The stability control of fractional order unified chaotic system with sliding mode control theory*

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This paper studies the stability of the fractional order unified chaotic system with sliding mode control theory. The sliding manifold is constructed by the definition of fractional order derivative and integral for the fractional order unified chaotic system. By the existing proof of sliding manifold, the sliding mode controller is designed. To improve the convergence rate, the equivalent controller includes two parts: the continuous part and switching part. With Gronwall's inequality and the boundness of chaotic attractor, the finite stabilization of the fractional order unified chaotic system is proved, and the controlling parameters can be obtained. Simulation results are made to verify the effectiveness of this method.

Keywords: fractional order unified chaotic system, sliding mode control, stability analysis

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1. Introduction

The fractional order derivative and integral are the fundamental theories of fractional order dynamical systems. It has been found that the behaviour of many physical systems can be properly described by fractional calculus, such as electromagnetic waves, heat conduction, visco-elastic systems, and so on.[1−3]

For integer order systems, the stability has been studied fully by researchers. However, these theories cannot be used to the fractional order systems directly. Although the problem of stability is a very essential and crucial issue for fractional order systems, due to the complexity of the relations, it has been discussed and investigated only in some recent literature.[4−6] In Ref. [7], chaotic behaviour of fractional order Chen system was discussed, and it was found that chaos exists in the fractional order Chen system with order less than 3. Linear feedback control of chaos in this system had also been studied.[7] With passive equivalence theory, the fractional order unified chaotic system could be stabilized.[8] Ahmad *et al.* studied the fractional order "jerk" model, and stabilized the system by backstepping method.[9] However, those methods substitute the integer order system controller into the counterpart fractional order system directly, and the stability area is decreased. For fractional order Chua chaotic system which has different system parameters and system order, the synchronization control was also designed.[10] This method is based on the nonlinear state feedback. Using nonlinear controller, the nonlinear terms of the nonlinear systems are removed, and nonlinear systems are transformed into linear systems. However, the controller is complicated. A sliding mode control design for fractional order linear systems with input and state time-delay is proposed.[11] But they do not give the control method for fractional order nonlinear systems. By extending this method, we study the stability of fractional order unified chaotic system with sliding mode control theory in this paper.

2. Fractional order unified chaotic system

In 1963, Lorenz found a chaotic attractor in a simple three-dimensional autonomous system. Then Chen and Ueta found a similar but non-equivalent chaotic attractor, which is known to be the dual of the Lorenz system. Recently, Lü system was reported which bridges the gap between the Lorenz system and Chen system. In a recent work of Lü *et al.*, they introduced a unified chaotic system that contains the Lorenz and the Chen systems as two dual systems at the two extremes of its parameter spectrum. The new

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system represents the continued transition from the Lorenz to Chen system and is chaotic over the entire spectrum of the key system parameter.[12] Fractional order unified chaotic system also represents the fractional order Lorenz chaotic system family.[13] Via fractional order unified chaotic system, the control of different chaotic attractor becomes easy among Lorenz system families.

The fractional order unified chaotic system is described as follows:[13]

$$\begin{cases} D_a^\alpha x_1 = (25\theta + 10)(x_2 - x_1), \\ D_a^\alpha x_2 = (28 - 35\theta)x_1 - x_1 x_3 + (29\theta - 1)x_2, \\ D_a^\alpha x_3 = x_1 x_2 - \frac{8+\theta}{3} x_3, \end{cases} \quad (1)$$

where x is state variable, $\theta \in [\,0\ 1\,]$, $0 < \alpha < 1$. Obviously, the system is Lorenz system when $\theta = 0$, and it is Chen system when $\theta = 1$. It is a continuous chaotic system when θ varies from 0 to 1.

The equilibria of the system are (0, 0, 0), $(\sqrt{72 - 7\theta - 2\theta^2}, \sqrt{72 - 7\theta - 2\theta^2}, 27 - 6\theta)$ and $(-\sqrt{72 - 7\theta - 2\theta^2}, -\sqrt{72 - 7\theta - 2\theta^2}, 27 - 6\theta)$. The equilibria are different when θ varies from 0 to 1. There is an unstable equilibrium.

System (1) can be described as follows:

$$D^\alpha x = Ax + f(x) + Bu, \quad (2)$$

where

$$A = \begin{bmatrix} -(25\theta + 10) & 25\theta + 10 & 0 \\ 28 - 35\theta & 29\theta - 1 & 0 \\ 0 & 0 & -\dfrac{8+\theta}{3} \end{bmatrix}$$

is a linear matrix, $f(x) = \begin{bmatrix} 0 \\ x_1 x_3 \\ x_1 x_2 \end{bmatrix}$ is a nonlinear term, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the control matrix, u is the controller.

3. Sliding mode control

3.1. Sliding manifold

Sliding mode control, also named variable structure control, is a kind of control method for controlled systems with strong robust property.[14] This method forces the trajectory to reach the sliding manifold in a limit time, and moves along the manifold. It has been used in a lot of systems.

Definition 1 System given by system (1) is finitely stable with respect to $\{0,\ J,\ \delta,\ \varepsilon\}$, if and only if

$$\|\varphi\| < \delta$$

implies

$$\|x(t)\| < \varepsilon, \quad \forall t \in J,$$

where δ is a positive real number and $\varepsilon \in R, \delta < \varepsilon$.

According to the definition of fractional order derivative and integral, sliding manifold of the system can be found. In this paper, we use the definition of fractional order integral and derivative by Riemann–Liouviller.

Definition 2[1,2] Let $f : [a, b] \to R$, and $f \in L^1[a, b]$. The left-sided Riemann–Liouville fractional integral of order α is defined as follows:

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) \mathrm{d}\tau, \quad (3)$$

where $\alpha > 0$, $a < t < b$, n is an integer satisfying $n - 1 < \alpha < n$, $\Gamma(\cdot)$ is Gamma function, and $\Gamma(\alpha) = \int_0^\infty v^{\alpha - 1} \mathrm{e}^{-v} \mathrm{d}v$.

Definition 3[1,2] The left-sided Riemann–Liouville fractional derivative of function $f : [a, b] \to R$ is defined as

$$D_a^\alpha f(t) = D^m I_a^{m-\alpha} f(t), \quad (4)$$

where $m = [\alpha] + 1$, $D^m = \mathrm{d}^m / \mathrm{d}t^m$, $\alpha > 0, a < t < b$.

Suppose the initial value of the system is zero, and the Laplace transform of system (1) is as follows:

$$L\left\{D_a^\alpha x(t)\right\} = s^\alpha L\left\{x(t)\right\}. \quad (5)$$

Theorem 1 For fractional order unified chaotic system (2), if the controller is as follows:

$$\begin{aligned} u = & -\frac{1}{k_2}\rho s - \frac{1}{k_2}\mu \operatorname{sgn}(s) \\ & + \left(\frac{k_1(25\theta + 10)}{k_2} - 28 + 35\theta\right) x_1 \\ & - \left(\frac{k_1(25\theta + 10)}{k_2} + 29\theta - 1\right) x_2 \\ & + \frac{(8+\theta)k_3}{3k_2} x_3 - x_1 x_3 - \frac{k_3}{k_2} x_1 x_2, \end{aligned} \quad (6)$$

and there is a proper control parameter $K = [\,k_1\ k_2\ k_3\,]$, then the following sliding manifold exists

$$s(t) = K \cdot I^\alpha x = k_1 I^\alpha x_1 + k_2 I^\alpha x_2 + k_3 I^\alpha x_3 = 0. \quad (7)$$

Proof For system (1), we define controller as

$$u = u_{\mathrm{eq}} + u_{\mathrm{d}}. \quad (8)$$

From Eq. (7), we have

$$\begin{aligned} \dot{s}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[KI^\alpha x(t)] = K \cdot D^\alpha x(t) \\ &= KAx + Kf(x) + KBu = 0, \end{aligned} \quad (9)$$

then

$$u_{\rm eq} = -(KB)^{-1}KAx - (KB)^{-1}Kf(x). \quad (10)$$

In order to decrease chattering of the system which is a fixed disadvantage of sliding mode control theory, controller includes two parts: continuous part and switching part. Choose

$$u_{\rm d} = -(KB)^{-1}\rho s - (KB)^{-1}\mu \cdot {\rm sgn}(s), \quad (11)$$

where constant $\rho > 0, \mu > 0$, and ${\rm sgn}(\cdot)$ is a signal function. From Eqs. (10) and (11), we have

$$\begin{aligned} u = & -(KB)^{-1}\rho s - (KB)^{-1}\mu \cdot {\rm sgn}(s) \\ & - (KB)^{-1}KAx - (KB)^{-1}Kf(x). \end{aligned} \quad (12)$$

So we can obtain the controller u which is described as Eq. (6). Substituting controller u to $\dot{s}$, we have

$$\dot{s} = -\rho s - \mu\,{\rm sgn}(s). \quad (13)$$

Then $s\dot{s} = -\rho s^2 - \mu s \cdot {\rm sgn}(s) \leq 0$. The sliding manifold s exists.

3.2. Controller design

According to the controller (12), the closed-loop system can be described as

$$D^\alpha x = A_1 x - A_2\rho s - A_2\mu\,{\rm sgn}(s) + B_1 f(x), \quad (14)$$

where $A_1 = A - B(KB)^{-1}KA$, $A_2 = B(KB)^{-1}$, $B_1 = I - B(KB)^{-1}K$, I is the identity matrix and its rank is 3.

When system trajectories reach sliding manifold, they will move on the sliding manifold, and the closed-loop system is

$$D^\alpha x = A_1 x + B_1 f(x). \quad (15)$$

For closed-loop controlled system, the system matrix is as follows:

$$\begin{aligned} A_1 = A - B(KB)^{-1}KA &= \begin{bmatrix} -(25\theta+10) & 25\theta+10 & 0 \\ \dfrac{k_1}{k_2}(25\theta+10) & -\dfrac{k_1}{k_2}(25\theta+10) & -\dfrac{k_3}{k_2}\dfrac{\theta+8}{3} \\ 0 & 0 & -\dfrac{\theta+8}{3} \end{bmatrix}; \\ B_1 = I - B(KB)^{-1}K &= \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{k_1}{k_2} & 0 & -\dfrac{k_3}{k_2} \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (16)$$

Theorem 2 Fractional order closed-loop system $D^\alpha x = A_1 x + B_1 f(x)$ satisfying initial condition is finitely stable, if the following condition is satisfied

$$\delta\frac{t^\alpha}{\Gamma(\alpha+1)}\mu_{\max}^{B_1} \cdot \exp\left(\frac{t^\alpha \mu_{\max}^{A_1}}{\Gamma(\alpha+1)}\right) \leq \varepsilon, \quad (17)$$

where $\mu_{\max}(\cdot)$ is the largest singular value of matrix $(\cdot)$. It implies $\|x(t)\| \leq \varepsilon$. At this time, the closed-loop system on sliding manifold is stable.

Proof With the definition of fractional order integral, we have

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t \frac{A_1}{(t-\tau)^{1-\alpha}}x(\tau){\rm d}\tau \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{B_1}{(t-\tau)^{1-\alpha}}f(x(\tau)){\rm d}\tau. \end{aligned} \quad (18)$$

Because chaotic attractor is bounded, we have $\|f(x)\| \leq M$, where M is a positive constant. Applying Gronwall's inequality[15] and condition (16), it is easy to obtain

$$\begin{aligned} \|x(t)\| &\leq \frac{\mu_{\max}^{A_1}}{\Gamma(\alpha)}\int_0^t \frac{\|x(\tau)\|}{(t-\tau)^{1-\alpha}}{\rm d}\tau + \frac{M\cdot t^\alpha}{\Gamma(\alpha+1)}\|B_1\| \\ &\leq \delta\frac{t^\alpha}{\Gamma(\alpha+1)}\mu_{\max}^{B_1} \cdot \exp\left(\frac{t^\alpha \mu_{\max}^{A_1}}{\Gamma(\alpha+1)}\right) \\ &\leq \varepsilon, \end{aligned}$$

where $\delta > 0$ and $\delta < \varepsilon$.

Theorem 3 Assume that there exists a finite constant $\eta > 0$ such that $\|s(t)\| \leq \eta\|x(t)\|$, then if the following condition is satisfied:

$$\delta\frac{\mu_{\max}^{B_{\rm eq}}}{\Gamma(\alpha+1)}t^\alpha \exp\left(\frac{\mu_{\max}^{A_{\rm eq}}t^\alpha}{\Gamma(\alpha+1)}\right) \leq \varepsilon, \quad (19)$$

the closed-loop system $D^\alpha x = A_1 x - A_2\rho s - A_2\mu\,{\rm sgn}(s) + B_1 f(x)$ is stable, i.e. $\|x(t)\| \leq \varepsilon$, for all $t \geq 0$ and for any finite initial condition where $\mu_{\max}^{A_{\rm eq}} = \mu_{\max}^{A_1} + \rho\mu_{\max}^{A_2}$, $\mu_{\max}^{B_{\rm eq}} = M\mu_{\max}^{B_1} + \mu\mu_{\max}^{A_2}$, and ε is a positive constant.

Proof When the system trajectories get close to the sliding manifolds, we have

$$\begin{aligned} x(t) = \frac{1}{\Gamma(\alpha)}\Bigg(& \int_0^t \frac{A_1}{(t-\tau)^{1-\alpha}} x(\tau)\mathrm{d}\tau \\ & + \int_0^t \frac{B_1}{(t-\tau)^{1-\alpha}} f(x(\tau))\mathrm{d}\tau \\ & - \int_0^t \frac{A_2}{(t-\tau)^{1-\alpha}} \rho s(\tau)\mathrm{d}\tau \\ & - \int_0^t \frac{A_2}{(t-\tau)^{1-\alpha}} \mathrm{sgn}(s(\tau))\mathrm{d}\tau \Bigg). \end{aligned} \quad (20)$$

Because chaotic attractor is bounded, we have $\|f(x)\| \leq M$. From $\|s(t)\| \leq \eta \|x(t)\|$, then

$$\begin{aligned} \|x(t)\| \leq & \frac{t^\alpha}{\Gamma(\alpha+1)}(\|A - B(KB)^{-1}KA\| \\ & + \rho\|B(KB)^{-1}\|) \cdot \int_0^t \frac{\|x(\tau)\|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau \\ & + \frac{M \cdot t^\alpha}{\Gamma(\alpha+1)}\|I - B(KB)^{-1}K\| \\ & + \frac{t^\alpha}{\Gamma(\alpha+1)}\|B(KB)^{-1}\mu\|. \end{aligned} \quad (21)$$

Applying Gronwall's inequality and condition (19), it is easy to obtain

$$\begin{aligned} \|x(t)\| \leq & \frac{t^\alpha}{\Gamma(\alpha+1)}\mu_{\max}^{A_{\mathrm{eq}}} \cdot \int_0^t \frac{\|x(\tau)\|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau \\ & + \frac{t^\alpha}{\Gamma(\alpha+1)}\mu_{\max}^{B_{\mathrm{eq}}} \\ \leq & \delta\frac{t^\alpha}{\Gamma(\alpha+1)}\mu_{\max}^{B_{\mathrm{eq}}} \cdot \exp\left(\frac{t^\alpha \mu_{\max}^{A_{\mathrm{eq}}}}{\Gamma(\alpha+1)}\right) \\ \leq & \varepsilon, \end{aligned} \quad (22)$$

where constant $\delta > 0$, and $\delta < \varepsilon$.

The theorem is proved. The closed-loop system is stable if the condition (19) is satisfied when we choose proper matrix K.

3.3. Simulation results

For closed-loop controlled system, the eigenvalues of A_1 are $\lambda_1 = 0$, $\lambda_2 = -(8+\theta)/3$, and $\lambda_3 = -(k_1/k_2 + 1)(25\theta + 10)$. In order to obey the stability condition of fractional order systems,[16] k_1 and k_2 must satisfy $k_1/k_2 > -1$.

According to inequalities (16) and (19), we can choose different K, and test the effectiveness of this method by simulation.

Suppose the initial point of the fractional order system is (1, −1, −2) and system fractional order $\alpha = 0.9$. Choose $\rho = 1$ and $\mu = 0.6$. Figure 1 shows the closed-loop system trajectories when $\theta = 0.5$. Figure 2 shows the closed-loop system trajectories when $\theta = 0$. Figure 3 shows the closed-loop system trajectories when $\theta = 1$.

From these output results, it can be found that the system states $x \to 0$ as $t \to \infty$ under the controller u. However, changing control parameter K can vary the convergence rate.

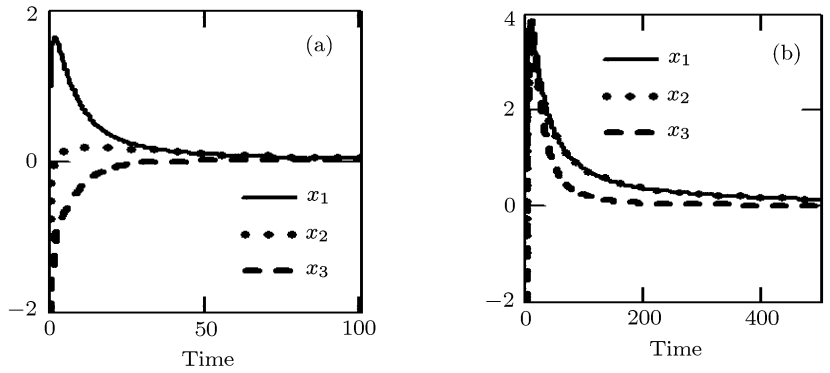


Fig. 1. The closed-loop system trajectories when $\theta = 0.5$, $\alpha = 0.9$, $\rho = 1$ and $\mu = 0.6$: (a) the system state variables x_1, x_2, x_3 when control parameters $k_1 = 2$, $k_2 = 1$, and $k_3 = 1$, the time step-length is 0.25; (b) the system state variables x_1, x_2, x_3 when parameters $k_1 = -1$, $k_2 = -0.01$, and $k_3 = -0.01$, the time step-length is 0.1.

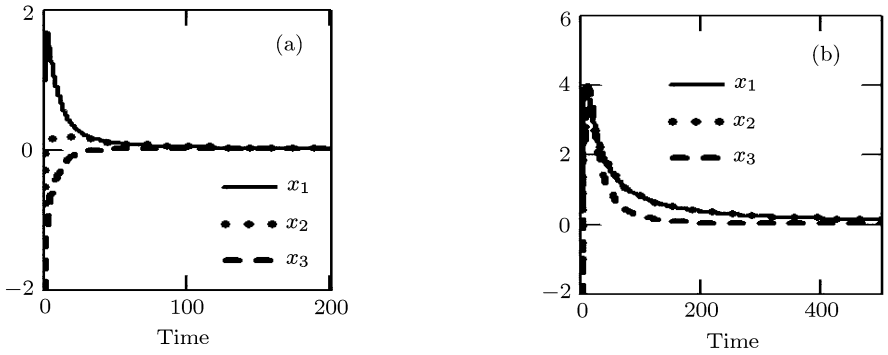


Fig. 2. The closed-loop system trajectories when $\theta = 0$, $\alpha = 0.9$, $\rho = 1$ and $\mu = 0.6$: (a) the system state variables x_1, x_2, x_3 when control parameters $k_1 = 2$, $k_2 = 1$, and $k_3 = 1$, the time step-length is 0.25; (b) the system state variables x_1, x_2, x_3 when control parameters $k_1 = -1$, $k_2 = -0.01$, and $k_3 = -0.01$, the time step-length is 0.1.

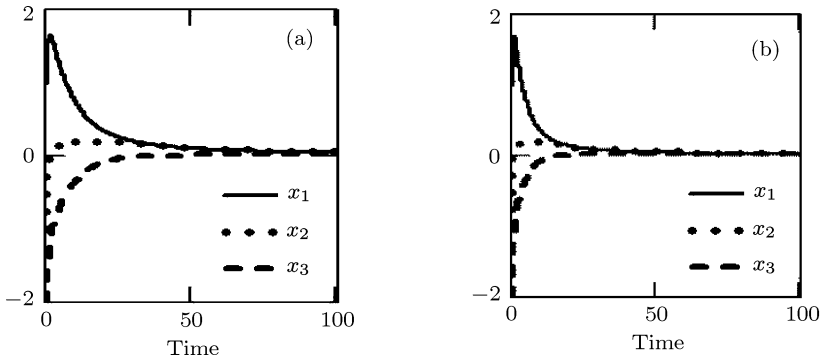


Fig. 3. The closed-loop system trajectories when $\theta = 1$, $\alpha = 0.9$, $\rho = 1$ and $\mu = 0.6$: (a) the system state variables x_1, x_2, x_3 when control parameters $k_1 = 2$, $k_2 = 1$, and $k_3 = 1$, the time step-length is 0.25; (b) the system state variables x_1, x_2, x_3 when control parameters $k_1 = -1$, $k_2 = -0.01$, and $k_3 = -0.01$, the time step-length is 0.1.

4. Conclusion

This paper shows that the fractional order unified chaotic system can be stabilized with the sliding mode control method. In order to decrease the chattering and improve the convergence rate, the equivalent controller includes two parts: the continuous part and switching part. By Gronwall's inequality, we can find the chosen ranges of the values of control parameters.

Chattering of the system is a fixed disadvantage in sliding mode control. How to decrease the effectiveness of chattering of the system is an important problem. One way is to use the saturation function instead of signal function. Extension of these methods is one of the future research topics. The related topic that can be considered as future work is to find the sliding mode control for some nonlinear fractional order systems.

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