Controlling electric field distribution by graded spherical core-shell metamaterials*

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(Received 1 January 2010; revised manuscript received 18 April 2010)

This paper investigates analytically the electric field distribution of graded spherical core-shell metamaterials, whose permittivity is given by the graded Drude model. Under the illumination of a uniform incident optical field, the obtained results show that the electrical field distribution in the shell region is controllable and the electric field peak's position inside the spherical shell can be confined in a desired position by varying the frequency of the optical field as well as the parameters of the graded dielectric profiles. It has also offered an intuitive explanation for controlling the local electric field by graded metamaterials.

Keywords: graded metamaterial, plasmon resonance, optical electric field

PACC: 7820, 7730, 7230

1. Introduction

Metamaterials attracted much attention because of their unique applications in optics and acoustics.[1−5] For instance, through the design of artificial materials,[6−8] one can achieve the negative dielectric permittivity, magnetic permeability and refractive index due to the plasmon resonance in a narrow frequency range. In particular, the core-shell metamarerials were designed to control the electromagnetic field to realize cloaking at microwave frequency experimentally.[9−11] The electromagnetic cloaking is based on the concept of transformation optics first proposed by Pendry *et al.*[9] and Leonhardt.[12] Similar to the transformation optics, with the form-invariant acoustic equation, the acoustic cloaking was observed for a linear surface wave in a fluid.[3] For more information about this topic see Refs. [13]–[18] and references therein. Actually, the metamaterial structures and the form-invariant constitutive equations indicate that the material properties can vary gradually in space and hence terminology, graded metamaterials.[19] Based on graded core-shell metamaterials, it is possible to hide an object from the detection by electromagnetic waves. Recently, graded metamaterials were investigated by means of spectral representation for cloaking and to control the local electric field in a graded cylinder.[20−22] To investigate the properties of graded metamaterials, it is true that the analytic solutions of electric field can help to control the field of graded metamaterials by varying the graded parameters. However, exact results are rare in graded matamaterial systems and the existence offers much insight. In addition, many applications deal with structures with spherical shape, which admits analytical solutions. In such a case, we consider a spherical core-shell graded metamaterial structure, with a shell of graded metamaterial having a permittivity given by the graded Drude model[20,23] and a homogeneous dielectric core embedded in an isotropic host. For a power-law profile of the plasmon frequency in the graded Drude model, the potentials and electric field distributions are solved analytically in terms of the hyper-geometric functions. Furthermore, the electric field can be controlled in the graded shell region by tuning the graded parameters and incident field frequencies.

*Project supported by the National 863 Project of China (Grant Nos. 2009AA09Z102 and 2008AA09A403) and the RGC General Research Fund of the Hong Kong SAR Government.

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2. Analytical potential

For a spherical particle with an isotropic graded permittivity $\varepsilon_\alpha(r)$ in spherical coordinates (r,θ,φ), the Maxwell's equation in the quasi-static limit can be reduced to the following equation when an external uniform electric field E_0 is applied along the z-direction,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\varepsilon_\alpha(r)\frac{\partial \Phi_\alpha}{\partial r}\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\varepsilon_\alpha(r)\frac{\sin\theta}{r}\frac{\partial \Phi_\alpha}{\partial\theta}\right] = 0, \quad (1)$$

where Φ_α denotes the potential of α-type medium (α=c, s, h represent the core, shell and host materials, respectively) and the local electric field is given by $\boldsymbol{E}_\alpha = -\boldsymbol{\nabla}\Phi_\alpha$.

Considering the graded shell with dielectric function $\varepsilon_{\rm s}(r)$, the potential $\Phi_{\rm s}$ can be expressed as $\Phi_{\rm s}(r,\theta) = \sum_{n=0}^{\infty} R_n(r){\rm P}_n(\cos\theta)$, where ${\rm P}_n(\cos\theta)$ is the n-th order Legendre polynomials. From Eq. (1), the radial functions $R_n(r)$ satisfy the following equation:

$$\frac{\partial}{\partial r}\left[r^2\varepsilon_{\rm s}(r)\frac{\partial R_n(r)}{\partial r}\right] - n(n+1)\varepsilon_{\rm s}(r)R_n(r) = 0. \quad (2)$$

There are two solutions $R_n^+(r)$ and $R_n^-(r)$ in Eq. (2), which are regular at the origin and infinity, respectively. By virtue of the continuous boundary conditions of the potentials and normal electric displacements, we obtain the potentials $\Phi_{\rm c}$, $\Phi_{\rm h}$ and $\Phi_{\rm s}$ in the core, host and shell regions, respectively,

$$\Phi_{\rm c}(r,\theta) = -Ar{\rm P}_1(\cos\theta)E_0, \quad r<a, \quad (3)$$

$$\Phi_{\rm h}(r,\theta) = (-r+b^3Br^{-2}){\rm P}_1(\cos\theta)E_0, \quad r>b, \quad (4)$$

$$\Phi_{\rm s}(r,\theta) = [A_1R_1^+(r)+B_1R_1^-(r)]{\rm P}_1(\cos\theta)E_0, \quad a\le r\le b, \quad (5)$$

where a and b denote inner and outer radii of the spherical shell, respectively. The coefficients in potentials are

$$\begin{aligned}
&A_1 = -3bT_1^-(a)/T(a,b),\\
&B_1 = 3bT_1^+(a)/T(a,b),\\
&A = -a^{-1}[A_1R_1^+(a)+B_1R_1^-(a)],\\
&B = [F(b)-1]/[F(b)+2],\\
&T(a,b) = T_2^+(b)T_1^-(a) - T_2^-(b)T_1^+(a),\\
&T_1^\pm(a) = R_1^\pm(a) - a\frac{\varepsilon_{\rm s}(a)}{\varepsilon_{\rm c}}\frac{\partial}{\partial r}R_1^\pm(a),\\
&T_2^\pm(b) = 2R_1^\pm(b) + b\frac{\varepsilon_{\rm s}(b)}{\varepsilon_{\rm h}}\frac{\partial}{\partial r}R_1^\pm(b),\\
&F(b) = \frac{b\varepsilon_{\rm s}(b)}{\varepsilon_{\rm h}R_1(b)}\frac{\partial R_1(b)}{\partial r},\\
&R_1(r) = A_1R_1^+(r)+B_1R_1^-(r).
\end{aligned}$$

Here, note that $F(b)$ is the equivalent permittivity of graded spherical core-shell structure.[20,21]

As an example, we investigate a structure of spherical shell permittivity profile given by the graded Drude model[20,23] $\varepsilon_{\rm s}(r,\omega) = 1-\omega_{\rm p}^2(r)/\omega(\omega+{\rm i}\,\Gamma)$, where ω is the angular frequency of the external optical electric field, $\omega_{\rm p}(r)$ (Γ) is the spatially varying plasmon frequency (relaxation rate), and r is the radial variable in the spherical coordinates. For a graded plasmon frequency, $\omega_{\rm p}^2(r) = \omega_{\rm p}^2(0)(1-hr^k)$, the analytical solutions of the graded spherical core–shell structure will be derived to study the local electric field. Without loss of generality, we further normalize the external field frequencies ω and the relaxation rate Γ with $\omega_{\rm p}(0)$. The normalized Drude model is rewritten as $\varepsilon_{\rm s}(r) = 1-(1-hr^k)/\omega(\omega+{\rm i}\,\Gamma)$. By introducing a transformation $R_n(r) = r^s g(z)$ into Eq. (2), where $z = -hr^k/(\omega^2+{\rm i}\,\omega\Gamma-1)$, we obtain the hypergeometric equation,

$$\begin{aligned}
&z(1-z)\frac{{\rm d}^2g(z)}{{\rm d}z^2}\\
&+[(2s+k+1)/k-(2s+2k+1)z/k]\frac{{\rm d}g(z)}{{\rm d}z}\\
&-\frac{s}{k}g(z) = 0,
\end{aligned} \quad (6)$$

if we let $s(s+1)-n(n+1)=0$, i.e. $s_1=n$, $s_2=-(n+1)$. Thus, for $n=1$, the two radial potential functions $R_1^+(r) = r{\rm F}(\alpha_1,\beta_1,\gamma_1,z)$ and $R_1^-(r) = r^{-2}{\rm F}(\alpha_{-1},\beta_{-1},\gamma_{-1},z)$ are obtained, respectively, where $\alpha_{\pm1} = [k\pm3-\sqrt{(k+1)^2+8}]/2k$, $\beta_{\pm1} = [k\pm3+\sqrt{(k+1)^2+8}]/2k$, $\gamma_{\pm1} = (\pm3+k)/k$. The function ${\rm F}(\alpha,\beta,\gamma,z)$ is the hyper-geometric function, which is analytic in the whole complex plane except at the singular points.[24] Therefore, we have analytically derived the potential solutions of graded spherical core-shell system with the graded Drude model. These solutions can be used to discuss the local electric field distribution spatially.

3. Numerical discussion

With the above formulas, we have calculated the modulus $|\boldsymbol{E}|$ of the corresponding local electric field shown in Fig. 1, where the spatial distributions of electric field in the spherical core–shell system ($0 \leq r \leq 1.2$ and $E_0 = 1$) are dependent on the reduced frequency $\omega/\omega_{\rm p}(0)$. It can be easily found that, with increasing $\omega/\omega_{\rm p}(0)$, the peak position of the electric field modulus shifts toward the core region with the compensation of reducing $|\boldsymbol{E}|$. The similar results can be obtained by decreasing graded parameter k

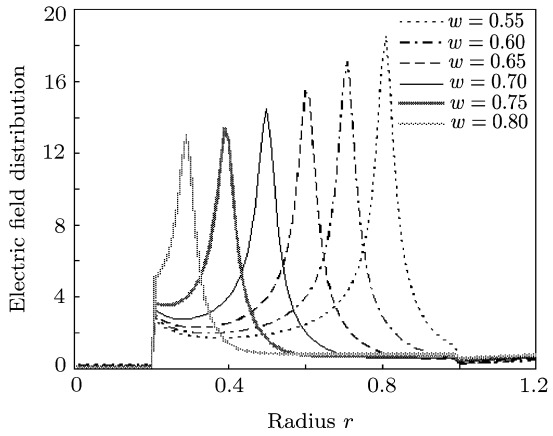


Fig. 1. The normalized electric field modulus $|\boldsymbol{E}|/E_0$ versus the reduced frequency parameter $w = \omega/\omega_{\rm p}(0)$ along the z axis ($\theta = 0$), where the shell inner radius $a = 0.2$, outer radius $b = 1$, with the host region ranging from $r = 1$ to $r = 1.2$. Other parameters: $k = 0.65$ and $h = 0.8$, $\Gamma/\omega_{\rm p}(0) = 0.02$, host and spherical core dielectric constants are $\varepsilon_{\rm h} = 1$ and $\varepsilon_{\rm c} = 20$, respectively.

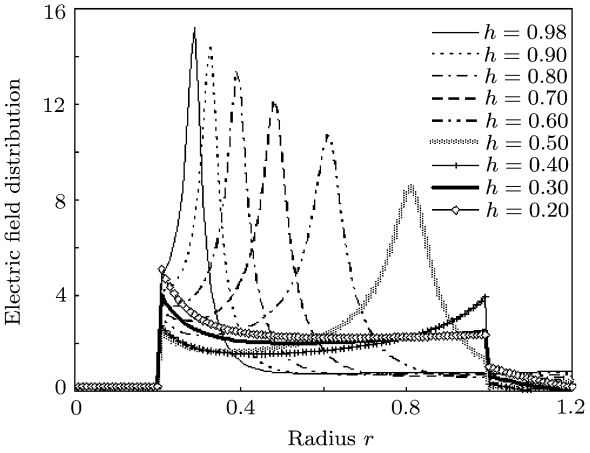


Fig. 2. The normalized electric field modulus $|\boldsymbol{E}|/E_0$ versus the graded parameter h along the z axis, where the reduced frequency parameter $\omega/\omega_{\rm p}(0) = 0.75$ and other parameters are given in the legend of Fig. 1

(figure omitted). This can be understood from the fact that the graded Drude model has a decreasing plasmon resonant frequency from $r = a$ to $r = b$ ($0 \leq a < b \leq 1$) with increasing the reduced frequency parameters $\omega/\omega_{\rm p}(0)$ and decreasing the graded parameter k. Meanwhile, we can also see that the peak position is also controlled by tuning the parameter h as shown in Fig. 2, where the peak position shifts toward the core with an increasing electric field modulus.

To determine the peak's spatial position of electric field distribution, we have investigated the resonant condition $\omega/\omega_{\rm p}(r) = 1$ of the external field frequency ω and graded plasmon frequency $\omega_{\rm p}(r)$ of metamaterial shell. The exact peak's positions predicted by the resonant condition are compared with those of numerical results obtained by analytical electric fields in Fig. 3. Excellent agreement is obtained and the peak's position can indeed be determined correctly by the resonant condition. The peak's positions of electric field distributions result from the localization of the electric field at the interfaces between the metallic and dielectric materials. From the graded Drude model, we know that graded permittivity can change from metallic-like ($\omega/\omega_{\rm p}(r) < 1$) to dielectric-like ($\omega/\omega_{\rm p}(r) > 1$) with increasing the frequency ω. Therefore, the peak's position of electric field is predicted by the resonant condition in our model. In addition, the graded metamaterials indicate that the transition between metallic-like and dielectric-like materials can occur in the shell region and the electric field peak's positions can be exactly controlled by tuning the external field frequency and the graded parameters.

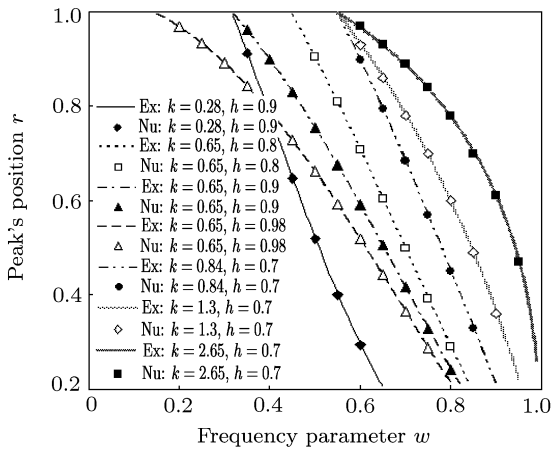


Fig. 3. The peak's positions of the normalized electric field modulus $|\boldsymbol{E}|/E_0$ predicted by the resonance condition $\omega/\omega_{\rm p}(r) = 1$ are compared with numerical results for various frequency parameters $w = \omega/\omega_{\rm p}(0)$ and graded parameters, where "Ex" and "Nu" represent the exact results and numerical results, respectively, and other parameters are given in the legend of Fig. 1.

4. Conclusions

In summary, the control of electric field has been discussed for a graded spherical core–shell structure by means of analytical solutions. The results show that the electric field distribution can be controlled by tuning the external field frequency and the parameters of graded plasmon frequency. The electric field peak's position can be predicted by the interface between the metallic-like and dielectric-like media. Although the mechanism of controlling electric field is illustrated with a graded Drude model, it will be useful as a basis of investigating the electrical field control for more general graded metamaterials when the graded metamaterials have both metallic-like and dielectric-like properties, which depend on the graded plasmon frequency parameters and incident field frequencies. For the graded metamaterials of arbitrary graded profiles and complex constitutive structures, numerical methods like finite-element or finite-difference methods must be used for more practical applications. Furthermore, the precise and controllable position of the enhanced field may be used in optical design, for example, in electric field concentration devices, displaying and printing technologies. To extend the practical applications of graded metamaterials in material technology, the mechanism of controlling electric field in other graded models should be investigated further. In addition, it is also instructive to extend our analytical methods for discussing the electric field distribution of metamaterials with the arbitrary structures and graded profiles.[25−29]

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