Function projective synchronization between fractional-order chaotic systems and integer-order chaotic systems

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This paper investigates the function projective synchronization between fractional-order chaotic systems and integer-order chaotic systems using the stability theory of fractional-order systems. The function projective synchronization between three-dimensional (3D) integer-order Lorenz chaotic system and 3D fractional-order Chen chaotic system are presented to demonstrate the effectiveness of the proposed scheme.

Keywords: fractional-order chaotic systems, chaotic systems of integer orders, function projective synchronization, stability theory of fractional-order systems

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1. Introduction

Chaotic phenomena can be found in many scientific and engineering fields such as biological systems, electronic circuits, power converters, chemical systems, and so on. Since the pioneering work of Pecora and Carroll[1] who proposed a successful method to synchronize two identical chaotic systems, chaos synchronization has gained a lot of attention in a variety of research fields over the last two decades.[1−6] Because chaos synchronization can be applied in vast areas of physics and engineering science, and in particular in secure communication.[7,8] A variety of approaches[9−12] have been proposed for the synchronization of chaotic systems which include complete synchronization, anti-synchronization, generalized synchronization, lag synchronization and projective synchronization (PS), etc. Amongst all kinds of chaos synchronization, PS has been extensively investigated in recent years because it can obtain faster communication with its proportional feature. In application to secure communications, the proportional feature can be used to extend binary digital to M-nary digital communication[13] for getting faster communication.

The PS is characterized in that the drive and response system could be synchronized up to a scaling factor. Recently, Chen *et al.*[14−16] put forward a new PS, which is called function projective synchronization (FPS). The FPS means that the master and slave systems could be synchronized up to a scaling function, but not a constant. The FPS is the more general definition of PS. It is obvious that the unpredictability of the scaling function in FPS can additionally enhance the security of communication.

However, many existing PS methods and FPS methods mainly focus on integer-order chaotic systems or fractional-order chaotic systems. To the best of our knowledge, there is little information about the FPS between fractional-order chaotic systems and integer-order chaotic systems, available from the literature. Motivated by the above discussion, in this paper, the FPS between fractional-order chaotic systems and integer-order chaotic systems are investigated based on the stability theory of fractional-order systems. The FPS between the three-dimensional (3D) integer-order Lorenz chaotic system and the 3D fractional-order Chen chaotic system is used to demonstrate the effectiveness of the proposed scheme.

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2. Function projective synchronization scheme

Now, consider the n-dimensional integer-order chaotic system, i.e., the drive system is

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ are the state vectors, $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))^{\mathrm{T}}$ are continuous nonlinear vector functions.

Given the following n-dimensional fractional-order chaotic system

$$\mathrm{d}^{q_r}\boldsymbol{y}/\mathrm{d}t^{q_r} = \boldsymbol{g}(\boldsymbol{y}), \tag{2}$$

where fractional order q_r is $0 < q_r < 1$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ are the state vectors, $\boldsymbol{g}(\boldsymbol{y}) = (g_1(\boldsymbol{y}), g_2(\boldsymbol{y}), \ldots, g_n(\boldsymbol{y}))^{\mathrm{T}}$ are continuous nonlinear vector functions. When a controller $\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x}) \in R^n$ is added to the original system (2), we yield the response system

$$\mathrm{d}^{q_r}\boldsymbol{y}/\mathrm{d}t^{q_r} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x}). \tag{3}$$

The controller $\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x})$ will be designed.

For the drive system (1) and the response system (3), there is FPS if there exists a controller $\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x})$ such that

$$\lim_{t\to+\infty} e_i = \lim_{t\to+\infty} y_i - \kappa_i(\boldsymbol{x})x_i = 0 \quad (i = 1, 2, \ldots, n), \tag{4}$$

where $\kappa_i(\boldsymbol{x})$ $(i = 1, 2, \ldots, n)$ are continuous functions, and $e_i = y_i - \kappa_i(\boldsymbol{x})x_i$ $(i = 1, 2, \ldots, n)$ is called the FPS error between systems (1) and (3).

Based on the idea of tracking control, in order to achieve the goal of Eq. (4), we can let $\kappa_i(\boldsymbol{x})x_i$ be a reference signal. The FPS between integer-order drive system (1) and fractional-order response chaotic system (3) belongs to the problem of tracking control, i.e. the output signal $\boldsymbol{y}$ in system (3) follows the reference signal $\boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x}$ ultimately. Here $\boldsymbol{\kappa}(\boldsymbol{x}) = \mathrm{diag}(\kappa_1(\boldsymbol{x}), \kappa_2(\boldsymbol{x}), \ldots, \kappa_n(\boldsymbol{x}))$.

Now, let the controller $\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x})$ be

$$\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{\theta}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}), \tag{5}$$

where $\boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}) \in R^n$ is a vector control function, and will be designed. The $\boldsymbol{\theta}(\boldsymbol{x}) \in R^n$ is a compensation controller, and $\boldsymbol{\theta}(\boldsymbol{x}) = \mathrm{d}^{q_r}(\boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x})/\mathrm{d}t^{q_r} - \boldsymbol{g}(\boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x})$.

According to Eq. (5), the response system (3) can be rewritten as

$$\mathrm{d}^{q_r}\boldsymbol{e}/\mathrm{d}t^{q_r} = \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}). \tag{6}$$

It indicates that the FPS between drive system (1) and response system (3) is transformed into the following problem: choose a control law $\boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x})$ or a vector control function $\boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x})$ such that system (6) asymptotically converges to zero.

Let $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x}$ be the FPS error between drive system (1) and the response system (3). So, equation (6) can be rewritten as

$$\mathrm{d}^{q_r}\boldsymbol{e}/\mathrm{d}t^{q_r} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}), \tag{7}$$

where $\boldsymbol{A}\boldsymbol{e} = \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{\kappa}(\boldsymbol{x})\boldsymbol{x})$, and $\boldsymbol{A} \in R^{n\times n}$.

Now, we can choose control vector $\boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{B}\boldsymbol{e}$, and $\boldsymbol{B} \in R^{n\times n}$. So, equation (7) can be rewritten as

$$\mathrm{d}^{q_r}\boldsymbol{e}/\mathrm{d}t^{q_r} = (\boldsymbol{A} + \boldsymbol{B})\boldsymbol{e}. \tag{8}$$

Therefore, we can yield the following result. If $(\boldsymbol{A} + \boldsymbol{B})$ is a stable matrix, then the FPS between response system (3) and drive system (1) can be achieved. Because $(\boldsymbol{A} + \boldsymbol{B})$ is a stable matrix, so all eigenvalues of matrix $\boldsymbol{A} + \boldsymbol{B}$ satisfy $\mathrm{Re}(\lambda) < 0$. It implies $|\arg\lambda| > \pi/2 > q_r\pi/2$. According to the stability theory of fractional-order systems,[17] the equilibrium point $\boldsymbol{e} = \boldsymbol{0}$ in Eq. (8) is asymptotically stable. So

$$\lim_{t\to+\infty} e_i = \lim_{t\to+\infty} y_i - \kappa_i(\boldsymbol{x})x_i = 0 \quad (i = 1, 2, \ldots, n). \tag{9}$$

Equation (9) implies that the FPS between response system (3) and drive system (1) can be achieved.

3. Applications

To illustrate the effectiveness of the proposed FPS scheme, the FPS between the 3D integer-order Lorenz chaotic system and 3D fractional-order Chen chaotic system is considered and the numerical simulations are performed.

The famous Lorenz system,[18] the first chaotic attractor model in a 3D integer-order autonomous system, is given by

$$\begin{cases} \mathrm{d}x_1/\mathrm{d}t = 10(x_2 - x_1), \\ \mathrm{d}x_2/\mathrm{d}t = 28x_1 - x_1x_3 - x_2, \\ \mathrm{d}x_3/\mathrm{d}t = x_1x_2 - 8x_3/3. \end{cases} \tag{10}$$

The chaotic attractor is shown in Fig. 1.

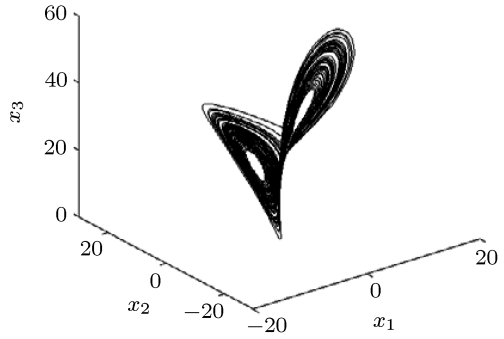


Fig. 1. Chaotic attractors of the Lorenz system (10).

The Chen system,[19] introduced by Chen and Ueta in 1999, is similar but not topologically equivalent to the Lorenz system and is described by

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = a(y_2 - y_1), \\ \mathrm{d}y_2/\mathrm{d}t = (c-a)y_1 - y_1y_3 + cy_2, \\ \mathrm{d}y_3/\mathrm{d}t = y_1y_2 - by_3. \end{cases}$$

Chen system exhibits chaotic behaviour at the parameters (a, b, c) = (35, 3, 28). Its corresponding fractional-order Chen system is described as follows:

$$\begin{cases} \mathrm{d}^{q_r}y_1/\mathrm{d}t^{q_r} = a(y_2 - y_1), \\ \mathrm{d}^{q_r}y_2/\mathrm{d}t^{q_r} = (c-a)y_1 - y_1y_3 + cy_2, \\ \mathrm{d}^{q_r}y_3/\mathrm{d}t^{q_r} = y_1y_2 - by_3. \end{cases}$$

Mohammad S T and Mohammad H[20] pointed out that the fractional-order Chen system exhibits chaotic behaviour for fractional-order $q_r \geq 0.83$. When $q_r = 0.95$, the chaotic attractor is shown in Fig. 2.

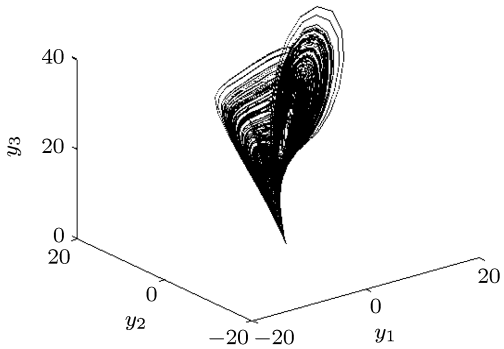


Fig. 2. Chaotic attractors of the fractional-order Chen system.

Consider the 3D integer-order Lorenz chaotic system (10) as drive system, and the 3D fractional-order Chen chaotic system with $q_r = 0.95$ as response system. The response system is

$$\begin{pmatrix} \mathrm{d}^{q_r}y_1/\mathrm{d}t^{q_r} \\ \mathrm{d}^{q_r}y_2/\mathrm{d}t^{q_r} \\ \mathrm{d}^{q_r}y_3/\mathrm{d}t^{q_r} \end{pmatrix} = \begin{pmatrix} a(y_2 - y_1) \\ (c-a)y_1 - y_1y_3 + cy_2 \\ y_1y_2 - by_3 \end{pmatrix} + \boldsymbol{\phi}(\boldsymbol{y}, \boldsymbol{x}). \tag{11}$$

According to the above-mentioned analyses, we can yield

$$\boldsymbol{A} = \begin{pmatrix} -a & a & 0 \\ c - a - y_3 & c & -\kappa_1(\boldsymbol{x})x_1 \\ y_2 & \kappa_1(\boldsymbol{x})x_1 & -b \end{pmatrix}.$$

Now, choose

$$\boldsymbol{B} = \begin{pmatrix} 0 & -c + y_3 & -y_2 \\ 0 & -d - c & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (d > 0),$$

and a positive-definite matrix $\boldsymbol{P} = \mathrm{diag}(1, 1, 1)$, and a positive-definite matrix $\boldsymbol{Q} = \mathrm{diag}(2a, 2d, 2b)$. We will have $\boldsymbol{P}(\boldsymbol{A} + \boldsymbol{B}) + (\boldsymbol{A} + \boldsymbol{B})^{\boldsymbol{H}}\boldsymbol{P} = \mathrm{diag}(-2a, -2d, -2b) = -\boldsymbol{Q}$. So, $(\boldsymbol{A} + \boldsymbol{B})$ is a stable matrix. According to the above-mentioned discussion, the FPS between response system (11) and drive system (10) can be achieved.

For example, choose $d = 5$ and

$$\begin{aligned} \boldsymbol{\kappa}(\boldsymbol{x}) &= \mathrm{diag}(\kappa_1(\boldsymbol{x}), \kappa_2(\boldsymbol{x}), \kappa_3(\boldsymbol{x})) \\ &= \mathrm{diag}(x_3 + x_1, 1 + 0.1x_1x_2, 1 + x_2). \end{aligned}$$

The corresponding numerical result is shown in Fig. 3, in which the initial conditions are $\boldsymbol{x} = (0, 1, 2)^{\mathrm{T}}$ for drive system (10), and $\boldsymbol{y} = (1, 3, 5)^{\mathrm{T}}$ for response system (11), respectively. Where $\varepsilon = \left(\sum_{i=1}^{3}(y_i - \kappa_i(\boldsymbol{x})x_i)^2\right)^{1/2}$.

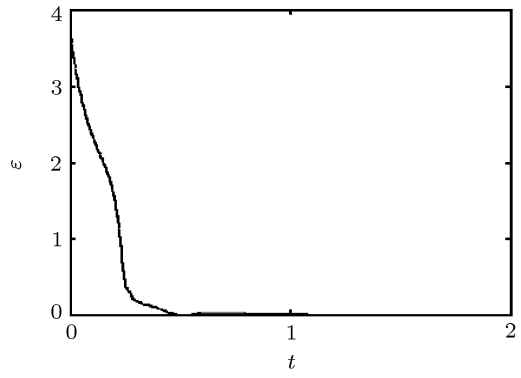


Fig. 3. The FPS result between 3D integer-order Lorenz chaotic system (10) and 3D fractional-order Chen chaotic system (11).

4. Conclusions

In summary, we have investigated the FPS between fractional-order chaotic systems and integer order chaotic systems. Based on the stability criterion of the fractional-order system and tracking control, a synchronization approach is proposed. This proposed scheme is simple and theoretically rigorous. Finally, the FPS between the 3D Lorenz chaotic system of integer order and the 3D fractional-order Chen chaotic system are used to demonstrate the effectiveness of the proposed scheme.

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