General solution of the modified Korteweg-de-Vries equation in the lattice hydrodynamic model*

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Traffic congestion is related to various density waves, which might be described by the nonlinear wave equations, such as the Burgers, Korteweg-de-Vries (KdV) and modified Korteweg-de-Vries (mKdV) equations. In this paper, the mKdV equations of four different versions of lattice hydrodynamic models, which describe the kink–antikink soliton waves are derived by nonlinear analysis. Furthermore, the general solution is given, which is applied to solving a new model — the lattice hydrodynamic model with bidirectional pedestrian flow. The result shows that this general solution is consistent with that given by previous work.

Keywords: traffic flow, lattice hydrodynamic model, mKdV equation

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1. Introduction

Over the past decades, traffic dynamics has attracted considerable attention from physicists, engineers and mathematicians as a typical example of non-linear dynamics. In order to describe the traffic characteristics better, various traffic models have been proposed, such as car-following models, cellular automaton models and hydrodynamic models.[1−7] Recently, the research indicates that it is quite important to describe the non-linear phenomenon and characteristics.[8,9] The lattice hydrodynamic model, which was proposed by Nagatani[10,11] is convenient for analysing the density wave in traffic flow. In 1998, Nagatani[10] proposed a simple version of the hydrodynamic models, in which he incorporated the idea of microscopic optimal model. Model A in Ref. [10] is described as

$$\partial_t \rho + \rho_0 \partial_x \rho v = 0, \tag{1}$$

$$\partial_t \rho v = a\rho_0 V(\rho(x+\delta)) - a\rho v, \tag{2}$$

where ρ_0 is the average density; τ is introduced to denote the delay time with which the vehicle velocity reaches the optimal velocity as the traffic flow varies, and it is related to the sensitivity a as $\tau = 1/a$; δ represents the average headway, which means $\delta = 1/\rho_0$; $\rho(x+\delta)$ is the local density at position $x+\delta$ at time t and it is related to the inverse of headway $h(x,t)$ as $\rho(x+\delta) = 1/h(x,t)$. The idea is that a driver adjusts the car velocity according to the observed headway $h(x,t)$ [or density ahead $\rho(x+\delta)$]. This is similar to the idea in the car following models with the optimal velocity $V(h(x,t))$. Equation (2) is modified with dimensionless space x. Here, let $\hat{x} = x/\delta$, and $\hat{x}$ is indicated as x.

Model B in Ref. [10] is a lattice version of model A with dimensionless space x. It is described by

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{3}$$

$$\partial_t \rho_j v_j = a\rho_0 V(\rho_{j+1}) - a\rho_j v_j, \tag{4}$$

where j indicates the j-th site on a one-dimensional lattice, $\rho_j(t)$ and $v_j(t)$ represent the local density and local velocity on site j at time t respectively, V is called the optimal velocity function.

By eliminating speed v in Eqs. (1) and (2) (Eqs. (3) and (4)), one obtains the density equations of models A and B as

$$\partial_t^2 \rho + a\partial_t \rho + a\rho_0^2 \partial_x V(\rho(x+1)) = 0, \tag{5}$$

$$\partial_t^2 \rho_j + a\partial_t \rho_j + a\rho_0^2 (V(\rho_{j+1}) - V(\rho_j)) = 0. \tag{6}$$

We adopt the same optimal velocity function as that used by Bando *et al.*[12] except for density variable:

$$V(\rho) = (v_{\max}/2)[\tanh(2/\rho_0 - \rho/\rho_0^2 - 1/\rho_c)$$

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$$+ \tanh(1/\rho_c)], \tag{7}$$

where $\rho_c = 1/h_c$, h_c is the safety distance. For simplicity, we set $v_{max} = 2$.

In 1999, Nagatani proposed another lattice hydrodynamic model.[11] For convenience, we revisited two conventional lattice versions of the continuum model on a single-lane highway. The models are described by

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{8}$$

$$\rho_j(t+\tau) v_j(t+\tau) = \rho_0 V(\rho_{j+1}(t)), \tag{9}$$

$$\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{10}$$

$$\rho_j(t+\tau) v_j(t+\tau) = \rho_0 V(\rho_{j+1}(t)). \tag{11}$$

The two models are model C (Eqs. (8), (9)) and model D (Eqs. (10), (11)). Model D is the difference equation of model C. In the above equations, the notations are the same as those in models A and B.

By eliminating the velocity in Eqs. (8), (9) and also in Eqs. (10), (11), we obtain the density equations for models C and D:

$$\partial_t \rho_j(t+\tau) + \rho_0^2[V(\rho_{j+1}(t)) - V(\rho_j(t))] = 0, \tag{12}$$

$$\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0^2[V(\rho_{j+1}(t)) - V(\rho_j(t))] = 0. \tag{13}$$

In this paper, based on the original lattice hydrodynamic models A, B, C and D, we plan to make the conclusion as to the four types of models through nonlinear methods. And try to derive the general forms of the mKdV equations by using the reductive perturbation method. Finally, we want to see whether they are applied to solving a new model — the lattice hydrodynamic model with bidirectional pedestrian flow.

2. The mKdV equation

By applying the linear stability method to the traffic models A–D, the neutral stability conditions for models A–D have been given respectively. For details readers may read Refs. [10] and [11]. It has been found that if $a > a_c$ ($a_c = 1$ for model A, $a_c = 2$ for model B, C, $a_c = 3$ for model D), the uniform flow is always stable irrespective of density. And there is a critical point at $\rho = \rho_c$, $a = a_c$. The neutral stability lines and critical points are shown in Fig. 1.

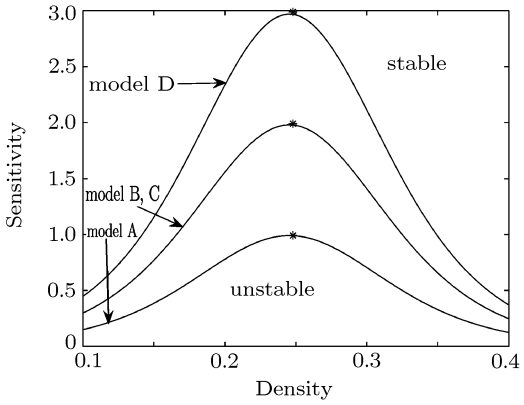


Fig. 1. Phase diagram in the density-sensitivity space. The solid lines represent the neutral stability curves for models A–D. The asterisks are the critical points. ($v_{max} = 2$, $\rho_c = 0.25$).

From the parabolic form of the neutral stability line near the critical point, it seems convenient to use slow scales for space x (or j) and time variable t.[10] For $0 < \varepsilon \ll 1$, we use the slow variables X and T defined as follows:[10]

$$X = \varepsilon(x + bt), \quad T = \varepsilon^3 t \tag{14}$$

for model A,

$$X = \varepsilon(j + bt), \quad T = \varepsilon^3 t \tag{15}$$

for models B, C, and D,

where b is a constant to be determined. Let

$$\rho(x,t) = \rho_c + \varepsilon R(X,T), \tag{16}$$

$$\rho_j(t) = \rho_c + \varepsilon R(X,T). \tag{17}$$

Inserting Eqs. (14)–(17) into Eqs. (5), (6), (12), and (13), respectively, and expanding to the fifth-order of ε, we obtain the expression (see Table 1):

Table 1. The coefficients f_i of the four models.

model	f_1	f_2	f_3	f_4	f_5	f_6	f_7
A	$b + \rho_c^2 V'$	$\frac{b^2}{a} + \rho_c^2 V'$	$\frac{\rho_c^2 V'}{2}$	$\frac{\rho_c^2 V'''}{6}$	$\frac{2b}{a}$	$\frac{\rho_c^2 V'}{6}$	$\frac{\rho_c^2 V'''}{6}$
B	$b + \rho_c^2 V'$	$\frac{b^2}{a} + \frac{\rho_c^2 V'}{2}$	$\frac{\rho_c^2 V'}{6}$	$\frac{\rho_c^2 V'''}{6}$	$\frac{2b}{a}$	$\frac{\rho_c^2 V'}{24}$	$\frac{\rho_c^2 V'''}{12}$
C	$b + \rho_c^2 V'$	$b^2\tau + \frac{\rho_c^2 V'}{2}$	$\frac{b^3\tau^2}{2} + \frac{\rho_c^2 V'}{6}$	$\frac{\rho_c^2 V'''}{6}$	$2b\tau$	$\frac{b^4\tau^3}{6} + \frac{\rho_c^2 V'}{24}$	$\frac{\rho_c^2 V'''}{12}$
D	$b + \rho_c^2 V'$	$\frac{3b^2\tau}{2} + \frac{\rho_c^2 V'}{2}$	$\frac{7b^3\tau^2}{6} + \frac{\rho_c^2 V'}{6}$	$\frac{\rho_c^2 V'''}{6}$	$3b\tau$	$\frac{5b^4\tau^3}{8} + \frac{\rho_c^2 V'}{24}$	$\frac{\rho_c^2 V'''}{12}$

$$\varepsilon^2 f_1 \partial_X R + \varepsilon^3 f_2 \partial_X^2 R + \varepsilon^4 (\partial_T R + f_3 \partial_X^3 R + f_4 \partial_X R^3) + \varepsilon^5 (f_5 \partial_X \partial_T R + f_6 \partial_X^4 R + f_7 \partial_X^2 R^3) = 0, \quad (18)$$

where $V' = \mathrm{d}V(\rho)/\mathrm{d}\rho$ at $\rho = \rho_\mathrm{c}$, $V''' = \mathrm{d}^3 V(\rho)/\mathrm{d}\rho^3$ at $\rho = \rho_\mathrm{c}$. Near the critical point $(\rho_\mathrm{c}, a_\mathrm{c})$, $\tau = (1+\varepsilon^2)\tau_\mathrm{c}$. Taking $b = -\rho_\mathrm{c}^2 V'$ and eliminating the second and third-order terms of ε from Eq. (18), we obtain the simple equation (see Table 2)

$$\varepsilon^4 [\partial_T R - g_1 \partial_X^3 R + g_2 \partial_X R^3] + \varepsilon^5 [g_3 \partial_X^2 R + g_4 \partial_X^4 R + g_5 \partial_X^2 R^3] = 0. \quad (19)$$

Table 2. The coefficients g_i of the four models.

model	g_1	g_2	g_3	g_4	g_5
A	$-\frac{\rho_\mathrm{c}^2 V'}{2}$	$\frac{\rho_\mathrm{c}^2 V'''}{6}$	a_c	$-\frac{5\rho_\mathrm{c}^2 V'}{6}$	$-\frac{\rho_\mathrm{c}^2 V'''}{6}$
B	$-\frac{\rho_\mathrm{c}^2 V'}{6}$	$\frac{\rho_\mathrm{c}^2 V'''}{6}$	$\frac{a_\mathrm{c}}{4}$	$-\frac{\rho_\mathrm{c}^2 V'}{8}$	$-\frac{\rho_\mathrm{c}^2 V'''}{12}$
C	$-\frac{\rho_\mathrm{c}^2 V'}{24}$	$\frac{\rho_\mathrm{c}^2 V'''}{6}$	$-\frac{\rho_\mathrm{c}^2 V'}{2}$	$-\frac{\rho_\mathrm{c}^2 V'}{48}$	$-\frac{\rho_\mathrm{c}^2 V'''}{12}$
D	$-\frac{\rho_\mathrm{c}^2 V'}{27}$	$\frac{\rho_\mathrm{c}^2 V'''}{6}$	$-\frac{\rho_\mathrm{c}^2 V'}{2}$	$-\frac{\rho_\mathrm{c}^2 V'}{54}$	$-\frac{\rho_\mathrm{c}^2 V'''}{12}$

Making the following transformations for Eq. (19):

$$T' = g_1 T, \quad R = \sqrt{\frac{g_1}{g_2}} R', \quad (20)$$

we obtain the standard mKdV equation with an $O(\varepsilon)$ correction term

$$\partial_{T'} R' - \partial_X^3 R' + \partial_X R'^3 + \varepsilon \frac{1}{g_1} \left[g_3 \partial_X^2 R' + g_4 \partial_X^4 R' + \frac{g_1 g_5}{g_2} \partial_X^2 R'^3 \right] = 0. \quad (21)$$

We ignore the $O(\varepsilon)$ term in Eq. (21), and obtain the mKdV equation with the kink–antikink soliton solution

$$R'_0(X, T') = \sqrt{c} \tanh \sqrt{\frac{c}{2}} (X - cT'). \quad (22)$$

Next, supposing that $R'(X, T') = R'_0(X, T') + \varepsilon R'_1(X, T')$, we take into account the $O(\varepsilon)$ correction. In order to determine the selected value of the propagation velocity c for the kink–antikink soliton solution (22), it is necessary to satisfy the solvability condition:

$$(R'_0, M[R'_0]) \equiv \int_\infty^\infty \mathrm{d}X R'_0 M[R'_0] = 0, \quad (23)$$

where

$$M[R'_0] = g_3 \partial_X^2 R' + g_4 \partial_X^4 R' + \frac{g_1 g_5}{g_2} \partial_X^2 R'^3$$

for Eq. (22). Performing the integration, we obtain the selected velocity

$$c = \frac{5 g_2 g_3}{2 g_2 g_4 - 3 g_1 g_5}. \quad (24)$$

The general kink–antikink soliton solutions of the headway ($-\rho_\mathrm{c}^2 V' = 1, V''' = 2$) are

$$\rho(x,t) = \rho_\mathrm{c} \pm \left(\frac{g_1 c}{g_2} \varepsilon^2 \right)^{1/2} \tanh \left\{ \left(\frac{c}{2} \varepsilon^2 \right)^{1/2} [x + (b - c g_1 \varepsilon^2) t] \right\}, \quad \text{for model A}, \quad (25)$$

$$\rho_j(t) = \rho_\mathrm{c} \pm \left(\frac{g_1 c}{g_2} \varepsilon^2 \right)^{1/2} \tanh \left\{ \left(\frac{c}{2} \varepsilon^2 \right)^{1/2} [j + (b - c g_1 \varepsilon^2) t] \right\}, \quad \text{for models B, C, and D}, \quad (26)$$

where $\varepsilon^2 = \tau/\tau_\mathrm{c} - 1$, $b = -\rho_\mathrm{c}^2 V' = 1$.

3. Lattice hydrodynamic model with bidirectional pedestrian flow

The lattice hydrodynamic model was first proposed by Nagatani.[10,11] Nagatani has extended the one-dimensional lattice hydrodynamic model to the two-dimensional lattice hydrodynamic model to investigate the two-way traffic like a two-dimensional traffic cellular automaton model proposed by Biham, Middelton and Levine (BML) model.[13] Meanwhile he has also derived the mKdV equation to describe traffic jam in the form of the kink–antikink density wave from the hydrodynamic model. In order to see whether there exists the kink–antikink density wave in pedestrian traffic. Tian *et al.*[14] extended the two-dimensional lattice hydrodynamic model to the two-dimensional bidirectional pedestrian flow in Fig. 2 (i.e. four-way traffic).

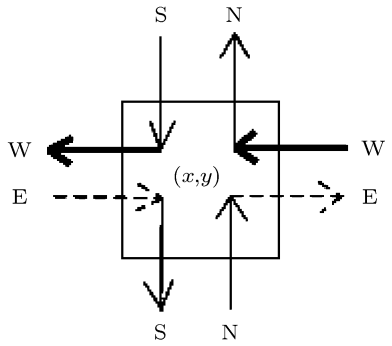


Fig. 2. The schematic diagram of the moving direction of the pedestrians, where E, W, N and S denote the eastbound, westbound, northbound and southbound pedestrians, respectively.

Four types of pedestrians are considered: one type of pedestrians (eastbound pedestrian) moves only to the positive x direction, one type of pedestrians (westbound pedestrian) moves only to the negative x direction, one type of pedestrians (northbound pedestrian) moves only to the positive y direction, and the other type of pedestrians (southbound pedestrian) moves only to the negative y direction. The fraction of eastbound and westbound pedestrians in all of the pedestrians is c, the fraction of eastbound pedestrian in eastbound and westbound pedestrians is c_1, and the fraction of northbound pedestrian in northbound and southbound pedestrians is c_2. The total average density is ρ_0. A two-dimensional lattice hydrodynamic equation with bidirectional pedestrian flow is obtained as

$$\begin{aligned}\rho(j,m,t+2\tau) \; - \; & \rho(j,m,t+\tau)+\tau(c_1c)^2\rho_0^2[V(\rho(j+1,m,t))-V(\rho(j,m,t))]\\ & + \tau[(1-c_1)c]^2\rho_0^2[V(\rho(j-1,m,t))-V(\rho(j,m,t))]\\ & + \tau[c_2(1-c)]^2\rho_0^2[V(\rho(j,m+1,t))-V(\rho(j,m,t))]\\ & + \tau[(1-c_2)(1-c)]^2\rho_0^2[V(\rho(j,m-1,t))-V(\rho(j,m,t))]=0. \end{aligned} \tag{27}$$

In the limit of $c=0, c_1=1$ or $c=1, c_2=1$, equation (27) reduces to the density equation of the one-dimensional lattice hydrodynamic equation.[11] In the limit of $c_1=1$ and $c_2=1$, equation (27) becomes the density equation of the two-dimensional lattice hydrodynamic equation with unidirectional flow.[13]

Then we use the reductive perturbation method to analyse the model. First, we derive the mKdV equation for the model. Inserting $X=\varepsilon(j+m+bt)$, $T=\varepsilon^3 t$, $\rho(j,m,t)=\rho_{\rm c}+\varepsilon R(X,T)$ into Eq. (27), expanding it to the fifth order of ε, we have Eq. (18) with the coefficients listed in Table 3.

Table 3. The c1oefficients f_i of the lattice hydrodynamic model with bidirectional pedestrian flow.

f_1	f_2	f_3	f_4	f_5	f_6	f_7
$b+g\rho_{\rm c}^2V'$	$\frac{3b^2\tau}{2}+\frac{f\rho_{\rm c}^2V'}{2}$	$\frac{7b^3\tau^2}{6}+\frac{g\rho_{\rm c}^2V'}{6}$	$\frac{\rho_{\rm c}^2V'''}{6}$	$3b\tau$	$\frac{5b^4\tau^3}{8}+\frac{f\rho_{\rm c}^2V'}{24}$	$\frac{f\rho_{\rm c}^2V'''}{12}$

In Table 3 $V'=[{\rm d}V(\rho)/{\rm d}\rho]|_{\rho=\rho_{\rm c}}$, $V'''=[{\rm d}^3V(\rho)/{\rm d}^3\rho]|_{\rho=\rho_{\rm c}}$,

$$g=(c_1c)^2-[(1-c_1)c]^2+[c_2(1-c)]^2-[(1-c_2)(1-c)]^2,$$
$$f=(c_1c)^2+[(1-c_1)c]^2+[c_2(1-c)]^2+[(1-c_2)(1-c)]^2.$$

Around the critical point $(\rho_{\rm c},\tau_{\rm c})$, $\tau=(1+\varepsilon^2)\tau_{\rm c}$, taking $b=-g\rho_{\rm c}^2V'$, we can eliminate the second- and the third-order terms of ε from Eq. (18). Then we obtain the simplified Eq. (19) with the coefficients presented in Table 4.

Table 4. The coefficients g_i of the lattice hydrodynamic model with bidirectional pedestrian flow.

g_1	g_2	g_3	g_4	g_5
$-\frac{\rho_{\rm c}^2V'(-7f^2+9g^2)}{54g}$	$\frac{g\rho_{\rm c}^2V'''}{6}$	$-\frac{g\rho_{\rm c}^2V'}{2}$	$\frac{f(-\rho_{\rm c}^2V')(-23f^2+27g^2)}{216g^2}$	$-\frac{f\rho_{\rm c}^2V'''}{12}$

Analogously, substituting the above coefficients of Eq. (27) into Eq. (24), we obtain the density wave velocity c of the mKdV equation for the same model

$$c=\frac{135g^2}{27g^2-22f^2}. \tag{28}$$

Similar to Eq. (25), the kink–antikink soliton solutions of the headway can be easily obtained as follows:

$$\rho(j,m,t) = \rho_{\rm c} \pm \left(\frac{g_1 c}{g_2}\varepsilon^2\right)^{1/2} \tanh\left\{\left(\frac{c}{2}\varepsilon^2\right)^{1/2} \times[j+m+(b-cg_1\varepsilon^2)t]\right\}, \quad (29)$$

where $b = -g\rho_{\rm c}^2 V'$, $\varepsilon^2 = \tau/\tau_{\rm c} - 1 = 3gb\tau/f - 1 = 3g^2\tau/f - 1$.

4. Summary

We have derived the mKdV equations of the four versions of one-dimensional lattice hydrodynamic models around the critical points. The traffic characteristics have been analysed with the nonlinear analysis and the general forms of the mKdV equations are presented. Meanwhile, the general solutions are given too. Then, we apply them to derive the mKdV equation of the two-dimensional lattice hydrodynamic model with bidirectional pedestrian flow, which indicates that the general solutions could simplify the complex calculating process. To our knowledge, until now, the general solution of the mKdV equation in the lattice hydrodynamic model has not been investigated.

References

[1] Chowdhury D, Santen L and Schadschneider A 2000 *Phys. Rep.* **329** 199

[2] Helbing D 2001 *Rev. Mod. Phys.* **73** 1067

[3] Nagatani T 2002 *Rep. Prog. Phys.* **65** 1331

[4] Gao Z Y, Jia B, Li X G and Zhao X M 2008 *Acta Phys. Sin.* **57** 4777 (in Chinese)

[5] Dai S Q, Dong L Y and Li Q D 2009 *Acta Phys. Sin.* **58** 7584 (in Chinese)

[6] Helbing D, Hennecke A, Shvetsov V and Treiber M 2001 *Trans. Res.* B **35** 183

[7] Huang H J, Wong S C, Jiang R and Tang T Q 2009 *Acta Phys. Sin.* **58** 975 (in Chinese)

[8] Komatsu T and Sasa S 1995 *Phys. Rev.* E **52** 5574

[9] Kurtze D A and Hong D C 1995 *Phys. Rev.* E **52** 218

[10] Nagatani T 1998 *Physica* A **261** 599

[11] Nagatani T 1999 *Physica* A **264** 581

[12] Bando M, Hasebe K, Nakayama A, Shibata A and Sugiyama Y 1995 *Phys. Rev.* E **51** 1035

[13] Nagatani T 1999 *Phys. Rev.* E **59** 4857

[14] Tian H H, He H D, Wei Y F, Yu X and Lu W Z 2009 *Physica* A **388** 2895