Relation between Debye series and generalized Lorentz–Mie theory of laser beam scattering by multilayer cylinder*

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With the recursive relations of Bessel function and applying inductive approach, the consistency between the scattering coefficient formula of Debye series and that of the generalized Lorentz–Mie theory (GLMT) in the scattering of a multilayer cylinder illuminated by a laser beam is proved in detail. As an application example, rainbow phenomenon in the scattering of a two-layer cylinder is discussed by the Debye series components. It can be concluded that the radius and the refractive index of a two-layer cylinder have an effect on the twin-primary rainbow, and the high-order Debye series components which are associated with the high-order rainbows are more complex than those of a homogeneous cylinder.

Keywords: laser beam scattering, a multilayer cylinder, rainbow phenomena

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1. Introduction

Particle-size analysis on the basis of light scattering, which has advantages of high speed, accurate measurement and non-contact, is widely used in the fields[1,2] of biomedicine, polymer science, atmosphere detection, etc. As an analytical solution, generalized Lorentz–Mie theory (GLMT),[3−6] which was provided by Gouesbet, has been applied to studying the scattering of laser beam by a sphere, or a cylinder, or an ellipsoid. The improved algorithm[7−10] of GLMT has also been presented to discuss the scattering by a multilayer sphere, or a multilayer cylinder or a multilayer ellipsoid. However, GLMT has some limitations to the description of the high order rainbow phenomena in the laser beam scattering of particles, especially for multilayer particles. In recent years, rainbow technique on the basis of light scattering by particles has become more and more popular with the development of grain size measurement. Both theory and experiments about rainbow phenomena and rainbow technique have been developed by many scholars.[11−19] However, most of them are effective only for homogeneous particles, and a little research on twin-primary rainbow metrology has been done.[20,21] The reason is that rainbows of inhomogeneous particles are more complex than those of homogeneous particles and are difficult to analyse precisely. Debye series, based on ray optics, can describe actions of different effects by particles on the incident wave, and it is beneficial to the analysis of high-order rainbows. With Debye series, the scatterings of a coated sphere,[21] a multilayer sphere,[22] and a multilayer cylinder[23−25] have been discussed.

Although the comparison between some numerical results obtained from GLMT and Debye series, which are two different methods used to treat the scattering of laser beam, has been given, the discussion about the consistency between mathematical formulas of the two methods has not been reported. In the following parts, for a transverse magnetic (TM) laser beam incidence, the derivation of this consistency will be given, and for an example, the rainbow phenomenon for a two-layer cylinder by Debye series components will be discussed.

The remainder of the present paper is organized as follows. In Section 2 the relations between Debye series and GLMT for a homogeneous cylinder are deduced. In Section 3, the consistency between the two methods for a multilayer cylinder is proved. With Debye series, rainbow intensities of a two-layer cylinder are calculated in Section 4. Some conclusions are drawn from the present study in Section 5.

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2. Relations between Debye series and GLMT of laser beam scattering by a homogeneous cylinder

When a homogeneous cylinder is illuminated by a laser beam, the scattering coefficients can be expressed with Debye series as follows:

$$a_{\rm l}^1 = \frac{1}{2}C_{\rm l}\Big(1 - R_{{\rm l},\varepsilon}^{212} - T_{{\rm l},\varepsilon}^{21}T_{{\rm l},\varepsilon}^{12} - T_{{\rm l},\varepsilon}^{21}\frac{R_{{\rm l},\varepsilon}^{121}}{1-R_{{\rm l},\varepsilon}^{121}}T_{{\rm l},\varepsilon}^{12}\Big) = \frac{1}{2}C_{\rm l}\Big[1 - R_{{\rm l},\varepsilon}^{212} - \sum_{p=1}^{\infty}T_{{\rm l},\varepsilon}^{21}(R_{{\rm l},\varepsilon}^{121})^{p-1}T_{{\rm l},\varepsilon}^{12}\Big], \tag{1}$$

$$b_{\rm l}^1 = \frac{1}{2}C_{\rm l}\Big(1 - R_{{\rm l},\mu}^{212} - T_{{\rm l},\mu}^{21}T_{{\rm l},\mu}^{12} - T_{{\rm l},\mu}^{21}\frac{R_{{\rm l},\mu}^{121}}{1-R_{{\rm l},\mu}^{121}}T_{{\rm l},\mu}^{12}\Big) = \frac{1}{2}C_{\rm l}\Big[1 - R_{{\rm l},\mu}^{212} - \sum_{p=1}^{\infty}T_{{\rm l},\mu}^{21}(R_{{\rm l},\mu}^{121})^{p-1}T_{{\rm l},\mu}^{12}\Big], \tag{2}$$

where the Fresnel coefficients $R_{{\rm l},i}^{212}, R_{{\rm l},i}^{121}, T_{{\rm l},i}^{12}, T_{{\rm l},i}^{21}$ have been considered in the previous paper;[15] the subscripts ε and μ denote the transverse electric (TE) mode and the TM mode, respectively; $C_{\rm l}$ is the laser beam shape coefficient. Substituting the expressions of the Fresnel coefficients into Eq. (2) produces the numerator and the denominator in Eq. (2) as follows:

$$\begin{aligned} &4\pi^2y^2\Big[m_1^2{\rm J}_{\rm l}(x){\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)'}(y) + {\rm J}'_{\rm l}(x){\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)}(y)\Big]. \\ &\quad\times\Big[-m_1{\rm J}'_{\rm l}(x){\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)}(y) - {\rm J}_{\rm l}(x){\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)'}(y) + 4/\pi^2x^2\Big], \end{aligned} \tag{3}$$

$$4\pi^2y^2\Big[m_1{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x) - {\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x)\Big]\Big[m_1{\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)'}(y) - {\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)}(y)\Big], \tag{4}$$

where $x = kr$, $y = m_1kr$, J is the Bessel function, and H is the Hankel function. Here, the wave number in free space is $k = 2\pi/\lambda$, m_1 and r are the refractive index and the radius of the cylinder, and λ is the wavelength of the incident wave. Thus

$$\begin{aligned} b_{\rm l}^1 &= C_{\rm l}\frac{m_1^2{\rm J}_{\rm l}(x){\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)'}(y) + {\rm J}'_{\rm l}(x){\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)}(y)}{m_1{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x) - {\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x)} \\ &\quad\times\frac{-m_1{\rm J}'_{\rm l}(x){\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)}(y) - {\rm J}_{\rm l}(x){\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)'}(y) + 4/\pi^2x^2}{m_1{\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)'}(y) - {\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)}(y)} \\ &= C_{\rm l}\frac{AA}{BB}. \end{aligned} \tag{5}$$

Assume that the numerator in Eq. (5) is AA, and the denominator is BB. With the relation

$${\rm J}_{\rm l}(\rho){\rm Y}'_{\rm l}(\rho) - {\rm J}'_{\rm l}(\rho){\rm Y}_{\rm l}(\rho) = \frac{2}{\pi\rho}, \tag{6}$$

we obtain

$$\begin{aligned} &m_1\Big[{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(2)}(y) - {\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(2)'}(y)\Big]\Big[{\rm J}'_{\rm l}(x){\rm H}_{\rm l}^{(1)}(x) - {\rm J}_{\rm l}(x){\rm H}_{\rm l}^{(1)'}(x)\Big] \\ &\quad= m_1\left[{\rm J}_{\rm l}(y){\rm Y}'_{\rm l}(y) - {\rm J}'_{\rm l}(y){\rm Y}_{\rm l}(y)\right]\left[{\rm J}_{\rm l}(x){\rm Y}'_{\rm l}(x) - {\rm J}'_{\rm l}(x){\rm Y}_{\rm l}(x)\right] \\ &\quad= \frac{4}{\pi^2x^2}. \end{aligned} \tag{7}$$

AA plus $m_1\Big[{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(2)}(y) - {\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(2)'}(y)\Big]\Big[{\rm J}'_{\rm l}(x){\rm H}_{\rm l}^{(1)}(x) - {\rm J}_{\rm l}(x){\rm H}_{\rm l}^{(1)'}(x)\Big] - 4/\pi^2x^2$ is

$$\begin{aligned} &AA + m_1\Big[{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(2)}(y) - {\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(2)'}(y)\Big]\Big[{\rm J}'_{\rm l}(x){\rm H}_{\rm l}^{(1)}(x) - {\rm J}_{\rm l}(x){\rm H}_{\rm l}^{(1)'}(x)\Big] - 4/\pi^2x^2 \\ &\quad= \left[m_1{\rm J}_{\rm l}(x){\rm J}'_{\rm l}(y) - {\rm J}'_{\rm l}(x){\rm J}_{\rm l}(y)\right]\Big[m_1{\rm H}_{\rm l}^{(1)}(x){\rm H}_{\rm l}^{(2)'}(y) - {\rm H}_{\rm l}^{(1)'}(x){\rm H}_{\rm l}^{(2)}(y)\Big]. \end{aligned} \tag{8}$$

Substituting Eq. (8) into Eq. (5), we have

$$b_{\rm l}^1 = C_{\rm l}\frac{AA}{BB} = C_{\rm l}\frac{{\rm J}'_{\rm l}(x){\rm J}_{\rm l}(y) - m_1{\rm J}_{\rm l}(x){\rm J}'_{\rm l}(y)}{{\rm J}_{\rm l}(y){\rm H}_{\rm l}^{(1)'}(x) - m_1{\rm J}'_{\rm l}(y){\rm H}_{\rm l}^{(1)}(x)}. \tag{9}$$

In GLMT, the scattering coefficient[8] for a homogeneous cylinder illuminated by a TM laser beam is

$$b_{\mathrm{l\,I}}^{t=1} = C_{\mathrm{l}}\frac{\mathrm{J_l}(x)}{\mathrm{H_l^{(1)}}(x)}\cdot\frac{\eta_1\mathrm{D_l^{(1)}}(x)-\mathrm{H_l^b}(m_1x)}{\eta_1\mathrm{D_l^{(3)}}(x)-\mathrm{H_l^b}(m_1x)} = C_{\mathrm{l}}\frac{\mathrm{J_l}(x)}{\mathrm{H_l^{(1)}}(x)}\cdot\frac{\mathrm{J_l'}(x)/\mathrm{J_l}(x)-m_1\mathrm{J_l'}(y)/\mathrm{J_l}(y)}{\mathrm{H_l^{(1)'}}(x)/\mathrm{H_l}(x)-m_1\mathrm{J_l'}(y)/\mathrm{J_l}(y)}$$
$$= C_{\mathrm{l}}\frac{\mathrm{J_l'}(x)\mathrm{J_l}(y)-m_1\mathrm{J_l'}(y)\mathrm{J_l}(x)}{\mathrm{H_l^{(1)'}}(x)\mathrm{J_l}(y)-m_1\mathrm{J_l'}(y)\mathrm{H_l}(x)}. \tag{10}$$

It is obvious that equation (9) is identical with Eq. (10). With a similar process, we can prove $a_{\mathrm{l\,II}}^{t=1} = a_{\mathrm{l}}^{1}$.

3. Relations between Debye series and GLMT of laser beam scattering by a multilayer cylinder

With the recursive relations, the scattering coefficients of an n-layer cylinder can be expanded in terms of Debye series[25] as

$$a_{\mathrm{l}} = \frac{1}{2}C_{\mathrm{l}}(1-Q_{\mathrm{l},\varepsilon}^{n+1}) = \frac{1}{2}C_{\mathrm{l}}\left[1-R_{\mathrm{l},\varepsilon}^{n+1,n,n+1}-(T_{\mathrm{l},\varepsilon}^{n+1,n}Q_{\mathrm{l},\varepsilon}^{n}T_{\mathrm{l},\varepsilon}^{n,n+1})/(1-R_{\mathrm{l},\varepsilon}^{n,n+1,n}Q_{\mathrm{l},\varepsilon}^{n})\right]$$
$$= \frac{1}{2}C_{\mathrm{l}}\left[1-R_{l,\varepsilon}^{n+1,n,n+1}-\sum_{p=1}^{\infty}T_{\mathrm{l},\varepsilon}^{n+1,n}Q_{\mathrm{l},\varepsilon}^{n}(Q_{\mathrm{l},\varepsilon}^{n}R_{\mathrm{l},\varepsilon}^{n,n+1,n})^{p-1}T_{\mathrm{l},\varepsilon}^{n,n+1}\right], \tag{11}$$

$$b_{\mathrm{l}} = \frac{1}{2}C_{\mathrm{l}}(1-Q_{\mathrm{l},\mu}^{n+1}) = \frac{1}{2}C_{\mathrm{l}}\left[1-R_{\mathrm{l},\mu}^{n+1,n,n+1}-(T_{\mathrm{l},\mu}^{n+1,n}Q_{\mathrm{l},\mu}^{n}T_{\mathrm{l},\mu}^{n,n+1})/(1-R_{\mathrm{l},\mu}^{n,n+1,n}Q_{\mathrm{l},\mu}^{n})\right]$$
$$= \frac{1}{2}C_{\mathrm{l}}\left[1-R_{\mathrm{l},\mu}^{n+1,n,n+1}-\sum_{p=1}^{\infty}T_{\mathrm{l},\mu}^{n+1,n}Q_{\mathrm{l},\mu}^{n}(Q_{\mathrm{l},\mu}^{n}R_{\mathrm{l},\mu}^{n,n+1,n})^{p-1}T_{\mathrm{l},\mu}^{n,n+1}\right], \tag{12}$$

where the recursive expression is $Q_{\mathrm{l},i}^{n} = R_{\mathrm{l},i}^{n,n-1,n} + T_{\mathrm{l},i}^{n,n-1}Q_{\mathrm{l},i}^{n-1}T_{\mathrm{l},i}^{n-1,n}\Big/\left(1-R_{\mathrm{l},i}^{n-1,n,n-1}Q_{\mathrm{l},i}^{n-1}\right)$, $(i=\varepsilon,\mu)$. $R_{\mathrm{l},i}^{j+1,j,j+1}$, $R_{\mathrm{l},i}^{j,j+1,j}$, $T_{\mathrm{l},i}^{j,j+1}$ and $T_{\mathrm{l},i}^{j+1,j}$ are the Fresnel coefficients[25] of a multilayer cylinder.

For convenience, we express the scattering coefficients of the two methods as $b_{\mathrm{l,GLMT}}^{t=n}$ and $b_{\mathrm{l,Debye\ series}}^{n}$, respectively. We assume that the equality $b_{\mathrm{l,GLMT}}^{t=j} = b_{\mathrm{l,Debye\ series}}^{j} = B_{\mathrm{GD}}$ $(j=2,3,4,\ldots)$ exists. What we should do is to prove the relation $b_{\mathrm{l,GLMT}}^{t=j+1} = b_{\mathrm{l,Debye\ series}}^{j+1}$.

For a cylinder with $j+1$ layers, the scattering coefficient[8] in GLMT is

$$b_{\mathrm{l,GLMT}}^{t=j+1} = C_{\mathrm{l}}\cdot\frac{\mathrm{J_l}(x_{j+1})}{\mathrm{H_l}^{(1)}(x_{j+1})}\cdot\frac{\eta_{j+1}\mathrm{D_l^{(1)}}(x_{j+1})-\mathrm{H_l}^{\mathrm{b}}(m_{j+1}x_{j+1})}{\eta_{j+1}\mathrm{D_l}^{(3)}(x_{j+1})-\mathrm{H_l}^{\mathrm{b}}(m_{j+1}x_{j+1})}, \tag{13}$$

where

$$H_{\mathrm{l}}^{\mathrm{b}}(m_{j+1}x_{j+1}) = \frac{\mathrm{J_l}(m_{j+1}x_{j+1})/\mathrm{Y_l}(m_{j+1}x_{j+1})\cdot\mathrm{D_l^{(1)}}(m_{j+1}x_{j+1})}{\mathrm{J_l}(m_{j+1}x_{j+1})/\mathrm{Y_l}(m_{j+1}x_{j+1})-B_l^{(j+1)}} - \frac{\mathrm{B_l}^{(j+1)}\cdot\mathrm{D_l^{(2)}}(m_{j+1}x_{j+1})}{\mathrm{J_l}(m_jx_j)/\mathrm{Y_l}(m_{j+1}x_{j+1})-B_{\mathrm{l}}^{(j+1)}}, \tag{14}$$

$$B_{\mathrm{l}}^{(j+1)} = \frac{\mathrm{J_l}(m_{j+1}x_j)}{\mathrm{Y_l}(m_{j+1}x_j)}\cdot\frac{\eta_j\mathrm{D_l^{(1)}}(m_{j+1}x_j)-\eta_{j+1}\mathrm{H_l^b}(m_jx_j)}{\eta_j\mathrm{D_l^{(2)}}(m_{j+1}x_j)-\eta_{j+1}\mathrm{H_l^b}(m_jx_j)}. \tag{15}$$

It should be noted that the expressions $m_{j+1}=1$, $\eta_{j+1}=1/m_{j+1}$, and $\eta_j = 1/m_j$ are true for Eq. (15). Therefore, equation (15) can be reduced to

$$B_{\mathrm{l}}^{(j+1)} = \frac{\mathrm{J_l}(x_j)}{\mathrm{Y_l}(x_j)}\frac{\mathrm{D_l^{(1)}}(m_{j+1}x_j)-m_j\mathrm{H_l^b}(m_jx_j)}{\mathrm{D_l^{(2)}}(m_{j+1}x_j)-m_j\mathrm{H_l^b}(m_jx_j)} = \frac{\mathrm{J_l}'(x_j)-m_j\mathrm{J_l}(x_j)\mathrm{H_l^b}(y_j)}{\mathrm{Y_l}'(x_j)-m_j\mathrm{Y_l}(x_j)\mathrm{H_l^b}(y_j)}. \tag{16}$$

The scattering coefficient for a cylinder with j layers is

$$B_{\mathrm{GD}} = b_{\mathrm{l,GLMT}}^{t=j} = C_{\mathrm{l}}\cdot B_{\mathrm{G}} = C_{\mathrm{l}}\cdot\frac{\mathrm{J_l}(x_t)}{\mathrm{H_l^{(1)}}(x_t)}\cdot\frac{\eta_t\mathrm{D_l^{(1)}}(x_t)-\mathrm{H_l^b}(m_tx_t)}{\eta_t\mathrm{D_l^{(3)}}(x_t)-\mathrm{H_l^b}(m_tx_t)}$$
$$= C_{\mathrm{l}}\frac{\mathrm{J_l'}(x_j)-m_j\mathrm{J_l}(x_j)\mathrm{H_l^b}(y_j)}{\mathrm{H_l^{(1)'}}(x_j)-m_j\mathrm{H_l^{(1)}}(x_j)\mathrm{H_l^b}(y_j)}, \tag{17}$$

where $B_{\rm G}$ is the scattering coefficient for the plane wave incidence. After some reductions, the relation between Eq. (16) and Eq. (17) can be shown as

$$B_{\rm l}^{j+1} = {\rm i}\frac{B_{\rm G}}{1-B_{\rm G}}. \tag{18}$$

Thus

$$\begin{aligned}
{\rm H}_{\rm l}^{\rm b}(m_{j+1}x_{j+1}) &= \frac{{\rm J_l}(y_{j+1})/{\rm Y_l}(y_{j+1})\cdot {\rm J_l'}(y_{j+1})/{\rm J_l}(y_{j+1})}{{\rm J_l}(y_{j+1})/{\rm Y_l}(y_{j+1}) - B_{\rm l}^{(j+1)}} - \frac{B_{\rm l}^{(j+1)}\cdot {\rm Y_l'}(y_{j+1})/{\rm Y_l}(y_{j+1})}{{\rm J_l}(y_{j+1})/{\rm Y_l}(y_{j+1}) - B_{\rm l}^{(j+1)}} \\
&= \frac{{\rm J_l'}(y_{j+1}) - {\rm i}\dfrac{B_{\rm G}}{1-B_{\rm G}}{\rm Y_l'}(y_{j+1})}{{\rm J_l}(y_{j+1}) - {\rm i}\,\dfrac{B_{\rm G}}{1-B_{\rm G}}{\rm Y_l}(y_{j+1})} \\
&= \frac{{\rm J_l}'(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)'}}(y_{j+1})}{{\rm J_l}(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)}}(y_{j+1})}.
\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (13) yields

$$\begin{aligned}
b_{\rm l,GLMT}^{t=j+1} &= C_{\rm l}\cdot\frac{{\rm J_l}(x_{j+1})}{{\rm H_l^{(1)}}(x_{j+1})}\cdot\frac{\eta_{j+1}{\rm D_l^{(1)}}(x_{j+1}) - {\rm H_l^b}(m_{j+1}x_{j+1})}{\eta_{j+1}{\rm D_l^{(3)}}(x_{j+1}) - {\rm H_l^b}(m_{j+1}x_{j+1})} \\
&= C_{\rm l}\cdot\frac{{\rm J_l}(x_{j+1})}{{\rm H_l^{(1)}}(x_{j+1})}\cdot\frac{\dfrac{{\rm J_l'}(x_{j+1})}{{\rm J_l}(x_{j+1})} - m_{j+1}\dfrac{{\rm J_l'}(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)'}}(y_{j+1})}{{\rm J_l}(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)}}(y_{j+1})}}{\dfrac{{\rm H_l^{(1)'}}(x_{j+1})}{{\rm H_l^{(1)}}(x_{j+1})} - m_{j+1}\dfrac{{\rm J_l'}(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)'}}(y_{j+1})}{{\rm J_l}(y_{j+1}) - B_{\rm G}{\rm H_l^{(1)}}(y_{j+1})}} \\
&= C_{\rm l}\cdot\Big\{[{\rm J_l'}(x_{j+1}){\rm J_l}(y_{j+1}) - m_{j+1}{\rm J_l}(x_{j+1}){\rm J_l'}(y_{j+1})] - B_{\rm G}[{\rm J_l'}(x_{j+1}){\rm H_l^{(1)}}(y_{j+1}) \\
&\quad - m_{j+1}{\rm J_l}(x_{j+1}){\rm H_l^{(1)'}}(y_{j+1})]\Big\}\times\Big\{[{\rm H_l^{(1)'}}(x_{j+1}){\rm J_l}(y_{j+1}) - m_{j+1}{\rm H_l^{(1)}}(x_{j+1}){\rm J_l'}(y_{j+1})] \\
&\quad - B_{\rm G}[{\rm H_l^{(1)'}}(x_{j+1}){\rm H_l^{(1)}}(y_{j+1}) - m_{j+1}{\rm H_l^{(1)}}(x_{j+1}){\rm H_l^{(1)'}}(y_{j+1})]\Big\}^{-1}.
\end{aligned} \tag{20}$$

By Eq. (12), we can obtain the following expression

$$Q_{{\rm l},\mu}^{j+1} = 1 - \frac{2b_{\rm l,Debye\ series}^{j}}{C_{\rm l}} = 1 - \frac{2B_{\rm GD}}{C_{\rm l}}. \tag{21}$$

As a result, for a cylinder with $j+1$ layers, the scattering coefficient by Debye series is

$$\begin{aligned}
b_{\rm l,Debye\ series}^{j+1} &= \frac{1}{2}C_{\rm l}\cdot\Big[1 - R_{{\rm l},\mu}^{j+2,j+1,j+2} - (T_{{\rm l},\mu}^{j+2,j+1}Q_{{\rm l},\mu}^{j+1}T_{{\rm l},\mu}^{j+1,j+2})/(1 - R_{{\rm l},\mu}^{j+1,j+2,j+1}Q_{{\rm l},\mu}^{j+1})\Big] \\
&= \frac{1}{2}C_{\rm l}\cdot\left[1 - R_{{\rm l},\mu}^{j+2,j+1,j+2} - \frac{T_{{\rm l},\mu}^{j+2,j+1}T_{{\rm l},\mu}^{j+1,j+2}(1-2B_{\rm GD}/C_{\rm l})}{1 - R_{{\rm l},\mu}^{j+1,j+2,j+1}(1-2B_{\rm GD}/C_{\rm l})}\right] \\
&= \frac{1}{2}C_{\rm l}\cdot\frac{1 - R_{{\rm l},\mu}^{j+2,j+1,j+2} - (R_{{\rm l},\mu}^{j+1,j+2,j+1} - R_{{\rm l},\mu}^{j+1,j+2,j+1}R_{{\rm l},\mu}^{j+2,j+1,j+2} + T_{{\rm l},\mu}^{j+2,j+1}T_{{\rm l},\mu}^{j+1,j+2})\Big(1 - \dfrac{2B_{\rm GD}}{C_{\rm l}}\Big)}{1 - R_{{\rm l},\mu}^{j+1,j+2,j+1}\Big(1 - \dfrac{2B_{\rm GD}}{C_{\rm l}}\Big)}.
\end{aligned} \tag{22}$$

Assuming

$$E = m_{j+1}{\rm H_l^{(2)'}}(y_{j+1}){\rm H_l^{(1)}}(x_{j+1}) - {\rm H_l^{(1)'}}(x_{j+1})H_{\rm l}^{(2)}(y_{j+1}), \tag{23}$$

$$A = m_{j+1}{\rm H_l^{(2)'}}(y_{j+1}){\rm H_l^{(2)}}(x_{j+1}) - {\rm H_l^{(2)'}}(x_{j+1}){\rm H_l^{(2)}}(y_{j+1}), \tag{24}$$

$$B = m_{j+1}{\rm H_l^{(1)'}}(y_{j+1}){\rm H_l^{(1)}}(x_{j+1}) - {\rm H_l^{(1)'}}(x_{j+1}){\rm H_l^{(1)}}(y_{j+1}), \tag{25}$$

$$CD = [{\rm H_l^{(2)'}}(y_{j+1}){\rm H_l^{(1)}}(y_{j+1}) - {\rm H_l^{(1)'}}(y_{j+1}){\rm H_l^{(2)}}(y_{j+1})]\cdot[{\rm H_l^{(2)'}}(x_{j+1}){\rm H_l^{(1)}}(x_{j+1}) - {\rm H_l^{(2)}}(x_{j+1}){\rm H_l^{(1)'}}(x_{j+1})], \tag{26}$$

thus, the Fresnel coefficients can be reduced to

$$R_{{\rm l},\mu}^{j+2,j+1,j+2} = -\frac{A}{E}, \qquad R_{{\rm l},\mu}^{j+1,j+2,j+1} = -\frac{B}{E}, \qquad T_{{\rm l},\mu}^{j+2,j+1}T_{{\rm l},\mu}^{j+1,j+2} = m_{j+1}\frac{CD}{E^2}. \tag{27}$$

Substituting Eq. (27) into Eq. (22), we have

$$b_{\mathrm{l,Debye\ series}}^{j+1} = C_{\mathrm{l}} \frac{1 + A/E - \left[-B/E - A \cdot B/E^2 + m_{j+1}CD/E^2\right]\left[1 - 2B_{\mathrm{G}}\right]}{1 + B \cdot \left[1 - 2B_{\mathrm{G}}\right]/E}$$
$$= C_{\mathrm{l}} \frac{\left[E^2 + A \cdot E + B \cdot E + A \cdot B - m_{j+1}CD\right] - 2\left[B \cdot E + A \cdot B - m_{j+1}CD\right]B_{\mathrm{G}}}{\left[E^2 + B \cdot E\right] - 2B \cdot E \cdot B_{\mathrm{G}}}. \quad (28)$$

With the relations of Bessel function, we have

$$E^2 + A \cdot E + B \cdot E + A \cdot B - m_{j+1}CD$$
$$= 4[m_{j+1}\mathrm{J_l}(x_{j+1})\mathrm{J_l'}(y_{j+1}) - \mathrm{J_l}(y_{j+1})\mathrm{J_l'}(x_{j+1})][m_{j+1}\mathrm{H_l^{(2)'}}(y_{j+1})\mathrm{H_l^{(1)}}(x_{j+1}) - \mathrm{H_l^{(2)}}(y_{j+1})\mathrm{H_l^{(1)'}}(x_{j+1})], \quad (29)$$

$$B \cdot E + A \cdot B - m_{j+1}CD = 2[m_{j+1}\mathrm{J_l}(x_{j+1})\mathrm{H_l^{(1)'}}(y_{j+1}) - \mathrm{J_l'}(x_{j+1})\mathrm{H_l^{(1)}}(y_{j+1})]$$
$$\times [m_{j+1}\mathrm{H_l^{(2)'}}(y_{j+1})\mathrm{H_l^{(1)}}(x_{j+1}) - \mathrm{H_l^{(2)}}(y_{j+1})\mathrm{H_l^{(1)'}}(x_{j+1})], \quad (30)$$

$$E^2 + B \cdot E = [m_{j+1}\mathrm{H_l^{(1)}}(x_{j+1})\mathrm{J_l'}(y_{j+1}) - \mathrm{H_l^{(1)'}}(x_{j+1})\mathrm{J_l}(y_{j+1})]$$
$$\times [m_{j+1}\mathrm{H_l^{(2)'}}(y_{j+1})\mathrm{H_l^{(1)}}(x_{j+1}) - \mathrm{H_l^{(2)}}(y_{j+1})\mathrm{H_l^{(1)'}}(x_{j+1})], \quad (31)$$

$$B \cdot E = [m_{j+1}\mathrm{H_l^{(1)}}(x_{j+1})\mathrm{H_l^{(1)'}}(y_{j+1}) - \mathrm{H_l^{(1)'}}(x_{j+1})\mathrm{H_l^{(1)}}(y_{j+1})]$$
$$\times [m_{j+1}\mathrm{H_l^{(2)'}}(y_{j+1})\mathrm{H_l^{(1)}}(x_{j+1}) - \mathrm{H_l^{(2)}}(y_{j+1})\mathrm{H_l^{(1)'}}(x_{j+1})]. \quad (32)$$

Substituting Eqs. (29)–(32) into Eq. (28) and making some reductions, we can obtain

$$b_{\mathrm{l,Debye\ series}}^{j+1} = C_{\mathrm{l}} \cdot \Big\{[\mathrm{J_l'}(x_{j+1})\mathrm{J_l}(y_{j+1}) - m_{j+1}\mathrm{J_l}(x_{j+1})\mathrm{J_l'}(y_{j+1})] - B_{\mathrm{G}}[\mathrm{J_l'}(x_{j+1})\mathrm{H_l^{(1)}}(y_{j+1})$$
$$- m_{j+1}\mathrm{J_l}(x_{j+1})\mathrm{H_l^{(1)'}}(y_{j+1})]\Big\} \times \Big\{[\mathrm{H_l^{(1)'}}(x_{j+1})\mathrm{J_l}(y_{j+1}) - m_{j+1}\mathrm{H_l^{(1)}}(x_{j+1})\mathrm{J_l'}(y_{j+1})]$$
$$- B_{\mathrm{G}}[\mathrm{H_l^{(1)'}}(x_{j+1})\mathrm{H_l^{(1)}}(y_{j+1}) - m_{j+1}\mathrm{H_l^{(1)}}(x_{j+1})\mathrm{H_l^{(1)'}}(y_{j+1})]\Big\}^{-1}$$
$$= b_{\mathrm{l,GLMT}}^{j+1}. \quad (33)$$

Equation (33) indicates $b_{\mathrm{l,Debye\ series}}^{j+1} = b_{\mathrm{l,GLMT}}^{j+1}$. Using the inductive approach, we can demonstrate $b_{\mathrm{l,Debye\ series}} = b_{\mathrm{l,GLMT}}$. With a similar process, the equality $a_{\mathrm{l,Debye\ series}} = a_{\mathrm{l,GLMT}}$ can also be proved. From Sections 2 and 3, it can be concluded that the mathematical formulas of scattering coefficients by Debye series are consistent with those of GLMT.

4. Rainbow phenomena for a two-layer cylinder by Debye series components

By comparison with GLMT, Debye series is convenient to simulate different-order rainbows of particles and explain rainbow phenomena. This section is devoted to discussing rainbow phenomena of a two-layer cylinder by Debye series components. The wavelength of the incident wave is chosen to be 0.6328 μm.

The primary rainbow can be simulated by the second-order Debye series component.[21] In theory, for the rainbow intensity of a homogeneous cylinder, there is only one extremum around the primary rainbow angle. But for the rainbow intensity of a two-layer cylinder, because of two possible one-internal reflective ray paths shown in Fig. 1, there are two extrema which are related to twin-primary rainbow.[21] For a two-layer cylinder, the ray transmission is similar to that for a coated sphere which has been discussed in Ref. [21]. The Debye series component for the first kind in Fig. 1 can be expressed as $T_{\mathrm{l}}^{32}T_{\mathrm{l}}^{21}T_{\mathrm{l}}^{12}R_{\mathrm{l}}^{232}T_{\mathrm{l}}^{21}T_{\mathrm{l}}^{12}T_{\mathrm{l}}^{23}$, while that for the other kind is given as $T_{\mathrm{l}}^{32}T_{\mathrm{l}}^{21}R_{\mathrm{l}}^{121}T_{\mathrm{l}}^{12}T_{\mathrm{l}}^{23}$. The variations of the Debye series component intensity with scattering angle for these two kinds are calculated and shown in Figs. 2–5.

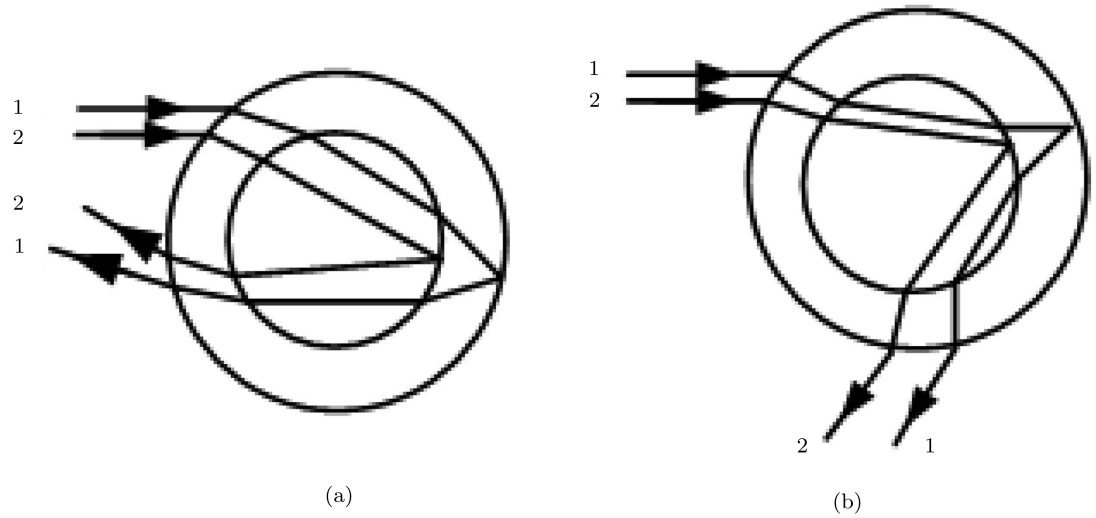


Fig. 1. Trajectories of two kinds of rays for twin-primary rainbow. (a) The refractive index of the internal layer is larger than that of the external layer. (b) The refractive index of the external layer is larger than that of the internal layer.

The angle of the rainbow is a function of the radius and the refractive index of a two-layer cylinder. With ray theory, an approximate formula of the two rainbow angles has been obtained in Ref. [21]. If θ^{R} is the scattering angle of the primary rainbow of a homogeneous sphere or cylinder with refractive index m_1, then the scattering angles θ_1^{R} and θ_2^{R} of the twin-primary rainbow are approximately expressed respectively as

$$\theta_1^{\mathrm{R}} \simeq \theta^{\mathrm{R}} + \frac{2(r_2 - r_1)}{r_1}\left[\left(\frac{4 - m_1^2}{3m_2^2 + m_1^2 - 4}\right)^{1/2} - \left(\frac{4 - m_1^2}{m_1^2 - 1}\right)^{1/2}\right], \tag{34}$$

$$\theta_2^{\mathrm{R}} \simeq \theta^{\mathrm{R}} + \frac{2(r_2 - r_1)}{r_1}\left[2\left(\frac{4 - m_1^2}{3m_2^2 + m_1^2 - 4}\right)^{1/2} - \left(\frac{4 - m_1^2}{m_1^2 - 1}\right)^{1/2}\right]. \tag{35}$$

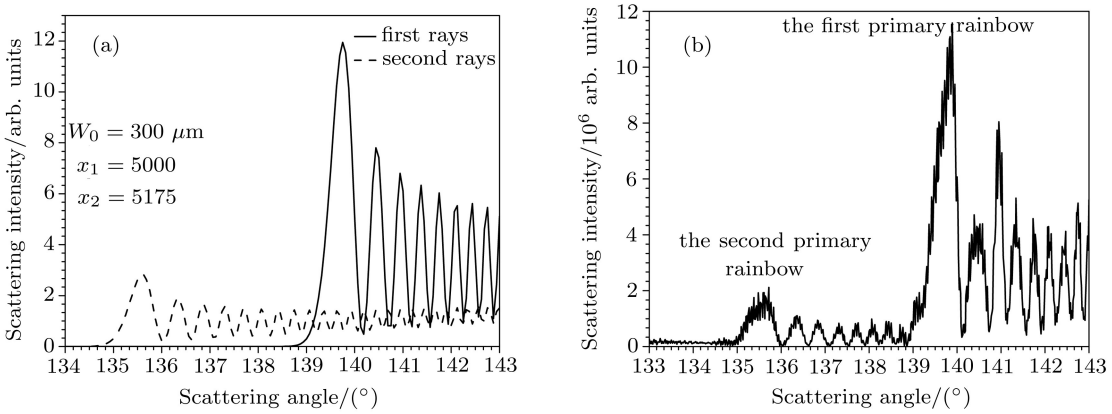


Fig. 2. Variations of scattering intensity of a two-layer cylinder with scattering angle. Panel (a) is for variations of the second order Debye series component with scattering angle, and panel (b) for variations of twin-primary rainbow intensity with scattering angle.

The Debye series component intensities and the total twin-primary rainbow intensities of the two-layer cylinder with size parameters $x_1 = 5000$ and $x_2 = 5175$ are shown in Fig. 2, respectively. The refractive indices of the cylinder are $m_1 = 1.333$ and $m_2 = 1.20$. The scattering angles corresponding to the extremal intensity are 135.625° and 139.75° in Fig. 2(a), while they are 135.48° and 139.72° in Fig. 2(b). Substituting $m_1 = 1.333$, $m_2 = 1.20$, $r_1 = 5000/k$, and $r_2 = 5175/k$ into Eq. (34) and Eq. (35), it can be obtained that the rainbow angle of the first ray occurs at $\theta_1^{\mathrm{R}} \simeq 135.5°$, and the rainbow angle of the second ray appears at $\theta_2^{\mathrm{R}} \simeq 139.5°$. The

difference in rainbow angle between Figs. 2(a) and 2(b), which is caused by the superimposition of other Debye series components, is very small. The neglecting of other kind of rays in formulas (34) and (35) leads to the difference in rainbow angle between the results obtained from the formula and Fig. 2(b).

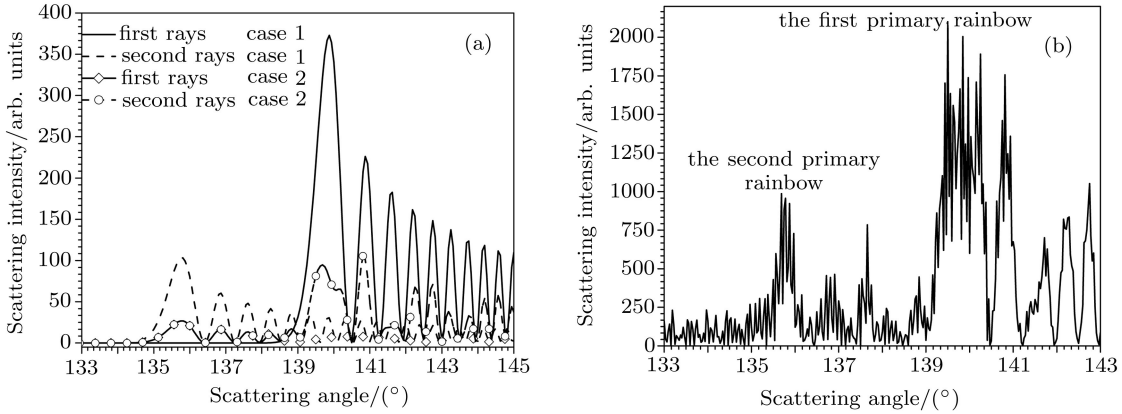


Fig. 3. Variations of scattering intensity of a two-layer cylinder. The parameters of the cylinder are $r_1 = 290$ μm, $r_2 = 300$ μm, $m_1 = 1.333$ and $m_2 = 1.20$. Panel (a) shows variations of the Debye series component with the waist radius of the incident laser beam, and panel (b) indicates variations of twin-primary rainbow intensity with scattering angle, with the waist radius of the incident laser beam $W_0 = 1000$ μm.

The effect of the waist radius of the incident laser beam on the twin-primary rainbow expressed by Debye series components is presented in Fig. 3(a). Varying with the waist radius, magnitudes of scattering intensities are modified, but values of rainbow angles are not shifted. In Fig. 3(b), the total scattering intensities for a laser beam as a function of scattering angle, which show the twin-primary rainbows corresponding to those in Fig. 3(a), are calculated. In the way similar to that in Fig. 2, the rainbow angles of the three conditions are 135.75° and 139.88°; 135.69° and 139.84°; 135.4° and 139.5°. An examination of the three results in Figs. 2 and 3 shows that the twin-primary rainbow of a two-layer cylinder can be simulated by the Debye series components.

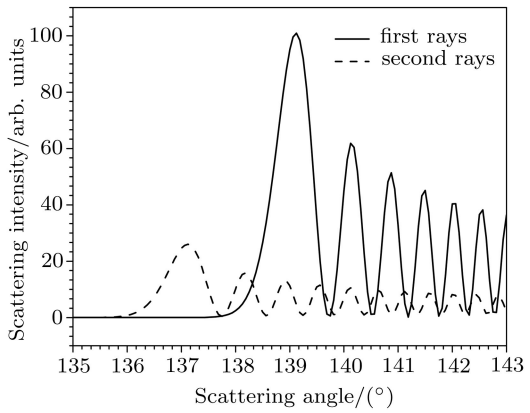


Fig. 4. Variations of the Debye series components with scattering angle. The parameters of the cylinder are $r_1 = 295$ μm, $r_2 = 300$ μm, $m_1 = 1.333$ and $m_2 = 1.20$.

Figure 4 shows the variations of the Debye series components with scattering angle. The radius of the inner layer is larger than that in Fig. 3, while other parameters are kept unchanged. The scattering angles corresponding to the two extremal intensities are $\theta_1^{\mathrm{R}} = 137.1°$, $\theta_2^{\mathrm{R}} = 139.1°$. By comparison with Fig. 3, it is indicated that $\theta_2^{\mathrm{R}} - \theta_1^{\mathrm{R}}$ diminishes with $r_2 - r_1$ decreasing. The radius of the two-layer cylinder has an effect on the scattering angle of the twin-primary rainbow.

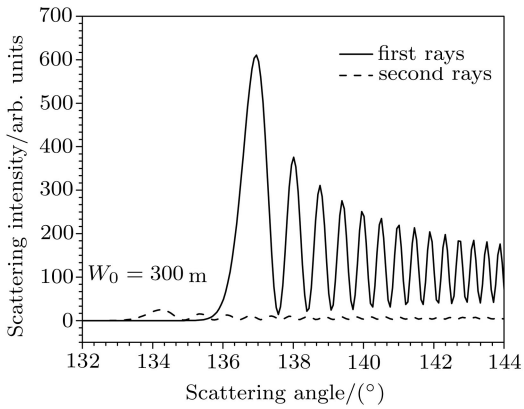


Fig. 5. Variations of the Debye series components with scattering angle. The parameters of the cylinder are $r_1 = 290$ μm, $r_2 = 300$ μm, $m_1 = 1.333$ and $m_2 = 1.55$.

The Debye series components of a cylinder with $r_1 = 290$ μm, $r_2 = 300$ μm, $m_1 = 1.333$, $m_2 = 1.55$ are shown in Fig. 5. The refractive index of the cylinder is different from that in Fig. 3 and corresponding to that

in Fig. 1(b), while the radius of the cylinder stays the same. A comparison of Fig. 5 with Fig. 3 leads to the conclusion that the refractive index also has influence on the scattering angle of the twin-primary rainbow.

After detailed analysis, we can have a conclusion that as the ratio of r_1/r_2 is less than 0.3, for the cylinder with refractive indices $m_1 = 1.333$ and $m_2 = 1.20$, the rainbow intensity of the first kind of rays will disappear. From Figs. 2–5, we conclude that the radius and refractive index of the two-layer cylinder play an important role in the twin-primary rainbow phenomena.

5. Conclusions

Based on the recursive relation and the inductive approach, the consistency between scattering coefficient formulas of Debye series and those of generalized Lorentz–Mie theory is proved. Using the Debye series components, rainbow phenomena in the scattering of a two-layer cylinder are discussed. It can be concluded that the radius and refractive index of the two-layer cylinder have an effect on twin-primary rainbow angle. In general, Debye series theory of light scattering by an inhomogeneous cylinder plays an important role in the rainbow technique and is beneficial to the size analysis of inhomogeneous cylinders.

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