A type of new conserved quantity deduced from Mei symmetry for Appell equations in a holonomic system with unilateral constraints*

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A type of new conserved quantity deduced from Mei symmetry of Appell equations for a holonomic system with unilateral constraints is investigated. The expressions of new structural equation and new conserved quantity deduced from Mei symmetry of Appell equations for a holonomic system with unilateral constraints expressed by Appell functions are obtained. An example is given to illustrate the application of the results.

Keywords: unilateral constraint, Appell equation, Mei symmetry, conserved quantity
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1. Introduction

Since 1918, A E Noether, a German female scientist, revealed the potential relation between the symmetry and the conserved quantity,[1] the study on both symmetry and conserved quantity in constrained mechanical systems have been becoming an important research direction. In 2000, Mei[2] for the time gave a kind of new symmetry to which the dynamical function in a mechanical system satisfies the original equation after an infinitesimal transformation, which is called Mei symmetry. From 2000 to 2007, the research results of Mei symmetry have been collected in the monographs of Mei and many Chinese scholars.[3,4] After 2008, Mei symmetry is still a hot aspect of theories of symmetries and conserved quantities for constrained mechanical systems.[5−14]

The Appell equation is one of the three famous mechanical systems in analytical forms, but for a long term there have been very few achievements for it. Mei[5] first deduced Noether conserved quantity indirectly from Noether symmetry according to form invariance.[15] The conserved quantities deduced indirectly from Noether symmetry according to form invariance for a variable mass holonomic system were obtained by Li *et al.*[16] The conserved quantity deduced indirectly from Noether symmetry and Lie symmetry according to form invariance for Appell equations in a rotational relativistic holonomic system was achieved by Luo.[17] The above research results provide a new idea to look for Mei conserved quantity of Appell equation, but there is no structural equation and Mei conserved quantity expressed directly by Appell function. In Ref. [18], structural equation and Mei conserved quantity expressed directly by Appell function for a mechanical system are first given.

In nature and practical engineering technology, there exist many unilateral constraints. Therefore, the study on the unilateral constraints is of important practical significance. In recent years, some research results have been obtained for symmetry and conserved quantity of unilateral constraint systems. In Ref. [19], Zhang studied form invariance for unilateral holonomic constrained systems, in Ref. [20] he presented symmetry and conserved quantity of unilateral holonomic systems in the phase space, and in Ref. [21] he studied non-Noether conserved quantity in non-holonomic constrained systems of unilateral non-Chetaev type. Li and Fang[21] studied Mei symmetry in variable mass systems with unilateral holonomic constraints. Jing *et al.*[22] have studied the perturbation of Lie symmetries and the generalised Hojman adiabatic invariants for variable mass systems with unilateral holonomic constraints.

This paper presents a new structural equation and a new Mei conserved quantity deduced from Mei sym-

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metry of Appell function for a unilateral holonomic system expressed by Appell function.

2. Mei symmetry and criteria of Appell equation for a unilateral holonomic system

Suppose that the ideal unilateral holonomic constraints for the system are

$$f_\beta(t,\boldsymbol{q}) \geq 0, \quad (\beta = 1,2,\ldots,g); \tag{1}$$

$Q_s = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ are the generalised forces, and the energy of acceleration for the system is $S = S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$, then Appell equations and constrain equations are written respectively in the form

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} = Q_s + \Lambda_s, \quad (s = 1,2,\ldots,n), \tag{2}$$

$$f_\beta = 0 \quad (\beta = 1,\ldots,g), \quad (\text{when} \quad f_\beta = 0); \tag{3}$$

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s, \quad (s = 1,\ldots,n), \tag{4}$$

$$f_\beta > 0 \quad (\beta = 1,\ldots,g), \quad (\text{when} \quad f_\beta > 0). \tag{5}$$

For each β, $\lambda_\beta \geq 0$, $f_\beta \geq 0$, $\lambda_\beta f_\beta = 0 (\beta = 1,\ldots,g)$ exists. In Eq. (2), λ_β are the multipliers, where

$$\Lambda_s = \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial q_s}. \tag{6}$$

Λ_s are generalised constrain forces corresponding to the generalised coordinate q_s. By virtue of Eqs.(2) and (4), we can solve all generalised accelerations as

$$\begin{aligned} &\ddot{q}_s = A_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (s = 1,2,\ldots,n), \quad (\text{when } f_\beta = 0);\\ &\ddot{q}_s = B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (s = 1,2,\ldots,n), \quad (\text{when} f_\beta > 0). \end{aligned} \tag{7}$$

Introduce the infinitesimal transformations of time and generalised coordinates as

$$\begin{aligned} &t^* = t + \varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad (s = 1,\ldots,n); \end{aligned} \tag{8}$$

where ε is an infinitesimal parameter and ξ_0, ξ_s are infinitesimal generators. Similarly, we introduce vector $\boldsymbol{X}^{(0)}$ of infinitesimal generators again

$$\boldsymbol{X}^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \tag{9}$$

as well as its first and second extended infinitesimal generators are

$$\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \left(\frac{\mathrm{d}\xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\mathrm{d}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

$$\boldsymbol{X}^{(2)} = \boldsymbol{X}^{(1)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\mathrm{d}\xi_0}{\mathrm{d}t}\right) - \ddot{q}_s \frac{\mathrm{d}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial \ddot{q}_s}, \tag{11}$$

where

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + A_s \frac{\partial}{\partial \dot{q}_s} + \dot{A}_s \frac{\partial}{\partial \ddot{q}_s},\\ &(\text{when} \quad f_\beta = 0); \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + B_s \frac{\partial}{\partial \dot{q}_s} + \dot{B}_s \frac{\partial}{\partial \ddot{q}_s},\\ &(\text{when} \quad f_\beta > 0). \end{aligned} \tag{13}$$

After carrying out the infinitesimal transformations (8), let dynamic functions S, Q_s, Λ_s of the system be S^*, Q_s^*, Λ_s^*; taking the Taylor expansion of S^*, Q_s^*, Λ_s^* with respect to (t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$), we can easily obtain

$$S^* = S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) + \varepsilon \boldsymbol{X}^{(2)}(S) + O(\varepsilon^2); \tag{14}$$

$$\begin{aligned} &Q_s^* = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) + \varepsilon \boldsymbol{X}^{(1)}(Q_s) + O(\varepsilon^2),\\ &(s = 1,\ldots,n); \end{aligned} \tag{15}$$

$$\begin{aligned} &\Lambda_s^* = \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) + \varepsilon \boldsymbol{X}^{(1)}(\Lambda_s) + O(\varepsilon^2),\\ &(s = 1,\ldots,n); \end{aligned} \tag{16}$$

$$\begin{aligned} &f_\beta^* = f_\beta(t,q) + \varepsilon \boldsymbol{X}^{(0)}(f_\beta) + O(\varepsilon^2),\\ &(\beta = 1,\ldots,g). \end{aligned} \tag{17}$$

Definition 1 If Appell equation (2) keeps its form invariant when dynamic functions S, Q_s, Λ_s are replaced by functions S^*, Q_s^*, Λ_s^* under the infinitesimal transformations (8), namely

$$\frac{\partial S^*}{\partial \ddot{q}_s} = Q_s^* + \Lambda_s^*, \quad (s = 1,\ldots,n)\,, \tag{18}$$

then the invariance is called Mei symmetry of Appell equation (2).

Definition 2 If constraint equations (3) keep its form invariant when function f_β is replaced by the function f_β^* under the infinitesimal transformations (8), namely

$$\begin{aligned} &f_\beta^* = f_\beta(t^*,\boldsymbol{q}^*) = 0, \quad (\beta = 1,\ldots,g),\\ &(\text{when} \quad f_\beta = 0), \end{aligned} \tag{19}$$

then the invariance is called Mei symmetry of constraint equations (3).

Definition 3 If Eqs. (4) and (5) satisfy the Mei symmetry, then the invariance is called the Mei symmetry of the system with unilateral holonomic constraints, which is under these constraints.

Definition 4 If Appell equation (4) keeps its form invariant when dynamic functions S, Q_s are replaced by the functions S^*, Q_s^* under the infinitesimal transformations (8), namely

$$\frac{\partial S^*}{\partial \ddot{q}_s} = Q_s^*, \quad (s = 1,2,\ldots,n), \tag{20}$$

then the invariance is called the Mei symmetry of the system with unilateral holonomic constraints, which is free from the constraints. Then, we have

$$f^*_\beta = f_\beta(t^*, \boldsymbol{q}^*) > 0, \quad (\beta = 1, \ldots, g), \quad (\text{when} \quad f_\beta > 0). \tag{21}$$

Definition 5 If Eqs. (2), (3) and (4) satisfy the Mei symmetry, then the invariance is called the Mei symmetry of Appell equation for the system with unilateral holonomic constraints.

3. Criteria of Mei symmetry

Taking notice of Eqs. (2)–(5), from Eqs. (14)–(21), we obtain

$$\frac{\partial}{\partial \ddot{q}_s}\boldsymbol{X}^{(2)}(S) = \boldsymbol{X}^{(1)}(Q_s + \Lambda_s), \quad (s = 1, \ldots, n); \tag{22}$$

$$\boldsymbol{X}^{(0)}(f_\beta) = 0, \quad (\beta = 1, \ldots, g) \quad (\text{when} \quad f_\beta = 0); \tag{23}$$

$$\frac{\partial}{\partial \ddot{q}_s}\boldsymbol{X}^{(2)}(S) = \boldsymbol{X}^{(1)}(Q_s), \quad (s = 1, \ldots, n); \tag{24}$$

$$f_\beta(t, \boldsymbol{q}) + \varepsilon \boldsymbol{X}^{(0)}(f_\beta) > 0, \quad (\beta = 1, \ldots, g) \quad (\text{when} \quad f_\beta > 0). \tag{25}$$

Equations (22), (23), (24) and (25) are called the criterion equations of the Mei symmetry for a holonomic system of Appell function with unilateral constraints. Then, we have

Criterion 1 If the infinitesimal generators ξ_0, ξ_s admit Eqs. (22) and (23), then the invariance of Eqs. (2) and (3) under the infinitesimal transformations (8) is called Mei symmetry of Appell equation for the system with unilateral holonomic constraints, which is under the constraints.

Criterion 2 If the infinitesimal generators ξ_0, ξ_s admit Eq. (24), then the invariance of Eq. (4) under the infinitesimal transformations (8) is called Mei symmetry of Appell equation for the system with unilateral holonomic constraints, which is free from the constraints.

Criterion 3 If the infinitesimal generators ξ_0, ξ_s admit Eqs. (22), (23), and (24), then the invariance of Eqs. (2), (3) and (4) under the infinitesimal transformations (8) is called Mei symmetry of Appell equation for the system with unilateral holonomic constraints.

4. New structural equation and new Mei conserved quantity deduced from Mei symmetry

Proposition If the infinitesimal generators ξ_0, ξ_s and the gauge function $G_{\rm M} = G_{\rm M}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of Mei symmetry satisfy

$$\begin{aligned}
&\frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial t} - \dot{q}_s E_s[\boldsymbol{X}^{(2)}(S)] + [\boldsymbol{X}^{(1)}(Q_s + \Lambda_s)]\frac{{\rm d}A_s}{{\rm d}t} + \frac{{\rm d}G_X}{{\rm d}t} = 0, \quad (\text{when} \quad f_\beta = 0);\\
&\frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial t} - \dot{q}_s E_s[\boldsymbol{X}^{(2)}(S)] + [\boldsymbol{X}^{(1)}(Q_s)]\frac{{\rm d}B_s}{{\rm d}t} + \frac{{\rm d}G_X}{{\rm d}t} = 0, \quad (\text{when} \quad f_\beta > 0),
\end{aligned} \tag{26}$$

then the Mei conserved quantities deduced by the Mei symmetries are expressed as follows:

$$I_X = \boldsymbol{X}^{(2)}(S) - \dot{q}_s \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial \dot{q}_s} + G_X = \text{const.} \tag{27}$$

In Eq. (26), $E_s = \frac{\rm d}{{\rm d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}$, is called generalised Euler arithmetic operators.

Proof First of all, when the system is under the constraints, the Mei conserved quantities deduced from the Mei symmetries of Appell equation (3) satisfy $f = q_2 - q_1 > 0$.

Using Eq. (14), we have

$$\begin{aligned}
\frac{{\rm d}I_X}{{\rm d}t} &= \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial t} + \dot{q}_s \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial q_s} + \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial \ddot{q}_s}\frac{{\rm d}A_s}{{\rm d}t} - \dot{q}_s \frac{\rm d}{{\rm d}t}\frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial \dot{q}_s} + \frac{{\rm d}G_X}{{\rm d}t}\\
&= \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial t} + \frac{{\rm d}G_X}{{\rm d}t} - \dot{q}_s E_s[\boldsymbol{X}^{(2)}(S)] + \frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial \ddot{q}_s}\frac{{\rm d}A_s}{{\rm d}t}.
\end{aligned}$$

Taking notice of the first equation of structural equation (26) and the criterion equations (22), we achieve

$$\frac{\mathrm{d}I_X}{\mathrm{d}t} = \left[\frac{\partial \boldsymbol{X}^{(2)}(S)}{\partial \ddot{q}_s} - \boldsymbol{X}^{(1)}(Q_s)\right]\frac{\mathrm{d}A_s}{\mathrm{d}t} = 0.$$

Thus, we have proved Eq. (27). In the same way, it can be proved in the situation that the constraints vanish.

5. An illustrative example

Consider the Appell function, generalised forces and constraint equation are respectively

$$S = \frac{1}{2}(\ddot{q}_1^2 + \ddot{q}_2^2 + \ddot{q}_3^2), \tag{28}$$

$$Q_1 = 2\dot{q}_1, \quad Q_2 = 0, \quad Q_3 = \dot{q}_3\,, \tag{29}$$

$$f = q_2 - q_1 \geq 0, \tag{30}$$

we try to study the Mei symmetry and the Mei conserved quantity.

First, we study the Mei symmetry of the Appell equation for the system with unilateral holonomic constraints. Consider the system is under the constraints. Substituting Eq. (32) into Eq. (3), we have

$$\ddot{q}_1 = 2\dot{q}_1 - \lambda, \quad \ddot{q}_2 = \lambda, \quad \ddot{q}_3 = \dot{q}_3, \tag{31}$$

because the system is under the constraints, then $f = q_2 - q_1 = 0$, so, from Eq. (31), we achieve

$$\lambda = \dot{q}_1. \tag{32}$$

Thus, when the system is under the constraints, from Eq. (31), we can also achieve the generalised acceleration as follows:

$$A_1 = \ddot{q}_1 = \dot{q}_1, \quad A_2 = \ddot{q}_2 = \dot{q}_1, \quad A_3 = \quad \ddot{q}_3 = \dot{q}_3. \tag{33}$$

When the system is free from the constraints, from Eqs. (4), (28) and (29), we can achieve the generalised acceleration as follows:

$$B_1 = \ddot{q}_1 = 2\dot{q}_1, \quad B_2 = \ddot{q}_2 = 0, \quad B_3 = \quad \ddot{q}_3 = \dot{q}_3. \tag{34}$$

From Eqs. (6) and (32), we can obtain

$$\Lambda_1 = -\dot{q}_1, \quad \Lambda_2 = \dot{q}_1, \quad \Lambda_3 = 0. \tag{35}$$

Take the infinitesimal generators ξ_0, ξ_s as follows:

$$\xi_0 = 1, \quad \xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = e^t. \tag{36}$$

By virtue of Eqs. (9), (10) and (11), we have

$$\boldsymbol{X}^{(2)}(S) = e^t\ddot{q}_3 = e^t\dot{q}_3, \tag{37}$$

$$\frac{\partial}{\partial \ddot{q}_1}[\boldsymbol{X}^{(2)}(S)] = \frac{\partial}{\partial \ddot{q}_2}[\boldsymbol{X}^{(2)}(S)] = 0, \tag{38}$$

$$\frac{\partial}{\partial \ddot{q}_3}[\boldsymbol{X}^{(2)}(S)] = \boldsymbol{X}^{(1)}(Q_3) = e^t, \tag{39}$$

$$\boldsymbol{X}^{(1)}(Q_1) = \boldsymbol{X}^{(1)}(Q_2) = \boldsymbol{X}^{(0)}(f) = 0, \tag{40}$$

$$\boldsymbol{X}^{(1)}(\Lambda_1) = \boldsymbol{X}^{(1)}(\Lambda_2) = \boldsymbol{X}^{(1)}(\Lambda_3) = 0. \tag{41}$$

Then, we know that the infinitesimal generators (36) satisfy the criterion equation (22),(23) and (24) are tenable. Hence, Appell equation for the unilateral holonomic system has Mei symmetry.

Finally, we study the Mei conserved quantities deduced from the Mei symmetries.

Take notice of the Eq. (29) and from the new structural equation (26), we have

$$\frac{\mathrm{d}G_X}{\mathrm{d}t} = e^t\dot{q}_3, \quad (\text{when} \quad f_\beta = 0); \tag{42}$$

$$\frac{\mathrm{d}G_X}{\mathrm{d}t} = e^t\dot{q}_3, \quad (\text{when} \quad f_\beta > 0); \tag{43}$$

$$G_{\boldsymbol{X}} = \frac{1}{2}e^t\dot{q}_3. \tag{44}$$

By means of Eq. (28), the new conserved quantity deduced from the Mei symmetry of Appell equations for a unilateral holonomic system is

$$I_X = \frac{1}{2}e^t\dot{q}_3 = \text{const.} \tag{45}$$

6. Conclusion

This paper presents a new structural equation and a new conserved quantity deduced from the Mei symmetry of Appell equation for a unilateral holonomic system. Not only its form is different from the structural equation and Mei conserved quantity of Mei symmetry of Appell equation found for a unilateral holonomic system,[23] but also it is more succinct. Hence, the results of the paper develop and prefect theories of Mei symmetries and Mei conserved quantities, and can spread research field of Mei symmetry of Appell equations for unilateral nonholonomic mechanical systems.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göttingen Math. Phys.* **KI**, **II** 235

[2] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[3] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[4] Luo S K and Zhang Y F 2008 *Advances in the Study of Dynamics of Constrained Systems* (Beijing: Science Press) (in Chinese)

[5] Ding N and Fang J H 2008 *Chin. Phys.* B **17** 1550

[6] Jia L Q, Xie J F and Luo S K 2008 *Chin. Phys.* B **17** 1560

[7] Zhang M J, Fang J H, Zhang X N and Lu K 2008 *Chin. Phys.* B **17** 1957

[8] Fang J H, Liu Y K and Zhang X N 2008 *Chin. Phys.* B **17** 1962

[9] Huang X H, Zhang X B and Shi S Y 2008 *Acta Phys. Sin.* **57** 6056 (in Chinese)

[10] Ge W K 2008 *Acta Phys. Sin.* **57** 6714 (in Chinese)

[11] Cai J L 2009 *Acta Phys. Sin.* **58** 22(in Chinese)

[12] Wang P, Fang J H and Wang X M 2009 *Chin. Phys.* B **18** 1312

[13] Cui J C, Zhang Y Y and Jia L Q 2009 *Chin. Phys.* B **18** 1731

[14] Cui J C, Jia L Q and Yang X F 2009 *J. Henan Normal University* (Natural Science) **37** 70 (in Chinese)

[15] Mei F X 2001 *Chin. Phys.* **10** 177

[16] Li R J, QiaoY F and Meng J 2002 *Acta Phys. Sin.* **51** 1 (in Chinese)

[17] Luo S K 2002 *Acta Phys. Sin.* **51** 712 (in Chinese)

[18] Zhang Y 2004 *Acta Phys. Sin.* **53** 331 (in Chinese)

[19] Zhang Y 2005 *Acta Phys. Sin.* **54** 4488 (in Chinese)

[20] Zhang Y 2006 *Acta Phys. Sin.* **55** 504 (in Chinese)

[21] Li H and Fang J H 2004 *Acta Phys. Sin.* **53** 2807 (in Chinese)

[22] Jing H X, Li Y C and Xia L L 2007 *Acta Phys. Sin.* **56** 3043 (in Chinese)

[23] Jia L Q, Zhang Y Y, Cui J C and Luo S K 2009 *Commun. Theor. Phys.* **52** 572