Punishment in optional public goods games*

Wang Zhen(王 震)[a], Xu Zhao-Jin(许照锦)[b], and Zhang Lian-Zhong(张连众)[a]†

[a] School of Physics, Nankai University, Tianjin 300071, China
[b] School of Science, Tianjin University of Technology, Tianjin 300384, China

(Received 17 May 2010; revised manuscript received 28 June 2010)

In this work, the optional public goods games with punishment are studied. By adopting the approximate best response dynamics, a micro model is given to explain the evolutionary process. Simultaneously, the magnitude of rationality is also considered. Under the condition of bounded rationality which provides a light to interpret phenomena in human society, the model leads to two types of equilibriums. One is the equilibrium without punishers and the other is the equilibrium including only punishers and cooperators. In addition, the effects of rationality on equilibriums are briefly investigated.

Keywords: public goods games, rationality, equilibrium
PACC: 0250, 0580

1. Introduction

How do human beings sustain collective cooperation among selfish individuals in the evolutionary games, especially in the public goods games (PGG)? This is an interesting question because cooperative behaviour works against Darwinian evolution, in which individuals do not invest and increase their fitness by exploiting the contributions of others. If such a behaviour spreads, no one will invest in common enterprise, which is well-known as the social dilemma.[1,2] A number of theories have been suggested to explain the emergence and abundance of cooperation. They include direct[3,4] and indirect[5,6] reciprocity, punishment theories,[7−10] spatially structured populations,[11−17] asymmetric payoff distribution[18,19] and compulsory and voluntary participation.[20−23] Most notably, a great many experimental evidences[24−35] have turned out that punishment is a very efficient tool to favour cooperation in PGG. However, other mechanisms will fail to interpret the persistent of cooperation if the repeated interactions and reputation effects are absent.

In a typically optional PGG, the sample group of N players is chosen from a large population continually. All the players within such a group must simultaneously decide whether to take part in the joint enterprise or not.[20−22] Loners who do not engage in the public goods games obtain a small fixed payoff σ independent of others' decisions. Those participants include cooperators who endow an investment into a common pool, and defectors who do not invest anything and exploit the contributions of cooperators. The amount of money in the pool is then multiplied by an interest rate r and divided equally among all the participants irrespective of their contributions. If only a single player joins in the PGG, it does no better or worse than the loners, and gains payoff σ. More recently, Fowler in Ref. [26] and Brandt *et al.* in Ref. [27] pointed out an advanced viewpoint that in optional PGG, there exist punishers in additional to the above three kinds of players. Punishers not only contribute to the joint enterprise, but punish all those participants who fail to make contributions, or who fail to punish defectors. If such a punishment mechanism emerges, how does the outcome of the public good develop?

In Ref. [26], it has been witnessed that with the cycle attracting to loners disappearing, punishment is easily established. Thus, this model can greatly further that a single punisher can invade a population of loners. Whereas, in Ref. [27], the replicator dynamics lead to the bistable behaviours. They consist of the equilibriums of cooperators and punishers and a periodically cyclic dominance without punishers. In this sense, the outcomes of the process relied on the initial conditions. It is also obvious that if any kind of player dies out during the cycle, they can not be recovered forever. However, note that the former literature is directly confirmed for large populations, the later

*Project supported by the National Natural Science Foundation of China (Grant No. 10672081), and the Center for Asia Studies of Nankai University (Grant No. 2010-5).
†Corresponding author. E-mail: zhanglz@nankai.edu.cn
© 2010 Chinese Physical Society and IOP Publishing Ltd

denotes a micro phenomenon of infinite populations. Thus, it is explicit that various modeling approaches lead to different conclusions.

In this present work, we introduce another model based on the approximate best response theory and the magnitude of rationalities which is more natural in social populations. It shows that different rationality will lead to different results. At the same time, we focus on that the emergence of punishers is vital for the evolution of cooperation, and punishment is stable.

2. Method and model

We consider a simple case that N players are randomly selected from a large population. The players, who optionally participate in PGG, consist of cooperators (who do not punish), defectors and punishers (who contribute, and pay a cost γ to incur a punishment β on the population of defectors and a cost $\alpha\gamma$ to incur a punishment $\alpha\beta$ on the population of cooperators), and loners relying on a fixed payoff σ. Assuming $x_{\rm c}$, $x_{\rm d}$, $x_{\rm p}$ and $x_{\rm l}$ denote the frequencies of cooperators, defectors, punishers and loners (with the normalization condition $x_{\rm c}+x_{\rm d}+x_{\rm p}+x_{\rm l}=1$) respectively. For simplicity and without loss of generality, we set each cooperator's investment equals to 1. According to the calculation in Ref. [27], their average payoffs are $P_{\rm c}$, $P_{\rm d}$, $P_{\rm l}$ and $P_{\rm p}$, whose explicit expressions are given below:

$$\begin{aligned}
P_{\rm c} &= \sigma x_{\rm l}^{N-1} + (r-1)(1-x_{\rm l}^{N-1}) - r x_{\rm d} F(x_{\rm l}) \\
&\quad - \alpha\beta x_{\rm p}(N-1)\left[1-(1-x_{\rm d})^{N-2}\right], \\
P_{\rm d} &= \sigma x_{\rm l}^{N-1} + r(x_{\rm c}+xp)F(x_{\rm l}) - \beta x_{\rm p}(N-1), \\
P_{\rm p} &= \sigma x_{\rm l}^{N-1} + (r-1)(1-x_{\rm l}^{N-1}) - r x_{\rm d} F(x_{\rm l}) \\
&\quad - \alpha\gamma x_{\rm p}(N-1)\left[1-(1-x_{\rm d})^{N-2}\right] \\
&\quad - \gamma x_{\rm d}(N-1), \\
P_{\rm l} &= \sigma
\end{aligned}$$

with

$$F(x_{\rm l}) := \frac{1}{1-x_{\rm l}}\left(1-\frac{1-x_{\rm l}^{N}}{N(1-x_{\rm l})}\right), \tag{1}$$

where $\beta > 1 > \alpha > 0$ is a generic choice of parameters and r $((1+\sigma) < r < N)$ denotes the interest rate of the games.

In this model, the evolution of the frequencies is described by the approximate best response dynamics[36,37]

$$\dot{x_i} = \frac{C_i(\boldsymbol{X})}{\sum_j (C_j \boldsymbol{X})} - x_i, \tag{2}$$

with

$$\begin{aligned}
C_i(\boldsymbol{X}) &= \max\left[0, P_i(\boldsymbol{X}) - \max_j P_j(\boldsymbol{X}) + K\right], \\
&\quad (i,j = {\rm c},\ {\rm d},\ {\rm p},\ {\rm l})
\end{aligned} \tag{3}$$

where $\boldsymbol{X} = (x_{\rm c}, x_{\rm d}, x_{\rm l}, x_{\rm p})$ denotes the strategy profile, and K characterizes the magnitude of rationality. In the case of $K \to 0$, players are perfectly rational, where palyers always adopt the best choices, and the final equilibrium is Nash equilibrium (NE). When $K > 0$, it denotes bounded rationality and introduces some irrational factors into the systems. This is not completely in accordance with NE, but it is more similar to the situation of social populations. $K \to \infty$ implies that players are completely irrational, and their choices are stochastic. Thus, the magnitude of rationality has obvious effect on the equilibrium in our model. Here, we mainly focus our analysis on the situations of bounded rationality.

3. Numerical results

First let us analyze the results of simulations performed in Ref. [27] and our model. Figures 1(a) and 1(b) show the evolution of games on the face of $x_{\rm P} = 0$. In Ref. [27], it has a neutrally stable point M which is a NE. More importantly, this point is surrounded by rock-scissors-paper cycles, which has been confirmed in Refs. [20] and [21]. However, our model exhibits an attracting point O, which is an equilibrium in the case of bounded rationality. It clearly demonstrates that all the strategies will finally converge to this point, irrespective of their initial states. This outcome resulting from the approximate best response theory is also suitable for finite populations. Moreover, a single strategy has the ability of being recovered even if it is invaded in the micro populations, which can not be effectively solved in Ref. [27].

Another main difference between the model in Ref. [27] and our model is about the fixed points on $x_{\rm c}x_{\rm p}$ edge. In Ref. [27], the $x_{\rm c}x_{\rm p}$ edge consists of fixed points. From Fig. 2(a), we know that under the same background, initial states even existing very tiny differences will end up with different points of $x_{\rm c}x_{\rm p}$ edge. In this case, the evolutionary process depends on the initial conditions and is unstable. Conversely, this phenomenon can be greatly improved in our model. Note that Fig. 2(b) illustrates only a fixed point on the $x_{\rm c}x_{\rm p}$ edge, which is another equilibrium under the same condition as Fig. 1(b). All the states with $T_{\rm p} < x_{\rm p} \leq 1$ will reach this point, where $T_{\rm p} := \frac{N-r}{N-1}\frac{\gamma}{N}$

obtained by numerical simulations is robust to a wide range of parameters. The result suggests that it will be convenient to depict the evolutionary results under this updating rule.

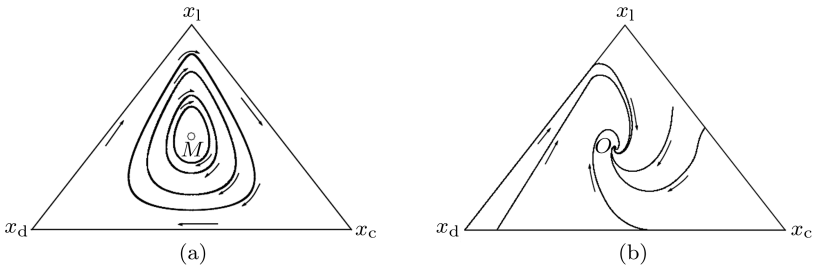


Fig. 1. Imitations of population evolution for the optional PGG in Ref. [27] (a) and our model (b). The black arrows show the varying states of cooperators, defectors and loners. The stationary points M and O denote different equilibrium states. (parameters: $N = 5$, $r = 3$, $\alpha = 0.1$, $\beta = 1.2$, $\gamma = 1.0$ and $K = 0.4$)

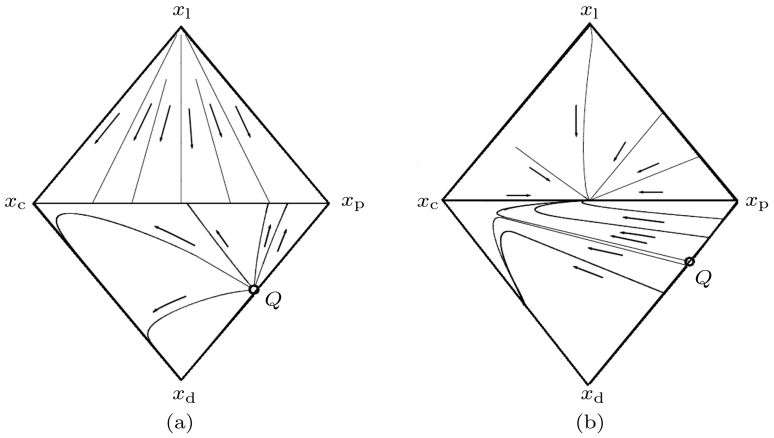


Fig. 2. Population dynamics for punishers engaging in PGG from Ref. [27] (a) and our model (b). Symbols indicate the results of individual-based simulation. Note the different values of point Q in both models. (parameters: $N = 5$, $r = 3$, $\alpha = 0.1$, $\beta = 1.2$, $\gamma = 1.0$ and $K = 0.4$)

Of particular interest, our model also has several characteristics in common with the model in Ref. [27]. For example, both models manifest the existence of two kinds of equilibriums. With the initial states, they may converge to the equilibrium on $x_c x_p$ edge, or to another equilibrium state on the face $x_p = 0$. Due to technological difficulties, it is not probable to explicitly distinguish the regions of two equilibriums in the interior of state space S_4, namely, all strategies are present (see Fig. 3). According to a great quantity of simulation results, the fraction of initial states converging to fixed point O (see Fig. 1(b)) is shown by the black region in Fig. 3. In this sense, the threshold value of x_p is given by $\frac{T_p}{\beta-\gamma}\frac{N-1}{r}$, which is corresponding to the value of point Q in Fig. 2(b). Of course, the other states in simplex S_4 will converge to another equilibrium point on $x_c x_p$ edge, which is shown by the gray region and has been confirmed through lots of simulations as well. Thus, two equilibrium points in our model are attracting points from the interior of simplex S_4. Notice, the later case yields the same payoffs for all the players, but their payoffs are different in the former.

Last, several predictions might be made in our model. With different magnitudes of rationality, the equilibriums will vary. For example, under the condition of perfect rationality, the fixed point M in Fig. 4 is an NE, which is in accordance with that point in Fig. 1(a). When the relevantly appropriate increasing of the magnitude of rationality is present, namely, bounded rationality appears in the system, the curve shown in Fig. 4 illustrates the variation of equilibriums. Along the same line, it is obvious that in the case of complete irrationality, there exists one and only one equilibrium, namely, the centre of state space S_4. In this sense, all the individual choices are stochastic; their payoffs are dominated by irrationality.

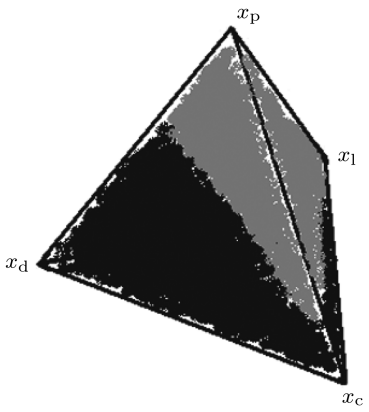


Fig. 3. The simulation of strategies evolution for our model in the interior of state space S_4, that is, all the strategies are present initially. The gray region denotes the initial states converging to the equilibrium point on $x_{\rm c}x_{\rm p}$ edge; the initial states that lead to the equilibrium point on the face $x_{\rm p} = 0$ are marked by black region. (parameters: $N = 5$, $r = 3$, $\alpha = 0.1$, $\beta = 1.2$, $\gamma = 1.0$ and $K = 0.4$)

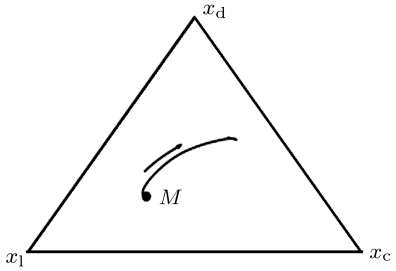


Fig. 4. The solid line shows the trajectory of equilibriums on the face $x_{\rm p} = 0$ in our model, when the magnitude of rationality is appropriately increasing. The point M denotes NE for perfect rationality, and it is the same as that point in Fig. 1(a). (The scale of simplex is multiplied by 1/2 for clear illustration)

4. Conclusion

To sum up, this model has some important conclusions. First, it demonstrates: how PGG will evolve to two equilibrium points, at which there exist punishers or not. In contrast, Ref. [27] shows that the final equilibriums are either many closed cycles or a series of equilibrium points consisting of cooperators and punishers. Thus, in the long run, the evolutionary process is more stable in our model. Besides, the emergence of punishers is important to solve social dilemmas. The effect of punishment on the PGG in our model is evident, which does not lead to the domination of punishers as Ref. [26].

In addition, our model reveals the effects of rationality on the equilibriums. Note when the magnitude of rationality varies, the equilibriums will be variable, namely, the equilibriums rely on the distribution of rationality. Because most individuals are bounded rationality in real life, the case of bounded rationality is mainly presented here.

Finally, the model studied here corresponds to the scenarios, in which the size of selected group is small. However, the total numbers of population are large, in which manner the evolution of the population-society is a challenge. In this sense, this model based on micro phenomenon is advantageous to provide an explanation for the evolution of population society.

References

[1] Hardin G 1968 *Science* **62** 1243
[2] Hardin G 1998 *Science* **280** 682
[3] Axelrod R and Hamilton W D 1981 *Science* **211** 1390
[4] Trivers R L 1971 *Q. Rev. Biol.* **46** 35
[5] Nowak M A and Sigmund K 1998 *Nature* **393** 573
[6] Nowak M A and Sigmund K 2005 *Nature* **437** 1291
[7] Fehr E and Gachter S 2002 *Nature* **415** 137
[8] Clutton-Brock T H and Parke G A 1995 *Nature* **373** 209
[9] Boyd R, Gintis R, Bowles S and Richerson P J 2003 *Proc. Natl. Acad. Sci. USA* **100** 3531
[10] Fehr E and Fischbacher U 2003 *Nature* **425** 785
[11] Zhang H F, Yang H X, Du W B, Wang B H and Cao X B 2010 *Physica* A **389** 1099
[12] Hauert C and Doebel M 2004 *Nature* **428** 643
[13] Szabó G and Hauert C 2002 *Phys. Rev. Lett.* **89** 118101
[14] Cao X B, Du W B and Rong Z H 2010 *Physica* A **389** 1273
[15] Guan J Y, Wu Z X and Wang Y H 2007 *Phys. Rev.* E **76** 056101
[16] Shi D M, Yang H X, Hu M B, Du W B, Wang B H and Cao X B 2009 *Physica* A **388** 4646
[17] Du W B, Cao X B, Zhao L and Hu M B 2009 *Physica* A **388** 4509
[18] Du W B, Cao X B, Yang H X and Hu M B 2010 *Chin. Phys.* B **19** 010204
[19] Du W B, Cao X B and Hu M B 2009 *Physica* A **388** 5005
[20] Hauert C, De Monte S, Hofbauer J and Sigmund K 2002 *Science* **296** 1129
[21] Hauert C, De Monte S, Hofbauer J and Sigmund K 2002 *J. Theor. Biol.* **218** 187
[22] Semmann D, Krambeck H J and Milinski M 2003 *Nature* **425** 390
[23] Peng D, Yang H X, Wang W X, Chen G R and Wang B H 2010 *Eur. Phys. J.* B **73** 455
[24] Fehr E and Gachter S 2000 *Am. Econ. Rev.* **90** 980
[25] Boyd R and Richerson P J 1992 *Ethol. Sociobiol.* **13** 171
[26] Fowler J 2005 *Proc. Natl. Acad. Sci. USA* **102** 7047

[27] Brandt H, Hauert C and Sigmund K 2005 *Proc. Natl. Acad. Sci. USA* **103** 495
[28] Henrich J and Boyd R 2001 *J. Theor. Biol.* **208** 79
[29] Sigmund K, Hauert C and Nowak M A 2001 *Proc. Natl. Acad. Sci. USA* **98** 10757
[30] de Quervain D J, Fischbacher U, Treyer V, Schellhammer M, Schnyder U, Buck A and Fehr E 2004 *Science* **305** 1254
[31] Traulsen A, Hauert C, De Silva H, Nowak M A and Sigmund K 2009 *Proc. Natl. Acad. Sci. USA* **106** 709
[32] Gintis H 2000 *J. Theor. Biol.* **206** 169
[33] Gintis H 2003 *J. Theor. Biol.* **220** 407
[34] Bowles S and Gintis H 2004 *Theor. Popul. Biol.* **65** 17
[35] Brandt H, Hauert C and Sigmund K 2003 *Proc. R. Soc.* B **270** 1099
[36] Hofbauer J 2000 *Selection* **1** 81
[37] Nash J 1950 Ph.D Thesis, Department of Mathematics, Princeton University