Operators' s-parameterized ordering and its classical correspondence in quantum optics theory*

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In reference to the Weyl ordering $x^m p^n \to (1/2)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l$, where X and P are coordinate and momentum operator, respectively, this paper examines operators' s-parameterized ordering and its classical correspondence, finds the fundamental function-operator correspondence

$$\left(\frac{1-s}{2}\right)^{(n+m)/2} \mathrm{H}_{m,n}\left(\sqrt{\frac{2}{1-s}}\alpha^*, \sqrt{\frac{2}{1-s}}\alpha\right) \to a^{\dagger m} a^n,$$

and its complementary relation

$$\alpha^n \alpha^{*m} \to (-\mathrm{i})^{n+m} \left(\frac{1-s}{2}\right)^{(m+n)/2} : \mathrm{H}_{m,n}\left(\mathrm{i}\sqrt{\frac{2}{1-s}}a^\dagger, \mathrm{i}\sqrt{\frac{2}{1-s}}a\right) : ,$$

where $\mathrm{H}_{m,n}$ is the two-variable Hermite polynomial, $a, a^\dagger$ are bosonic annihilation and creation operators respectively, s is a complex parameter. The s'-ordered operator power-series expansion of s-ordered operator $\circledS a^{\dagger m} a^n \circledS$ in terms of the two-variable Hermite polynomial is also derived. Application of operators' s-ordering formula in studying displaced-squeezed chaotic field is discussed.

Keywords: s-ordered operator expansion formula, the IWSOP technique, two-variable Hermite polynomial

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1. Introduction

Operator ordering is important for converting the master equation of density operator to C-number equation in quantum optics theory.[1−3] In a recent paper[4] the authors derived two new quantum-mechanical formulas for describing no counts registered on a photonic detector when a light field's density operator ρ is known, their method is based on the generalized Wigner operator for s-parameterized quasiprobability distribution. Operators' s-parameterized rule (denoted by $\circledS \cdots \circledS$ in this work) was proposed by Cahill and Glauber,[5,6] however, the problem that what is the normally ordered expansion of $\circledS a^{\dagger m} a^n \circledS$ (or *vice versa*, what is the s-ordered operator expansion of $a^{\dagger m} a^n$), where $a^\dagger, a$ are creation and annihilation photon operators, respectively, has not been seen in the literature before, this problem is related to operators' s-parameterized ordering and its classical correspondence, which is essential for completely understanding operators' s-parameterized rule. We try to solve it by virtue of the technique of integration within s-parametric ordered product (IWSOP) of operators which unifies the technique of integration within normal ordered, antinormally ordered and Weyl ordered product of operators.[7,8]

Let us begin with recalling the Weyl ordering[9,10] of operators, which is characteristic of the fundamental function-operator correspondence relation

$$x^m p^n \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l, \qquad (1)$$

where $[X, P] = \mathrm{i}$, $(\hbar = 1)$, $x^m p^n$ is the quantity of classical coordinate and momentum. It is associated with the original definition of Weyl quantization scheme about operator ρ and its classical corresponding function $h(p, x)$,

$$\rho = \iint_{-\infty}^{\infty} \mathrm{d}p\mathrm{d}x \Delta(p, x) h(p, x), \qquad (2)$$

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where $\Delta(p,x)$ is the Wigner operator defined in the coordinate representation as

$$\Delta(p,x)=\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{2\pi}\,\mathrm{e}^{\mathrm{i}up}\left|x+\frac{u}{2}\right\rangle\left\langle x-\frac{u}{2}\right|. \quad (3)$$

The reciprocal relation of Eq. (2) is

$$\mathrm{Tr}\left[\rho\Delta(p,x)\right]=(2\pi)^{-1}h(p,x)=W(p,x), \quad (4)$$

where $W(p,x)$ is the Wigner function.[11,12] When $\rho=(1/2)^m\sum_{l=0}^{m}\binom{m}{l}X^{m-l}P^nX^l$, according to Eqs. (3) and (4), its classical correspondence is

$$\begin{aligned}
&2\pi\mathrm{Tr}\left[\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\binom{m}{l}X^{m-l}P^nX^l\Delta(x,p)\right]\\
&=\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{\mathrm{i}pu}\left\langle x-\frac{u}{2}\right|\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!\,(m-l)!}\\
&\quad\times X^{m-l}P^nX^l\left|x+\frac{u}{2}\right\rangle\\
&=x^m\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{\mathrm{i}pu}\left\langle x-\frac{u}{2}\right|P^n\left|x+\frac{u}{2}\right\rangle\\
&=x^m\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{\mathrm{i}pu}\int_{-\infty}^{\infty}\mathrm{d}p'\,\mathrm{e}^{-\mathrm{i}p'u}p'^n\\
&=x^m\int_{-\infty}^{\infty}\mathrm{d}p'\delta\left(p-p'\right)p'^n\\
&=x^mp^n. \quad (5)
\end{aligned}$$

So equation (1) characterizes the original definition of Weyl quantization scheme. In 1969 Glauber (Nobel prize winner in 2005 for his contribution to quantum optics) *et al.*[5,6] introduced the so-called s-parameterized ordering convention, which associates with every complex number s a unique way of ordering all products of the operators a and $a^\dagger$, i.e., normal ordering, Weyl ordering, and antinormal ordering correspond to the values $s=+1$, 0 and -1, respectively. Cahill and Glauber in their long paper[5,6] introduced the operators' s-ordered power-series expansions, their s-ordered $\left\{a^{\dagger n}a^m\right\}_s$ is originated from the following derivative

$$\left\{a^{\dagger n}a^m\right\}_s=\left.\frac{\partial^{n+m}D(\alpha,s)}{\partial\alpha^n\partial\left(-\alpha^*\right)^m}\right|_{\alpha=0}, \quad (6)$$

where

$$\begin{aligned}
&D(\alpha,s)=D(\alpha)\,\mathrm{e}^{s|\alpha|^2},\\
&D(\alpha)=\exp[\alpha a^\dagger-\alpha^*a],\ \left[a,a^\dagger\right]=1, \quad (7)
\end{aligned}$$

s is a complex number. Then the crucial elementary problem arises: like $(1/2)^m\sum_{l=0}^{m}\binom{m}{l}X^{m-l}P^nX^l$ corresponding to x^mp^n in Weyl quantization scheme, what is characteristic of the fundamental operator-function correspondence relation in the context of the s-parametric quantization? In the following we shall take the s-ordered expansion of $a^{\dagger m}a^n$ as being characteristic of the ingredient of the s-parameterized quantization scheme. In another word, we can directly explore s-ordered power-series expansion of $a^{\dagger m}a^n$, by introducing into phase-space the s-parameterized generalized Wigner operator

$$\begin{aligned}
\Delta_s(\alpha)&=\int\frac{\mathrm{d}^2\beta}{2\pi^2}\exp\left(\frac{s|\beta|^2}{2}-\beta\alpha^*+\beta^*\alpha\right)D\left(\beta\right)\\
&=\int\frac{\mathrm{d}^2\beta}{2\pi^2}\colon\exp\left(\frac{-\left(1-s\right)|\beta|^2}{2}+\beta a^\dagger-\beta^*a-\beta\alpha^*+\beta^*\alpha\right)\colon\\
&=\frac{1}{\left(1-s\right)\pi}\colon\exp\left[\frac{-2}{1-s}\left(a^\dagger-\alpha^*\right)\left(a-\alpha\right)\right]\colon, \quad (8)
\end{aligned}$$

where $:\ :$ denotes the normal ordering, and the s-parameterized quantization scheme (or viewed as the generalized Weyl correspondence between $\mathfrak{P}(\alpha)$ and ρ)

$$\rho=2\int\mathrm{d}^2\alpha\Delta_s(\alpha)\mathfrak{P}(\alpha). \quad (9)$$

By using the IWSOP technique with the properties that the order of Boson operators a and $a^\dagger$ within an s-ordered symbol can be permuted, even though $\left[a,a^\dagger\right]=1$, we have also derived the s-ordered operator expansion formula of density operators $\rho(a^\dagger,a)$,

$$\rho=\frac{2}{1-s}\int\frac{\mathrm{d}^2\beta}{\pi}\left\langle-\beta\right|\rho\left|\beta\right\rangle\text{Ⓢ}\exp\left[\frac{2}{s-1}\left(s|\beta|^2-\beta^*a+\beta a^\dagger-a^\dagger a\right)\right]\text{Ⓢ}, \quad (10)$$

where Ⓢ$\cdots$Ⓢ means s-ordering symbol in our notation, $|\beta\rangle$ is a coherent state[13,14] and the eigenvector of a,

$$\left|\beta\right\rangle=\exp\left[-\frac{|\beta|^2}{2}+\beta a^\dagger\right]\left|0\right\rangle,\quad a\left|\beta\right\rangle=\beta\left|\beta\right\rangle. \quad (11)$$

Equation (8) serves to convert ρ into its s-ordered form. In the light of Eq. (8) we have seen the s-ordered expansion of $\Delta_s(\alpha)$,

$$\begin{aligned}\Delta_s(\alpha) &= \frac{2}{(1-s)^2\pi}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|:\exp\left[\frac{-2}{1-s}\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]:|\beta\rangle\\ &\quad\times\circledS\exp\left[\frac{2}{s-1}\left(s|\beta|^2-\beta^*a+\beta a^\dagger-a^\dagger a\right)\right]\circledS\\ &= \frac{1}{2}\circledS\,\delta\left(a^\dagger-\alpha^*\right)\delta\left(a-\alpha\right)\circledS,\end{aligned}\tag{12}$$

which is easily to be remembered. Therefore

$$\circledS\,\mathfrak{P}(a^\dagger,a)\circledS=2\int\mathrm{d}^2\alpha\Delta_s(\alpha)\mathfrak{P}(\alpha).\tag{13}$$

As one can see shortly later, all our s-ordered expansion formulas are expressed in terms of the two-variable Hermite polynomials which appear more concise than some results in Ref. [5].

2. The s-ordered operator expansion of $a^{\dagger m}a^n$

For later uses, we first introduce the two-variable Hermite polynomial $\mathrm{H}_{m,n}$,[15,16]

$$\mathrm{H}_{m,n}(\xi,\eta)=\sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l\xi^{m-l}\eta^{n-l},\tag{14}$$

whose generating function is

$$\sum_{m,n=0}^{\infty}\frac{t^m\tau^n}{m!n!}\mathrm{H}_{m,n}(\xi,\eta)=\exp\left(-t\tau+t\xi+\tau\eta\right),\tag{15}$$

and its integral representation is[17]

$$(-1)^m\,\mathrm{e}^{\xi\eta}\int\frac{\mathrm{d}^2\beta}{\pi}\beta^{*m}\beta^n\exp\left(-|\beta|^2-\xi\beta+\eta\beta^*\right)=\mathrm{H}_{m,n}\left(\xi,\eta\right).\tag{16}$$

When we take $\rho=a^{\dagger m}a^n$ in Eq. (10), using $\langle-\beta|\,\beta\rangle=\exp\left(-2|\beta|^2\right)$ and Eq. (16) as well as the IWSOP technique, we derive

$$\begin{aligned}&a^{\dagger m}a^n\\ &=(-1)^m\frac{2}{1-s}\int\frac{\mathrm{d}^2\beta}{\pi}\beta^{*m}\beta^n\circledS\exp\left[\frac{-2}{1-s}|\beta|^2-\frac{2}{1-s}\left(\beta a^\dagger-\beta^*a-a^\dagger a\right)\right]\circledS\\ &=(-1)^m\left(\frac{1-s}{2}\right)^{(n+m)/2}\int\frac{\mathrm{d}^2\beta'}{\pi}\beta'^{*m}\beta'^n\circledS\exp\left[-|\beta'|^2-\beta'\sqrt{\frac{2}{1-s}}a^\dagger+\beta'^*\sqrt{\frac{2}{1-s}}a-\frac{2}{s-1}a^\dagger a\right]\circledS\\ &=\left(\frac{1-s}{2}\right)^{(n+m)/2}\circledS\,\mathrm{H}_{m,n}\left(\sqrt{\frac{2}{1-s}}a^\dagger,\sqrt{\frac{2}{1-s}}a\right)\circledS,\end{aligned}\tag{17}$$

where $\beta'=\sqrt{2/(1-s)}\,\beta$. Thus the classical correspondence of $a^{\dagger m}a^n$ according to Eqs. (12), (13) and (17) is

$$\left(\frac{1-s}{2}\right)^{(n+m)/2}\mathrm{H}_{m,n}\left(\sqrt{\frac{2}{1-s}}\alpha^*,\sqrt{\frac{2}{1-s}}\alpha\right)\to a^{\dagger m}a^n,\tag{18}$$

which is just the essence of the s-parameterized ordering. In particular, when $s=0$, it becomes the Weyl ordering $\left(\text{denoted by }\vdots\ \vdots\right)$,

$$\left(\frac{1}{2}\right)^{(n+m)/2}\vdots\,\mathrm{H}_{m,n}(\sqrt{2}a^\dagger,\sqrt{2}a)\,\vdots=a^{\dagger m}a^n,\tag{19}$$

when $s=-1$, it reduces to the antinormal ordering (denoted by $\vdots\ \ \vdots$)

$$\vdots\, \mathrm{H}_{m,n}\left(a^{\dagger}, a\right) \vdots = a^{\dagger m} a^{n}, \tag{20}$$

which is an appealing result.[18]

3. The normally ordered expansion of $\circledS a^{\dagger m} a^{n} \circledS$

As a result of Eqs. (8), (12) and (13), we now consider $\rho = \circledS\, a^{\dagger m} a^{n} \circledS$. By virtue of the IWOP technique,[19] we have

$$\begin{aligned}
\circledS\, a^{\dagger m} a^{n} \circledS &= \int \mathrm{d}^2\alpha \alpha^{n} \alpha^{*m} \circledS\, \delta\left(a^{\dagger}-\alpha^{*}\right) \delta\left(a-\alpha\right) \circledS \\
&= \frac{2}{1-s} \int \frac{\mathrm{d}^2\alpha}{\pi} \alpha^{n} \alpha^{*m} \colon \exp\left[\frac{-2|\alpha|^2}{1-s} + \frac{2}{1-s}\left(a^{\dagger}\alpha + \alpha^{*} a\right) - \frac{2a^{\dagger}a}{1-s}\right] \colon \\
&= \left(\frac{1-s}{2}\right)^{(m+n)/2} \int \frac{\mathrm{d}^2\alpha'}{\pi} \alpha'^{n} \alpha'^{*m} \colon \exp\left[-|\alpha'|^2 + \sqrt{\frac{2}{1-s}}\left(a^{\dagger}\alpha' + \alpha'^{*} a\right) - \frac{2a^{\dagger}a}{1-s}\right] \colon \\
&= (-\mathrm{i})^{n+m} \left(\frac{1-s}{2}\right)^{(m+n)/2} \colon \mathrm{H}_{m,n}\left(\mathrm{i}\sqrt{\frac{2}{1-s}} a^{\dagger}, \mathrm{i}\sqrt{\frac{2}{1-s}} a\right) \colon ,
\end{aligned} \tag{21}$$

where we have used

$$\mathrm{e}^{\xi\eta} \int \frac{\mathrm{d}^2\beta}{\pi} \beta^{*m} \beta^{n} \exp\left(-|\beta|^2 + \beta\xi + \beta^{*}\eta\right) = (-\mathrm{i})^{m+n}\, \mathrm{H}_{m,n}(\mathrm{i}\xi, \mathrm{i}\eta). \tag{22}$$

Note that equations (17) and (21) are complementary for the s-parametric quantization. Thus, it is seen from Eqs. (21), (12) and (13) that $\alpha^{n}\alpha^{*m}$ is quantized as

$$\alpha^{n}\alpha^{*m} \to (-\mathrm{i})^{n+m} \left(\frac{1-s}{2}\right)^{(m+n)/2} \colon \mathrm{H}_{m,n}\left(\mathrm{i}\sqrt{\frac{2}{1-s}} a^{\dagger}, \mathrm{i}\sqrt{\frac{2}{1-s}} a\right) \colon . \tag{23}$$

By comparing the two-variable Hermite polynomial in Eq. (14) with the Laguerre polynomial $\mathrm{L}_{n}^{(\mu)}(x)$ defined as

$$\mathrm{L}_{n}^{(\mu)}(x) = \sum_{l=0}^{n} \frac{(n+\mu)!}{(n-l)!\,(\mu+l)!} \frac{(-x)^{l}}{l!}, \tag{24}$$

we see the following relation

$$\mathrm{H}_{m,n}(\xi,\kappa) = \begin{cases} n!\,(-1)^{n}\, \xi^{m-n} \mathrm{L}_{n}^{(m-n)}\left(\xi\kappa\right), & m > n, \\ m!\,(-1)^{m}\, \kappa^{n-m} \mathrm{L}_{m}^{(n-m)}\left(\xi\kappa\right), & m < n. \end{cases} \tag{25}$$

Using Eq. (25), we see that for $m<n$, equation (21) is just[5]

$$\circledS\, a^{\dagger m} a^{n} \circledS = \left(\frac{1-s}{2}\right)^{m} m! \colon \mathrm{L}_{m}^{(n-m)}\left(\frac{2a^{\dagger}a}{s-1}\right) \colon a^{n-m}. \tag{26}$$

On the other hand, for $m>n$

$$\circledS\, a^{\dagger m} a^{n} \circledS = \left(\frac{1-s}{2}\right)^{n} n! a^{\dagger m-n} \colon \mathrm{L}_{n}^{(m-n)}\left(\frac{2a^{\dagger}a}{s-1}\right) \colon . \tag{27}$$

4. The s'-ordered operator expansion of $\circledS\, a^{\dagger m}a^n \circledS$

Using Eqs. (10) and (21), we can derive the s'-ordered expansion of $\circledS\, a^{\dagger m}a^n \circledS$ by virtue of the IWSOP technique, i.e.,

$$\begin{aligned}
\circledS\, a^{\dagger m}a^n \circledS &= \frac{2}{1-s'}\int \frac{\mathrm{d}^2\beta}{\pi}\,\langle -\beta|\circledS\, a^{\dagger m}a^n \circledS\,|\beta\rangle\,\circledS' \exp\left[\frac{2}{s'-1}\left(s'|\beta|^2-\beta^* a+\beta a^\dagger - a^\dagger a\right)\right]\circledS' \\
&= \frac{2\,(-\mathrm{i})^{n+m}}{1-s'}\left(\frac{1-s}{2}\right)^{(m+n)/2}\int \frac{\mathrm{d}^2\beta}{\pi}\mathrm{H}_{m,n}\left(-\mathrm{i}\sqrt{\frac{2}{1-s}}\beta^*, \mathrm{i}\sqrt{\frac{2}{1-s}}\beta\right) \\
&\quad \times \circledS' \exp\left[\frac{-2}{1-s'}|\beta|^2+\frac{2}{s'-1}\left(-\beta^* a+\beta a^\dagger - a^\dagger a\right)\right]\circledS' \\
&= \frac{2\,(-\mathrm{i})^{n+m}}{1-s'}\left(\frac{1-s}{2}\right)^{(m+n)/2}\frac{\partial^{m+n}}{\partial t^m \partial \tau^n}\int \frac{\mathrm{d}^2\beta}{\pi}\circledS' \exp\left[\frac{2}{s'-1}|\beta|^2\right. \\
&\quad \left. + \left(\mathrm{i}\tau\sqrt{\frac{2}{1-s}}+\frac{2a^\dagger}{s'-1}\right)\beta - \left(\mathrm{i}t\sqrt{\frac{2}{1-s}}+\frac{2a}{s'-1}\right)\beta^* - \frac{2a^\dagger a}{s'-1} - t\tau\right]_{t=\tau=0}\circledS' \\
&= (-\mathrm{i})^{n+m}\left(\frac{s'-s}{2}\right)^{(m+n)/2}\circledS'\mathrm{H}_{m,n}\left(\mathrm{i}\sqrt{\frac{2}{s'-s}}a^\dagger, \mathrm{i}\sqrt{\frac{2}{s'-s}}a\right)\circledS', \qquad (28)
\end{aligned}$$

where in the above derivation, we have used

$$\mathrm{H}_{m,n}(\xi,\eta) = \frac{\partial^{m+n}}{\partial t^m \partial \tau^n}\exp\left(-t\tau + t\xi + \tau\eta\right)|_{t=\tau=0}\,, \qquad (29)$$

and the integral formula[20]

$$\int \frac{\mathrm{d}^2 z}{\pi}\exp\left(\zeta\,|z|^2+\xi z+\eta z^*\right) = -\frac{1}{\zeta}\exp\left[-\frac{\xi\eta}{\zeta}\right], \quad \mathrm{Re}\,(\zeta) < 0. \qquad (30)$$

Especially, when $s'=1$, equation (28) becomes Eq. (21).

On the other hand, when $m > n$, using Eq. (25) we can see that equation (28) becomes[5]

$$\begin{aligned}
\circledS\, a^{\dagger m}a^n \circledS &= (-\mathrm{i})^{n+m}\left(\frac{s'-s}{2}\right)^{(m+n)/2} n!\,(-1)^n\,\circledS'\left(\mathrm{i}\sqrt{\frac{2}{s'-s}}a^\dagger\right)^{m-n}\mathrm{L}_n^{(m-n)}\left(\frac{2a^\dagger a}{s-s'}\right)\circledS' \\
&= n!\left(\frac{s'-s}{2}\right)^n \circledS' a^{\dagger m-n}\mathrm{L}_n^{m-n}\left(\frac{2a^\dagger a}{s-s'}\right)\circledS', \qquad (31)
\end{aligned}$$

while for $m < n$,

$$\circledS\, a^{\dagger m}a^n \circledS = m!\left(\frac{s'-s}{2}\right)^m \circledS' a^{n-m}\mathrm{L}_m^{n-m}\left(\frac{2a^\dagger a}{s-s'}\right)\circledS'. \qquad (32)$$

5. Application of Eq. (10)

Knowing the essence of the s-parameterized ordering of operators, we can apply Eq. (10) to obtain both the P-representation and the Wigner function of the squeezed chaotic field,[21] expressed as

$$\rho_s = S(r)\rho_c S^{-1}(r), \qquad (33)$$

in a unified way, where $S(r) = \exp[\mathrm{i}\,r(XP+PX)/2]$ is the squeezing operator. $X = (a+a^\dagger)/\sqrt{2}$, $P = (a-a^\dagger)/\mathrm{i}\sqrt{2}$, and

$$\rho_c = (1-\mathrm{e}^{-\hbar\omega/kT})\,\mathrm{e}^{-\hbar\omega a^\dagger a/kT}, \qquad (34)$$

(k is the Boltzmann constant, T denotes temperature) is qualified to be a density operator of chaotic field, due to $\mathrm{tr}\,\rho_c = 1$. By virtue of the IWOP technique we have converted ρ_s to its normally ordered Gaussian form:[22]

$$\rho_s = 2\sqrt{fg}\colon \exp\left(-fX^2-gP^2\right)\colon, \qquad (35)$$

where

$$f \equiv \frac{1}{(2\bar{n}+1)\,\mathrm{e}^{2r}+1} < \frac{1}{2}, \quad g \equiv \frac{1}{(2\bar{n}+1)\,\mathrm{e}^{-2r}+1} < 1 \tag{36}$$

and $\bar{n}$ denotes the average photon number of the chaotic field ρ_c in Eq. (34), i.e. $\bar{n} = \mathtt{tr}\left(\rho_c a^\dagger a\right) = (\,\mathrm{e}^{\hbar\omega/kT} - 1)^{-1}$. Using Eq. (10) and the integral formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp(\zeta\,|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left(\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right), \tag{37}$$

which holds for $\mathtt{Re}\,(\zeta \pm f \pm g) < 0$ and $\mathtt{Re}\left((\zeta^2 - 4fg)/(\zeta \pm f \pm g)\right) < 0$, yields

$$\begin{aligned}
\rho_s =& \frac{4\sqrt{fg}}{1-s} \int \frac{\mathrm{d}^2\beta}{\pi} \exp\left[-f\left(\frac{\beta-\beta^*}{\sqrt{2}}\right)^2 - g\left(\frac{\beta+\beta^*}{\mathrm{i}\sqrt{2}}\right)^2\right] \\
& \times \text{Ⓢ} \exp\left[\frac{-2}{1-s}|\beta|^2 - \frac{2a}{s-1}\beta^* + \frac{2a^\dagger}{s-1}\beta - \frac{2}{s-1}a^\dagger a\right] \text{Ⓢ} \\
=& \frac{2}{1-s}\sqrt{\frac{(s-1)^2\,fg}{G_s}} \text{Ⓢ} \exp\left[-\frac{(f+g-2fg+2fgs)\,a^\dagger a}{G_s} + \frac{(g-f)\left(a^{\dagger 2}+a^2\right)}{2G_s}\right] \text{Ⓢ}\,,
\end{aligned} \tag{38}$$

where

$$G_s \equiv (fs - f + 1)\,(gs - g + 1)\,. \tag{39}$$

As a special case, when $s = 0$, it reduces to the Weyl ordering

$$\rho_s = 2\sqrt{\frac{fg}{(1-f)\,(1-g)}}\,\vdots \exp\left[\frac{(g-f)\left(a^{\dagger 2}+a^2\right) - 2\,(f+g-2fg)\,a^\dagger a}{2\,(1-f)\,(1-g)}\right]\vdots\,. \tag{40}$$

Thus, ρ_s's Weyl correspondence is

$$\rho_s \to 2\sqrt{\frac{fg}{(1-f)\,(1-g)}} \exp\left[\frac{(g-f)\left(\alpha^{*2}+\alpha^2\right) - 2\,(f+g-2fg)\,|\alpha|^2}{2\,(1-f)\,(1-g)}\right]. \tag{41}$$

According to the Wigner theorem, for a given state $|\psi\rangle$, we have

$$\langle\psi|\,\rho_s\,|\psi\rangle = 4\sqrt{\frac{fg}{(1-f)\,(1-g)}} \int \mathrm{d}^2\alpha \exp\left[\frac{(g-f)\left(\alpha^{*2}+\alpha^2\right) - 2\,(f+g-2fg)\,|\alpha|^2}{2\,(1-f)\,(1-g)}\right] W_\psi\,(\alpha^*, \alpha)\,, \tag{42}$$

where $W_\psi\,(\alpha^*, \alpha)$ is just Wigner function of $|\psi\rangle$. On the other hand, when $s = -1$, it reduces to the antinormal ordering

$$\rho_s = 2\sqrt{\frac{fg}{(1-2f)\,(1-2g)}}\,\vdots \exp\left[\frac{(g-f)\left(a^{\dagger 2}+a^2\right) - 2\,(f+g-4fg)\,a^\dagger a}{2\,(1-2f)\,(1-2g)}\right]\vdots\,. \tag{43}$$

It follows that the P-representation of the squeezed chaotic field is expressed as

$$P(\alpha^*, \alpha) = 2\sqrt{\frac{fg}{(1-2f)\,(1-2g)}} \exp\left[\frac{(g-f)\left(\alpha^{*2}+\alpha^2\right) - 2\,(f+g-4fg)\,|\alpha|^2}{2\,(1-2f)\,(1-2g)}\right]. \tag{44}$$

6. Conclusions

In summary, in reference to the Weyl ordering $x^m p^n \to (1/2)^m \sum_{l=0}^m \binom{m}{l} X^{m-l} P^n X^l$, we have employed the IWSOP technique to reveal what is characteristic of the fundamental operator-function correspondence relation in the context of the s-parametric operator ordering, equations (17) and (21) are our main result. All our derived results are expressed in terms of the two-variable Hermite polynomial whose special cases lead to the formulas in Ref. [5]. The s-ordered operator-expansion formula of ρ in Eq. (10) plays a key role in our discussion. Through this approach, the essence of Glauber's s-parameterized operator ordering rule is more manifest.

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