the tube are neglected. In the long-wave approximation, the fluid properties are averaged over the tube radius and the motion of the fluid is one-dimensional flow.[3,6–8]

(ii) The Womersly number N_{W} and Reynolds number N_{R} of the fluid are large; the transient boundary layer is very thin as compared with the tube radius, and the effects of viscosity of the fluid are neglected.[8]

(iii) The tube is considered to be made of incompressible and isotropic material, whose constitutive relation is described by Kelvin–Vogit model.

2. Governing equation

2.1. Motion equation of the tube

A long viscoelastic tube, with radius R_0 and thickness H_0, is considered. Before being subject to the disturbance, the tube is assumed to be under a

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uniform inner pressure P_0, and corresponding to this static equilibrium state the radius and thickness of the tube are r_0 and h_0 respectively. In the course of fluid flow, a dynamical displacement field is superimposed on the static deformation. Let R and h be the radius and thickness of the tube after such a dynamic deformation. Based on the radial dynamic equilibrium of an element of the tube wall, we have

$$h\sigma_\theta = RP - h\rho_w R\frac{\partial^2 R}{\partial t^2}, \tag{1}$$

where ρ_w is the mass density of the tube, σ_θ is circumferential stress, t denotes the time parameter. The circumferential strain may be written as

$$\varepsilon_\theta = \frac{R - R_0}{R_0} = \frac{R}{R_0} - 1. \tag{2}$$

In the present work, viscous effect is taken into consideration and the constitutive relation is described by Kelvin–Vogit model

$$\sigma_\theta = K\varepsilon_\theta + \eta\frac{\partial \varepsilon_\theta}{\partial t}, \tag{3}$$

where K is the elastic coefficient, η is the viscous coefficient. Assuming that the tube is incompressible, we have

$$R_0 H_0 = Rh = r_0 h_0. \tag{4}$$

Introducing Eqs. (2), (3) and (4) into Eq. (1), the equilibrium equation governing the radial motion of the tube is obtained

$$\rho_w R\frac{\partial^2 R}{\partial t^2} - \frac{1}{R_0 H_0}PR^2 + \frac{\eta}{R_0}\frac{\partial R}{\partial t} + K\left(\frac{R}{R_0} - 1\right) = 0. \tag{5}$$

2.2. Fluid equations

The fluid is treated as an incompressible inviscid fluid. From the basic assumptions mentioned above, the variation of field parameters in radial coordinates can be disregarded. According to the mass conservation of the fluid, we can obtain the equation of continuity

$$\frac{\partial R}{\partial t} + \frac{R}{2}\frac{\partial V}{\partial x} + V\frac{\partial R}{\partial x} = 0, \tag{6}$$

where x is the axial coordinate. On the basis of the momentum theorem of the fluid, we obtain the equation of motion

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho_\mathrm{f}}\frac{\partial P}{\partial x} = 0, \tag{7}$$

where P and V are the pressure and flow velocity of the fluid, ρ_f is the mass density of the fluid. Then Eqs. (5), (6) and (7) provide adequate relations to determine the unknowns P, R and V.

3. Solving equations by reductive perturbation technique

Summing up Eqs. (5), (6), and (7), the nonlinear partial differential equations governing the propagation of nonlinear pressure wave in the solid–liquid coupled system can be written as

$$\begin{aligned}
&\frac{\partial R}{\partial t} + \frac{R}{2}\frac{\partial V}{\partial x} + V\frac{\partial R}{\partial x} = 0,\\
&\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho_\mathrm{f}}\frac{\partial P}{\partial x} = 0,\\
&\rho_w R\frac{\partial^2 R}{\partial t^2} - \frac{1}{R_0 H_0}PR^2 + \frac{\eta}{R_0}\frac{\partial R}{\partial t}\\
&+K\left(\frac{R}{R_0} - 1\right) = 0.
\end{aligned} \tag{8}$$

We adopt the long-wave approximation and solve Eq. (8) by employing the reductive perturbation technique.[9,10] Introduce the following Gardner–Morikawa transformation:

$$\begin{aligned}
&\xi = \varepsilon^\alpha (x - gt),\\
&\tau = \varepsilon^{\alpha+1} gt,
\end{aligned} \tag{9}$$

where ε is a small parameter measuring the weakness of dispersion and/or non-linearity, and α is a positive constant whose values will be specified later. Assuming that the field parameters can be expanded into the following asymptotic series of ε

$$\begin{aligned}
&R = r_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots,\\
&V = \varepsilon V_1 + \varepsilon^2 V_2 + \ldots,\\
&P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots.
\end{aligned} \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) and setting the coefficients of like powers of ε equal to zero, we obtain the following sets of differential equations.
$O(\varepsilon)$ order equations:

$$\begin{aligned}
&\frac{1}{2}r_0\frac{\partial V_1}{\partial \xi} - g\frac{\partial R_1}{\partial \xi} = 0,\\
&\frac{1}{\rho_\mathrm{f}}\frac{\partial P_1}{\partial \xi} - g\frac{\partial V_1}{\partial \xi} = 0,\\
&R_1 = \lambda P_1.
\end{aligned} \tag{11}$$

$O(\varepsilon^2)$ order equations:

$$g\frac{\partial R_1}{\partial \tau} + \frac{1}{2}r_0\frac{\partial V_2}{\partial \xi} + \frac{1}{2}R_1\frac{\partial V_1}{\partial \xi} + V_1\frac{\partial R_1}{\partial \xi} - g\frac{\partial R_2}{\partial \xi} = 0,$$

$$g\frac{\partial V_1}{\partial \tau}+V_1\frac{\partial V_1}{\partial \xi}+\frac{1}{\rho_{\rm f}}\frac{\partial P_2}{\partial \xi}-g\frac{\partial V_2}{\partial \xi}=0,$$
$$\rho_w r_0 g^2\varepsilon^{2\alpha-1}\frac{\partial^2 R_1}{\partial \xi^2}$$
$$-\frac{1}{R_0H_0}(r_0^2P_2+2r_0P_1R_1+P_0R_1^2+2r_0P_0R_2)$$
$$-\frac{\eta g}{R_0}\varepsilon^{\alpha-1}\frac{\partial R_1}{\partial \xi}+\frac{KR_2}{R_0}=0. \quad (12)$$

Integrating Eq. (11) with respect to ξ, and taking the integral constant to be zero (due to $\xi\to\infty$, the values of V_1, P_1, R_1 will approach zero), we have

$$R_1=\frac{r_0}{2g}V_1,\quad P_1=g\rho_{\rm f}V_1,\quad R_1=\lambda P_1,$$
$$g^2=\frac{r_0}{2\lambda\rho_{\rm f}},\quad \lambda=\frac{r_0^2}{H_0K-2r_0P_0}. \quad (13)$$

Let

$$V_1=U(\xi,\tau),\quad R_1=\frac{r_0}{2g}U,\quad P_1=g\rho_{\rm f}U, \quad (14)$$

where $U(\xi,\tau)$ is an unknown function to be solved. Introducing Eq. (14) into the set of Eq. (12), we obtain

$$2\frac{g^2}{r_0}\frac{\partial R_2}{\partial \xi}-g\frac{\partial V_2}{\partial \xi}-g\frac{\partial U}{\partial \tau}-\frac{3}{2}U\frac{\partial U}{\partial \xi}=0,$$
$$\frac{1}{\rho_{\rm f}}\frac{\partial P_2}{\partial \xi}-g\frac{\partial V_2}{\partial \xi}+g\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial \xi}=0,$$
$$R_2-\lambda P_2-b\lambda U^2-c\lambda\eta\varepsilon^{\alpha-1}\frac{\partial U}{\partial \xi}$$
$$+\frac{1}{2}a\lambda\varepsilon^{2\alpha-1}\frac{\partial^2 U}{\partial \xi^2}=0. \quad (15)$$

Here $a=(1/2)\rho_w gR_0H_0$, $b=(1/4g^2)(P_0+4g^2\rho_{\rm f})$, $c=H_0/2r_0$. Eliminating P_2, R_2 and V_2 between these equations, we have

$$\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}-\beta\varepsilon^{\alpha-1}\frac{\partial^2 u}{\partial \xi^2}+\gamma\varepsilon^{2\alpha-1}\frac{\partial^3 u}{\partial \xi^3}=0, \quad (16)$$

where $u=\alpha U$, $\alpha=(1/2g)(5/2-2b/\rho_{\rm f})$, $\beta=c\eta/2g\rho_{\rm f}$, $\gamma=a/4g\rho_{\rm f}$. From evolution equation (16), several equations with soliton solution may be deduced in some special cases.

4. Traveling-wave solutions

4.1. The KdV–Burgers equation

When $\alpha=1/2$ and $\eta=O(\varepsilon^{1/2})$, the evolution equation (16) reduces to the following Korteweg–de Vries–Burgers (KdVB) equation

$$\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}-\beta\frac{\partial^2 u}{\partial \xi^2}+\gamma\frac{\partial^3 u}{\partial \xi^3}=0. \quad (17)$$

The KdVB equation (17) includes three kinds of physical factors, i.e. nonlinearity, dispersion and dissipation.

4.1.1. The oscillating solitary wave solution and shock-wave solution

When the effect of the dispersion is stronger than that of dissipation, i.e. $0<\beta^2\le 4\gamma\sqrt{c_0^2+2A}$, the KdVB equation has the following traveling-wave solution

$$u(\xi,\tau)=\begin{cases}u_1+\dfrac{u_1-u_2}{2}{\rm e}^{(\beta/2\gamma)(\xi-c_0\tau)}\cos\delta(\xi-c_0\tau), & \xi\le 0;\\ u_2+\dfrac{3(u_1-u_2)}{2}\,{\rm sech}^2\sqrt{\dfrac{u_1-u_2}{8\gamma}}(\xi-c_0\tau), & \xi\ge 0;\end{cases} \quad (18)$$

where $\delta=\sqrt{(u_1-u_2)/2\gamma-(\beta/2\gamma)^2}$, $u_1=c_0+\sqrt{c_0^2+2A}$, $u_2=c_0-\sqrt{c_0^2+2A}$, c_0 is the traveling wave velocity, A is a constant. The qualitative analysis indicates that its phase orbit on the phase plane is a heteroclinic orbit from unstable focus point u_1 to saddle point u_2. Taking $u_1=3$, $u_2=2.8$, $\beta/2\gamma=0.1$, $c_0=3$, $\delta=0.1$, $(u_1-u_2)/8\gamma=0.1$, we shows the physical picture of Eq. (18) in Fig. 1, which represents oscillating solitary wave.

When the effect of the dissipation is stronger than that of dispersion, i.e. $\beta^2\ge 4\gamma\sqrt{c_0^2+2A}$, the KdVB equation has the shock-wave solution as follows:

$$u(\xi,\tau)=u_1-\frac{3\beta^2}{25\gamma}\left(1+\tanh\frac{\beta}{\gamma}(\xi-c_0\tau)\right)^2. \quad (19)$$

From the qualitative analysis, we know that its phase orbit on the phase plane is a heteroclinic orbit from unstable joint point to saddle point. The shock-wave is shown in Fig. 2 for $u_1=2$, $3\beta^2/25\gamma=1$, $\beta/\gamma=0.5$, $c_0=2$.

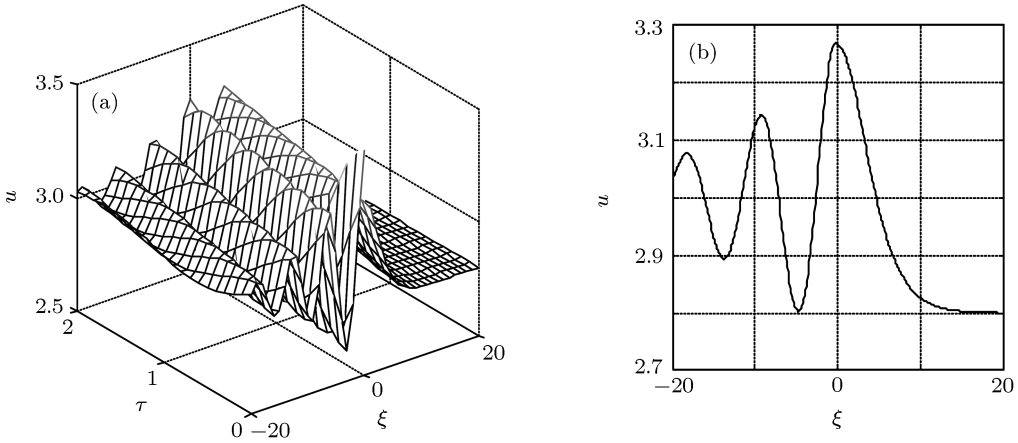


Fig. 1. The oscillating solitary wave solution of KdVB equation. (a) stereogram, (b) plane graph.

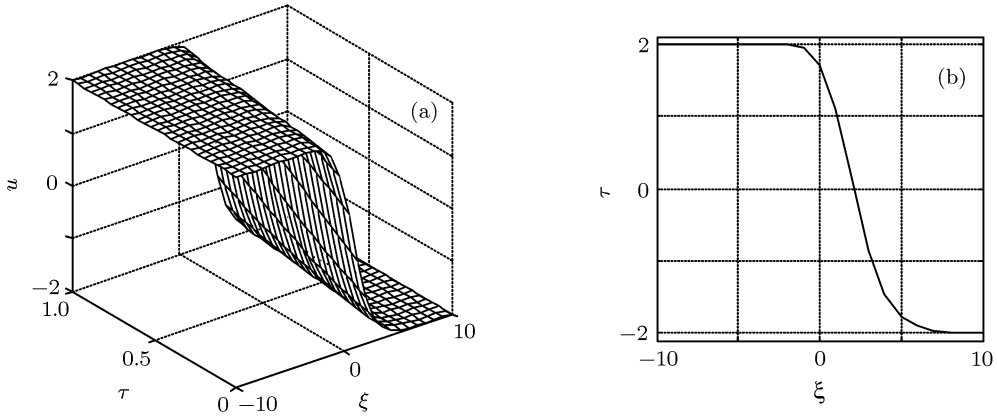


Fig. 2. The shock-wave solution of KdVB equation. (a) stereogram, (b) plane graph.

4.1.2. Hyperbolic function solution

We suppose a travelling wave solution to Eq. (17) of the form

$$u = u(\zeta); \quad \zeta = k(\xi - c\tau), \tag{20}$$

where k is wave number, c is wave velocity. Introducing Eq. (20) into (17), we have

$$-c\frac{\partial u}{\partial \zeta} + u\frac{\partial u}{\partial \zeta} - \beta k\frac{\partial^2 u}{\partial \zeta^2} + \gamma k^2\frac{\partial^3 u}{\partial \zeta^3} = 0. \tag{21}$$

Integrating Eq. (21) once with respect to ζ, we obtain

$$-cu + \frac{1}{2}u^2 - \beta k\frac{\partial u}{\partial \zeta} + \gamma k^2\frac{\partial^2 u}{\partial \zeta^2} = D, \tag{22}$$

where D is an integration constant. Expanding $u(\zeta)$ into a series of hyperbolic function

$$u(\zeta) = \sum_{j=0}^{n} A_j f^j(\zeta), \tag{23}$$

with $f(\zeta) = \tanh\zeta$ or $f(\zeta) = \coth\zeta$. According to the homogeneous balance principle, we have $n = 2$. Therefore, Eq. (22) has the following solution

$$u(\zeta) = A_0 + A_1 f(\zeta) + A_2 f^2(\zeta), \tag{24}$$

where A_0, A_1 and A_2 are some constants to be determined from the solution of Eq. (22). Completing some differentiating operation, finally we obtain for $f(\zeta) = \tanh\zeta$

$$u_t(\zeta) = \frac{3\beta^2}{25\gamma}(\operatorname{sech}^2\zeta - 2\tanh\zeta \pm 2). \tag{25}$$

And for $f(\zeta) = \coth\zeta$ we have

$$u_c(\zeta) = \frac{3\beta^2}{25\gamma}(\coth^2\zeta + 2\coth\zeta - 1 \pm 2). \tag{26}$$

Equations (25) and (26) are numerically simulated when the positive sign is taken, and $3\beta^2/25\gamma = 1$, $c_0 = 2$. The calculation results are shown in Figs. 3 and 4 respectively.

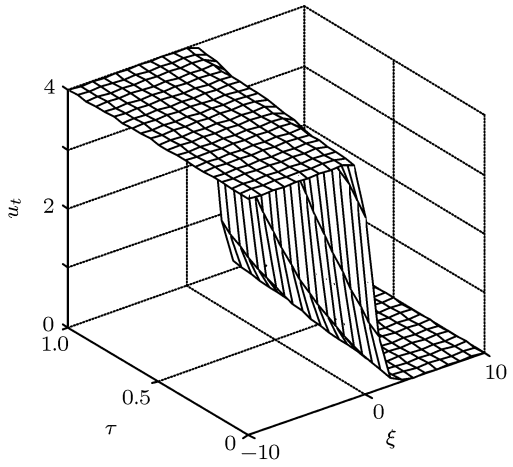


Fig. 3. The curves corresponding to Eq. (25).

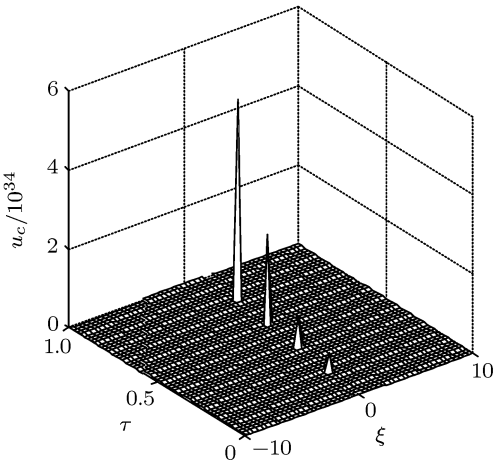


Fig. 4. The curves corresponding to Eq. (26).

4.2. Degeneration of KdVB

When $\alpha = 1$ and $\eta = O(\varepsilon^0)$, the coefficient of $\partial^3 u/\partial\xi^3$ is of the order of ε and the evolution equation reduces to the classical Burgers equation

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} - \beta\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{27}$$

As is well known, Burgers equation has a steady shock-wave solution

$$u(\xi,\tau) = \frac{u_1+u_2}{2} - \frac{u_1-u_2}{2}\tanh\frac{u_1-u_2}{4\beta}(\xi - c_0\tau). \tag{28}$$

The graph of shock-wave solution is illustrated in Fig. 5 when $(u_1+u_2)/2 = 2$, $(u_1-u_2)/2 = 1$, $(u_1-u_2)/4\beta = 0.5$, $c_0 = 2$.

When $\alpha = 1/2$ and $\eta = O(\varepsilon^1)$, the coefficient of $\partial^2 u/\partial\xi^2$ is of the order of $\varepsilon^{1/2}$ and the resulting evolution equation reduces to the well-known Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} + \gamma\frac{\partial^3 u}{\partial \xi^3} = 0. \tag{29}$$

The KdV equation has a steady solitary solution

$$u(\xi,\tau) = u_2 + (u_1-u_2)\,\mathrm{sech}^2\sqrt{\frac{u_1-u_2}{12\gamma}}(\xi - c_0\tau). \tag{30}$$

The graph of solitary wave is presented in Fig. 6 when $u_2 = 2$, $u_1 - u_2 = 1$, $(u_1-u_2)/12\gamma = 1$, $c_0 = 2$.

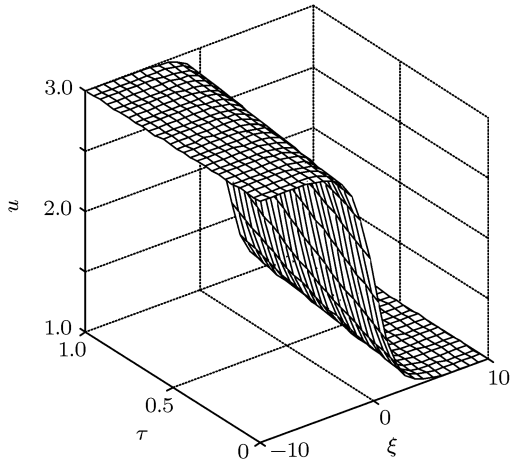


Fig. 5. The shock-wave solution of Burgers equation.

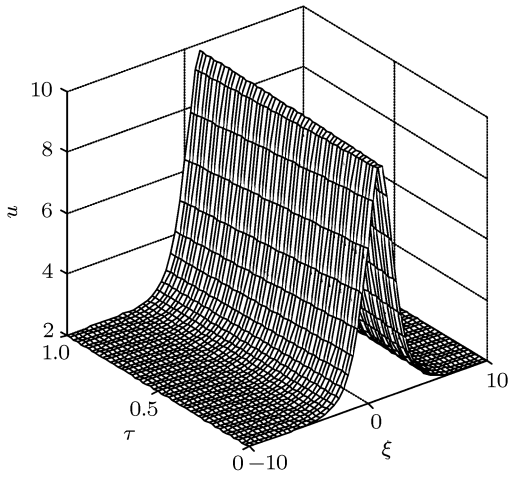


Fig. 6. The solitary solution of KdV equation.

5. Discussion and conclusion

(i) In the present paper, the propagation property of solitary waves in a thin viscoelastic tube filled with incompressible inviscid fluid is studied. The viscous effect of the coupled system is introduced by using the Kelvin–Vogit model to describe the constitutive equation of the tube wall. In the long-wave approximation the system of equations governing the propagation of

nonlinear pressure wave in a fluid-filled thin viscoelatic tube is transformed into KdV–Burgers equation by the reductive perturbation method. Under special conditions KdV–Burgers equation can degenerate into KdV and Burgers equations. The solitary-wave solution or shock-wave solution of the coupled system may exist, depending on that dispersion effect is relatively stronger or weaker than dissipation one. The physical pictures of these waves are drawn by numerical calculation.

(ii) The problems studied and the results given in this work are of significance in many practical applications, such as blood flow and pressure-feed tube etc. In what follows we briefly discuss the propagation of pressure pulses in large blood vessels. The intermittent ejection of blood into the aorta from the left ventricle of the heart produces flow and pressure pulses in the arterial tree.[11] Experimental studies of these pulses have revealed that they propagate with a characteristic pattern.

The change of pressure wave with distance from the aortic valve is shown in Fig. 7. First we see a shift of the profile to the right, suggesting the wave propagation. We also see a steepening and increase in amplitude, while the sharp dicrotic notch is gradually lost. The mean value of the pressure, averaged over the period of a heart beat, slightly decreases with increasing distance from the aortic valve. The fall in mean pressure is only 4 mmHg (0.5 kPa) (1 mmHg=133.322 Pa) in the whole length of the aorta.[12]

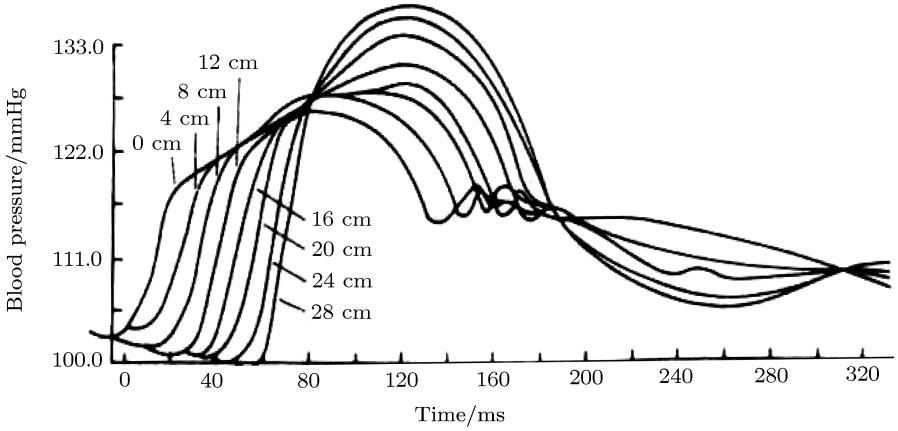


Fig. 7. Simultaneous blood pressure records made at a series of sites along the aorta in the dog, with distance measured from the beginning of the descending aorta, from Olson.[12]

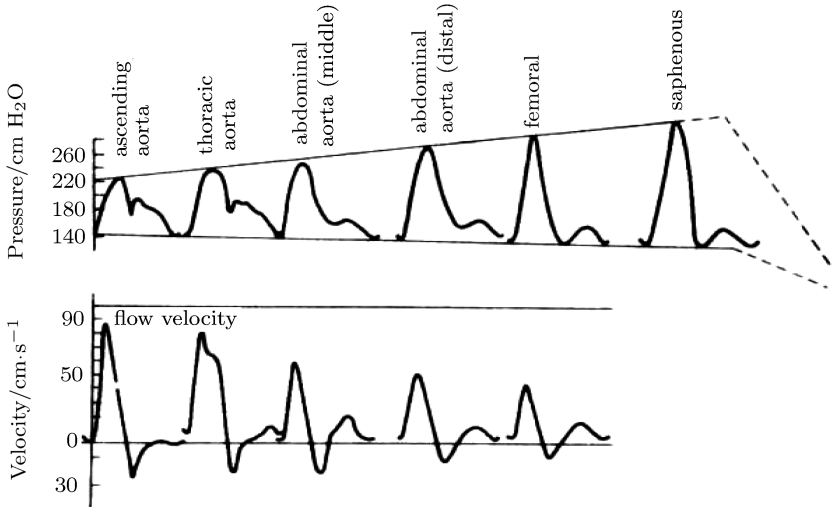


Fig. 8. Pressure and velocity waves at different sites in the arteries of a dog, from McDonald.[13]

In 1960 McDonald[13] measured simultaneously changes in amplitude and wave form of blood pressure at five sites from the ascending aorta to saphenous artery in a dog (Fig. 8). Figure 8 shows that the process of 'peaking'

and 'steepening' of pressure pulse continues into the branches of the aorta. Both the oscillation and the mean pressure decrease gradually downstream along the arterial tree until they reach the level of microcirculation.

The phenomena shown in Figs. 7 and 8 have attracted the interest of many researchers for a long time. References [3]–[5] assumed that blood is an ideal fluid and artery is an elastic tube and derived KdV equation from the dynamic equations for this system. The soliton solution of KdV equation can well interpret pulsatile character of arterial blood flow. In this paper we have introduced viscous effect into dynamic equations and obtained KdV–Burgers equation. The oscillating solitary wave solution of this equation under the condition of weak viscosity can even better explicate the oscillatory tail in pressure pulse. Besides the fall in mean pressure possibly comes from viscous effect. As is well known the messages carried in the arterial pulse waves are of important applications in clinical diagnosing. Therefore some results in our study are helpful for discovering diseases of the arteries and inferring the condition of the heart.

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