A hyperchaotic system stabilization via inverse optimal control and experimental research

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In this paper, some basic dynamical properties of a four-dimensional autonomous hyperchaotic system are investigated by means of Poincaré mapping, Lyapunov exponents and bifurcation diagram. The dynamical behaviours of this new hyperchaotic system are proved not only by performing numerical simulation and brief theoretical analysis but also by conducting an electronic circuit experiment. An efficient approaching is developed for global asymptotic stabilization of this four-dimensional hyperchaotic system. Based on the method of inverse optimal control for nonlinear systems, a linear state feedback is electronically implemented. It is remarkably simple as compared with other chaos control ways, like nonlinear state feedback.

Keywords: hyperchaos, inverse optimal control, numerical simulation, circuitry experiment

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1. Introduction

Hyperchaos with more than one positive Lyapunov exponent has attracted increasing attention, since Rossler first introduced the Rossler hyperchaotic system.[1] There exist some well-known hyperchaotic systems such as hyperchaotic Chua's circuit,[2] hyperchaotic Liu system,[3,4] and so on.[5−7] As a higher-dimensional chaotic system, the hyperchaotic system exhibits more complex and richer dynamical behaviours than a low-dimensional chaotic system. The controlling of this kind of complex dynamical systems has been studied with great interest in the recent years.[8−11] The problem is finally boiled down to solving the Hamilton–Jacobi–Bellman equation of which in general the solution is non-existent or non-unique. In order to avoid solving equation, we will consider the inverse optimal control. The basic idea of the inverse optimal control[12−14] is: for the nonlinear control systems, the steady state solution of the Hamilton–Jacobi–Bellman equation is to control Lyapunov function. In this way, solving the Hamilton–Jacobi–Bellman equation is converted to the searching for controlling the Lyapunov function of a closed-loop system.

In the present paper, a four-dimensional hyperchaotic system is introduced briefly. Then an analog electronic circuit and a hardware implementation are designed for realizing its dynamical behaviours. Finally, the inverse optimal control is used to achieve globally asymptotically stability of this four-dimensional hyperchaotic system by a designed controller.

2. A hyperchaotic system

The new hyperchaotic system, which is put forward by Liu,[15] is described by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(-x + yz), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = by - hxz + cv, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = kxy - gz, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -dy. \end{cases} \tag{1}$$

In the state-space form, where $X = (x, y, z, v)^{\mathrm{T}}$, and $f(X)$ is given by Eq. (1), the system can be defined by

$$\dot{X} = f(X). \tag{2}$$

With $a = 10$, $b = 2.5$, $c = 1$, $d = 2$, $h = 1$, $k = 4$ and $g = 4$, the hyperchaotic strange attractors are obtained as shown in Figs. 1 and 2, where the initial values are set to be (2.3, 2.6, 2.9, 3).

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Obviously, the system has a unique equilibrium points O (0, 0, 0, 0). The Jacobian matrix is equal to

$$J = \begin{bmatrix} -a & az & ay & 0 \\ -hz & b & -hx & c \\ ky & kx & -g & 0 \\ 0 & -d & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}. \quad (3)$$

Then let $|J - \lambda I| = 0$, the eigenvalues corresponding to O (0, 0, 0, 0) are

$$\lambda_1 = -10, \quad \lambda_2 = -4, \quad \lambda_{3,4} = 1.25 \pm 0.6614\,\mathrm{i}\,.$$

Therefore, the equilibrium point O (0, 0, 0, 0) is an unstable saddle-focus because $\lambda_{1,2} < 0$, are negative root, but $\lambda_{3,4}$ are a pair of complex conjugate eigenvalues with positive real part.

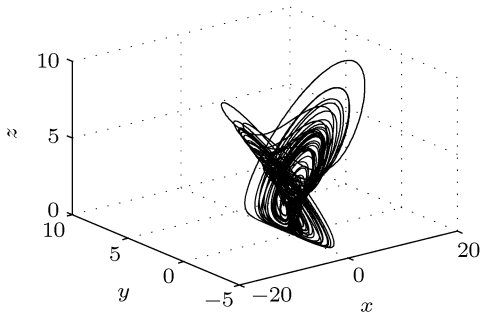


Fig. 1. Three-dimensional (x, y, z) view.

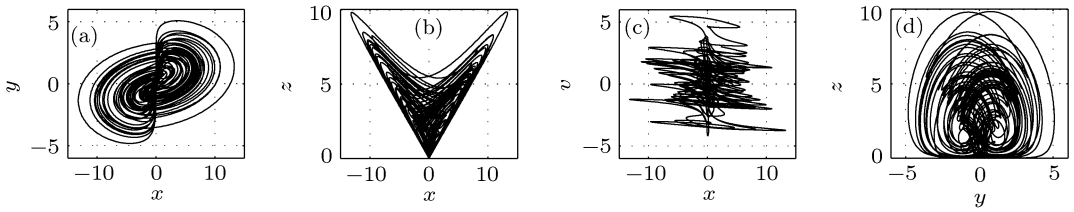


Fig. 2. Phase plane strange attractors: (a) x–y phase plane; (b) x–z phase plane; (c) x–v phase plane; (d) y–z phase plane.

The Poincaré mapping of this non-linear system is also analysed. It can be seen that the Poincaré mappings are the points in confusion as shown in Fig. 3.

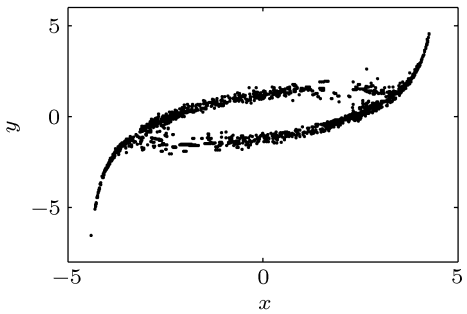


Fig. 3. Poincaré map of the x–y plane of the system.

The bifurcation diagram of x with increasing b is given in Fig. 4, and it shows abundant and complex dynamical behaviours.

According to chaos theory, the Lyapunov exponents measure the exponential rates of divergence and convergence of nearby trajectories in phase space of the system. In general, a four-dimensional autonomous system is defined as a hyperchaotic system with two positive Lyapunov exponents, implying that its dynamics is expanded in several different directions simultaneously.

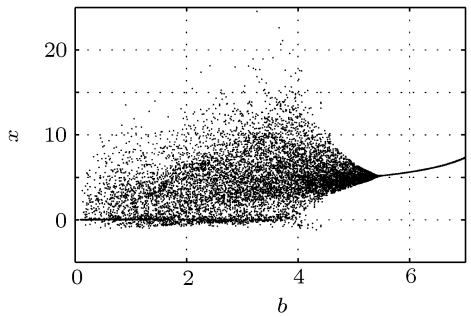


Fig. 4. Bifurcation diagram of x versus b with $a = 10$, $k = 4$, $d = 2$, $c = 1$, $h = 1$, $g = 4$.

The four Lyapunov exponents of this nonlinear system are respectively obtained from Fig. 5 as

$$\lambda_{\mathrm{L}1} = 0.444, \quad \lambda_{\mathrm{L}2} = 0.391, \quad \lambda_{\mathrm{L}3} = 0, \quad \lambda_{\mathrm{L}4} = -12.299.$$

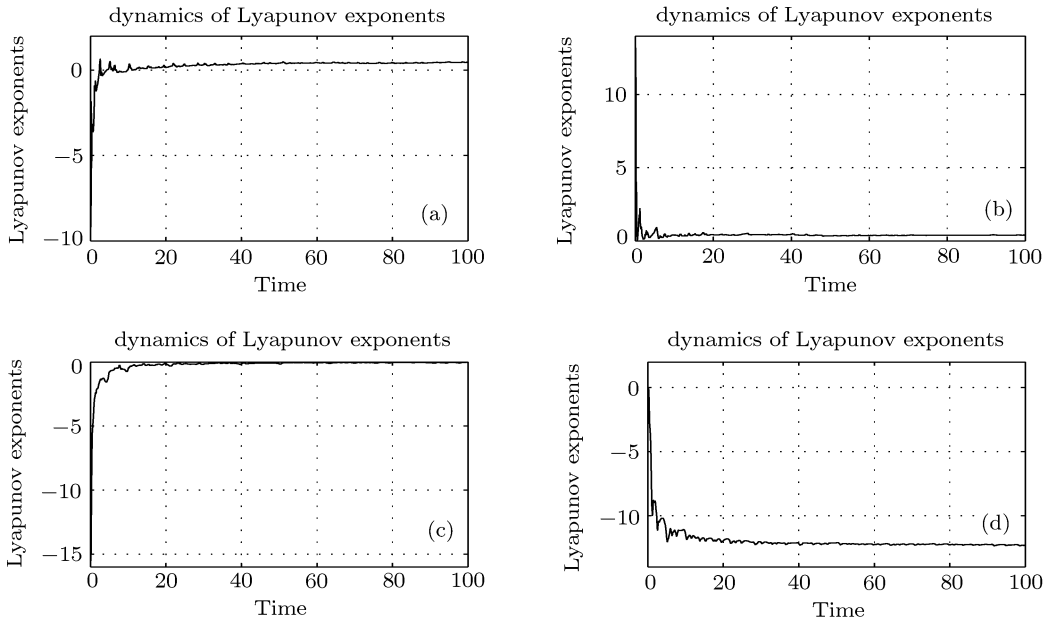


Fig. 5. Time evolution of Lyapunov exponents for $b = 2.5$. (a) Lyapunov exponents λ_{L1}; (b) Lyapunov exponents λ_{L2}; (c) Lyapunov exponents λ_{L3}; (d) Lyapunov exponents λ_{L4}.

Fix the parameters $a = 10$, $k = 4$, $d = 2$, $c = 1$, $h = 1$, $g = 4$ and let b vary. Lyapunov exponents of the system are shown in Fig. 6.

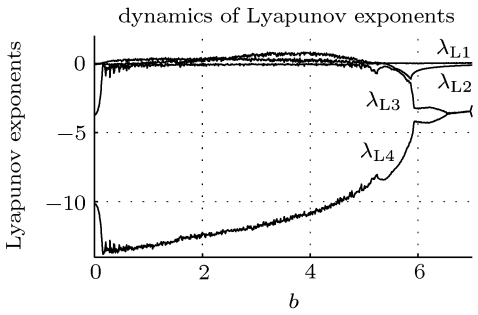


Fig. 6. Lyapunov exponents of the system for varying b.

(i) When $0 < b < 5.1$, $\lambda_{L1,2} > 0$, $\lambda_{L3} = 0$, $\lambda_{L4} < 0$, the system is in a hyperchaotic state;

(ii) When $5.1 < b < 5.4$, $\lambda_{L1,2} = 0$, $\lambda_{L3,4} < 0$, the system is in a quasi-periodic state;

(iii) When $5.4 \leq b$, $\lambda_{L1} = 0$, $\lambda_{L2,3,4} < 0$, the system is in a periodic state.

Figures 7 and 8 show the phase diagram with $b = 5.4$.

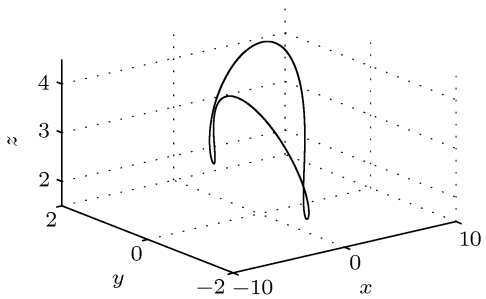


Fig. 7. Three-dimensional (x, y, z) view for varying b.

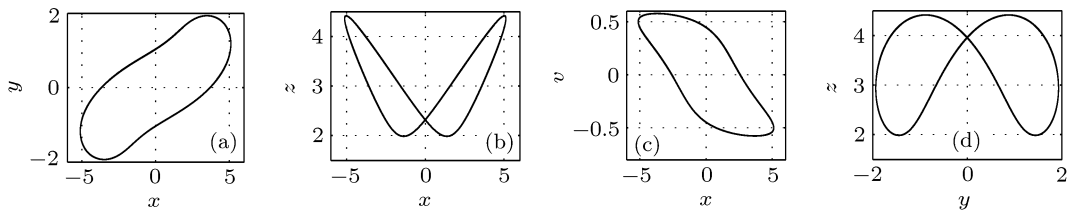


Fig. 8. Phase plane strange attractors for varying b: (a) x–y phase plane; (b) x–z phase plane; (c) x–v phase plane; (d) y–z phase plane.

We want to obtain the global asymptotical stability of this hyperchaotic system to have only one of its unstable equilibrium points (0, 0, 0, 0). Henceforth in the next section, the inverse optimal control method will be applied to achieve this objective.

3. Stabilization via inverse optimal control

This hyperchaotic system can be added by a linear state-feedback controller u to the second state, then the closed-loop system is given by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(-x+yz), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = by - hxz + cv + u, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = kxy - gz, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -dy. \end{cases} \tag{4}$$

First, find a Lyapunov function

$$V = x^2 + \frac{2a}{h}y^2 + \frac{a}{k}z^2 + \frac{2ac}{hd}v^2, \tag{5}$$

where $a > 0$, $c > 0$, $d > 0$, $h > 0$, $k > 0$. Its time derivative is

$$\dot{V} = 2\dot{x}x + \frac{4a}{h}\dot{y}y + \frac{2a}{k}\dot{z}z + \frac{4ac}{hd}\dot{v}v, \tag{6}$$

or, in another form

$$\dot{V} = \frac{\partial V}{\partial x}\dot{X} = \frac{\partial V}{\partial x}(f(X) + g(X)u) = L_{\mathrm{f}}V + uL_{\mathrm{g}}V, \tag{7}$$

where

$$\begin{cases} L_{\mathrm{f}}V = \dfrac{\partial V}{\partial x}f(X), \\ L_{\mathrm{g}}V = \dfrac{\partial V}{\partial x}g(X). \end{cases} \tag{8}$$

Now, substituting Eq. (4) into Eq. (6) yields

$$\begin{aligned} \dot{V} = {} & 2[a(-x+yz)]x + \frac{4a}{h}(by - hxz + cv + u)y \\ & + \frac{2a}{k}(kxy - gz)z + \frac{4ac}{hd}(-dy)v. \end{aligned} \tag{9}$$

After some simple calculations, one obtains

$$\dot{V} = -2ax^2 - \frac{2ag}{k}z^2 + \frac{4ab}{h}y^2 + \frac{4a}{h}uy. \tag{10}$$

And

$$\begin{cases} L_{\mathrm{f}}V = -2ax^2 - \dfrac{2ag}{k}z^2 + \dfrac{4ab}{h}y^2, \\ L_{\mathrm{g}}V = \dfrac{4a}{h}y. \end{cases} \tag{11}$$

Next, we can define a linear state-feedback control

$$u = -\beta R(X)^{-1}(L_{\mathrm{g}}V) = -(b + k_0)y, \tag{12}$$

where k_0 and β are positive constants, and $R(X)^{-1}$ is a positive-definite function of x

$$R(x)^{-1} = \frac{h}{4a\beta}(b + k_0). \tag{13}$$

Substituting Eq. (12) into Eq. (10) gives

$$\dot{V} = -2ax^2 - \frac{2ag}{k}z^2 - k_0y^2. \tag{14}$$

We can find $\dot{V} < 0$ for all $X \neq 0$. In other words, this closed-loop system can be globally asymptotically stabilized by the controller.

Finally, to confirm k_0, we define a cost functional as follows:

$$\begin{aligned} J(u) = \lim_{x\to\infty} \Big\{ & 2\beta V(X) \\ & + \int_0^t (l(X) + u^{\mathrm{T}}R(X)u)\mathrm{d}\tau \Big\}, \end{aligned} \tag{15}$$

where

$$l(X) = -2\beta L_{\mathrm{f}}V + \beta^2 R(X)^{-1}(L_{\mathrm{g}}V)^2. \tag{16}$$

Substituting Eq. (8) into Eq. (16) yields

$$\begin{aligned} l(X) = {} & -2\beta\Big(- 2ax^2 - \frac{2ag}{k}z^2 + \frac{4ab}{h}y^2 \Big) \\ & + \frac{4a}{h}\beta(b + k_0)y^2. \end{aligned} \tag{17}$$

On the basis of the basic idea of the inverse optimal control theory, we select

$$k_0 = b + 1. \tag{18}$$

So we obtain

$$l(X) = 4a\beta x^2 + \frac{4ag}{k}\beta z^2 + \frac{4a}{h}y^2, \tag{19}$$

which implies $l(X) > 0$ for all $X \neq 0$, and when $X \to \infty$, $l(X) \to \infty$. Substituting Eq. (12) into Eq. (7) we obtain

$$\dot{V} = L_{\mathrm{f}}V - \beta R(x)^{-1}(L_{\mathrm{g}}V)^2. \tag{20}$$

On both sides of the same equation multiplied by -2β, it can be written as

$$-2\beta\dot{V} = -2\beta L_{\mathrm{f}}V + 2\beta^2 R(X)^{-1}(L_{\mathrm{g}}V)^2. \tag{21}$$

Substituting Eq. (17) into Eq. (21) yields

$$-2\beta\dot{V} = l(X) + \beta^2 R(x)^{-1}(L_{\mathrm{g}}V)^2. \tag{22}$$

According to Eq. (12) we obtain

$$u^{\mathrm{T}}R(X)u = \beta^2 R(X)^{-1}(L_{\mathrm{g}}V)^2, \tag{23}$$

so that

$$l(x) + u^{\mathrm{T}} R(x) u = -2\beta \dot{V}. \tag{24}$$

Substituting Eq. (24) into Eq. (15) gives

$$J(u) = \lim_{x \to \infty} \left\{ 2\beta V(x) + \int_0^t (-2\beta \dot{V}) \mathrm{d}\tau \right\}, \tag{25}$$

thus, from the optimal control law, the minimum of the cost function is

$$J(u)_{\min} = 2\beta V(x(0)). \tag{26}$$

In the end, combining Eq. (12) and Eq. (18) we obtain

$$u = -(2b+1)y. \tag{27}$$

As shown in Figs. 9 and 10, when the initial values of state x, y, z and v are taken as (2.3, 2.6, 2.9, 3), $a = 10$, $b = 2.5$, $c = 1$, $d = 2$, $h = 1$, $k = 4$ and $g = 4$, the numerical simulation using Matlab proves that the chaotic orbit of the controlled system can quickly tend to the equilibrium point O (0, 0, 0, 0). In other words, this hyperchaotic system can be globally asymptotically stabilized by the designed controller.

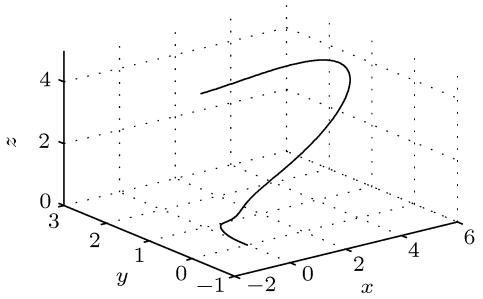


Fig. 9. Three-dimensional (x, y, z) plot of the controlled hyperchaotic system via inverse optimal control.

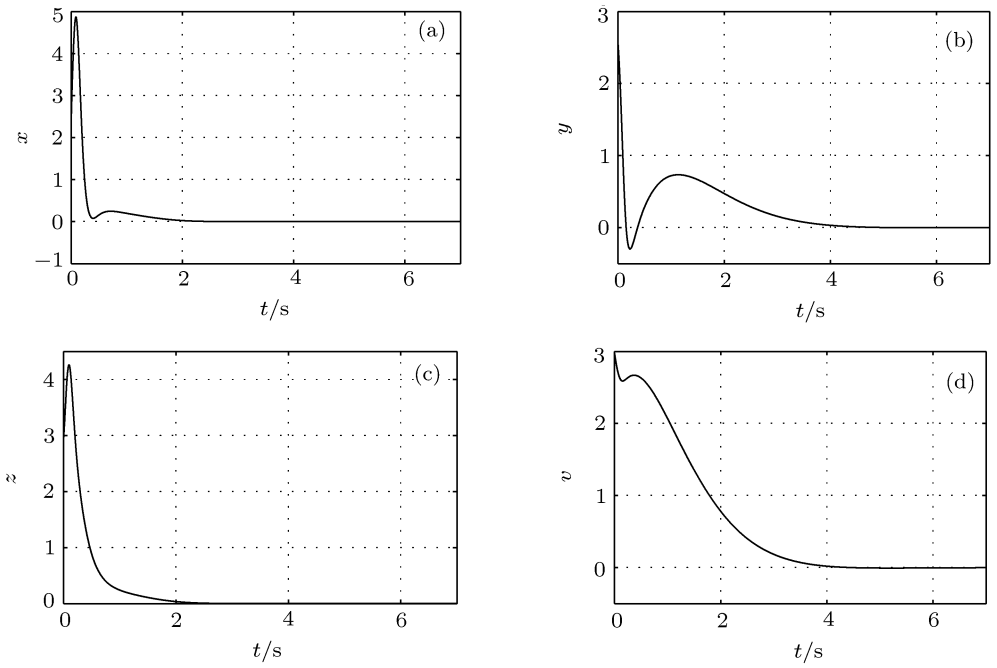


Fig. 10. Time evolution of stabilizing via inverse optimal control: (a) x phase plane; (b) y phase plane; (c) z phase plane; (d) v phase plane.

4. Circuitry for experimental research

In this section, an electronic oscillation circuit is designed to realize the new hyperchaotic system and the control of its global asymptotical stability.

The designed circuit, as shown in Fig. 11, consists of operational amplifiers, multipliers, resistors, and capacitors. Op-Amps is AD741 and AD633 is selected for multipliers; $C_1 = C_2 = C_3 = C_4 = 1$ μF, $R_{13} = R_{18} = R_{19} = 1$ kΩ, $R_5 = R_6 = R_{12} = 1$ kΩ, $R_7 = R_{16} = 40$ kΩ, $R_{21} = 2.5$ kΩ, $R_1 = R_3 = R_9 = R_{14} = 100$ kΩ, $R_2 = R_4 = R_8 = 10$ kΩ, $R_{10} = R_{11} = R_{15} = 10$ kΩ, $R_{17} = R_{20} = R_{22} = 10$ kΩ. The four state variables x, y, z and v are respectively obtained from the voltage outputs of V_{c1}, V_{c2}, V_{c3} and V_{c4}.

First of all, the circuit without controller is simulated by using Pspice. The circuitry simulation of the phase diagram is shown in Fig. 12. It is in agreement with the numerical simulation of Matlab, which was mentioned in Section 2.

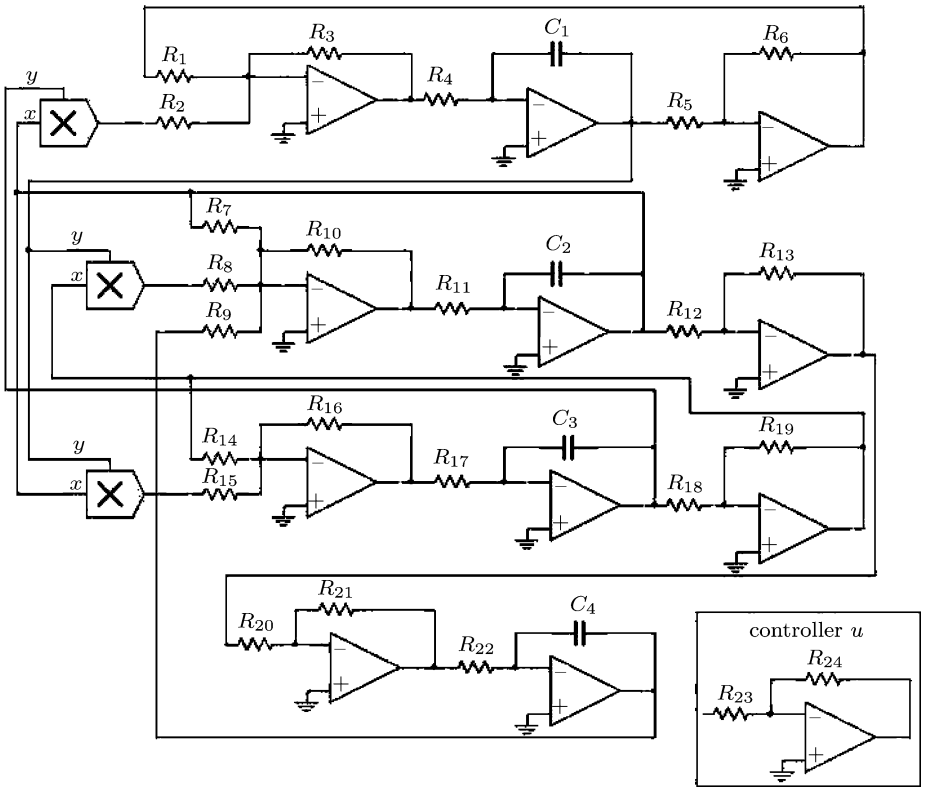


Fig. 11. Analog electronic implementation of hyperchaotic system. Op-Amps: AD741, multiplier: AD633.

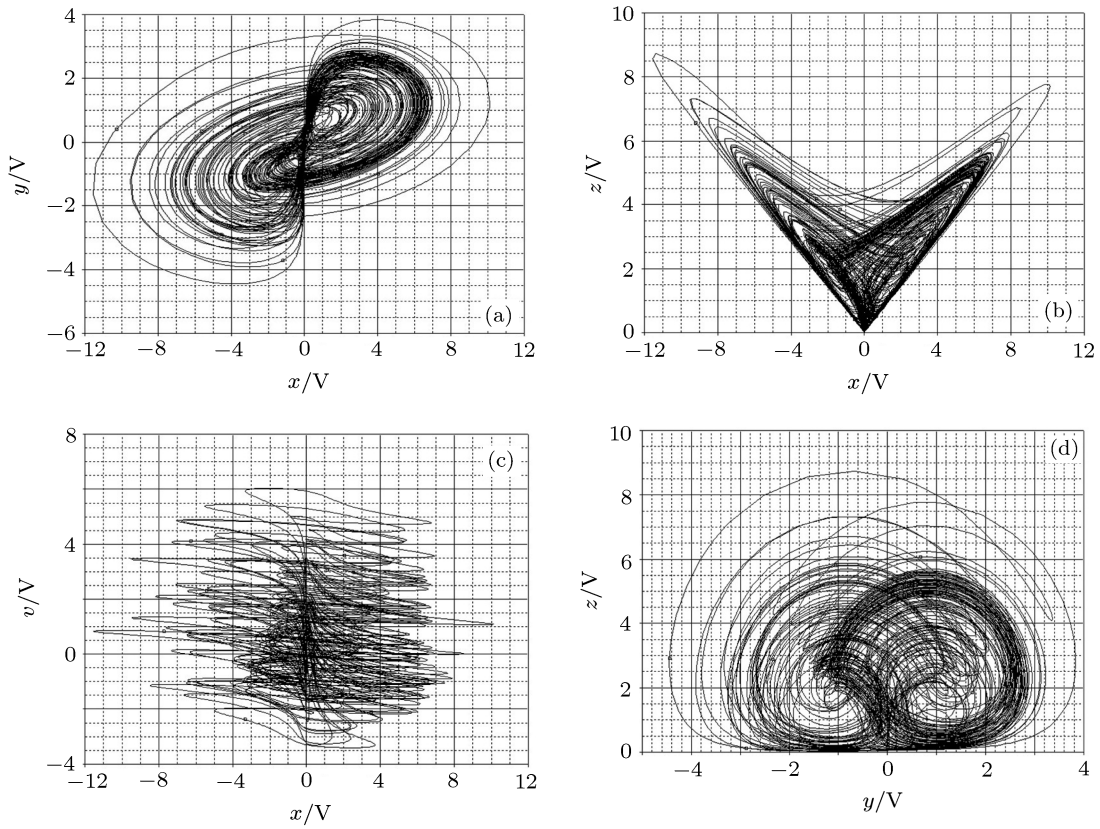


Fig. 12. Phase plane of hyperchaotic system by using Pspice: (a) x–y phase plane; (b) x–z phase plane; (c) x–v phase plane; (d) y–z phase plane.

When $t = 10$ s, using the controller, the hyperchaotic system is stabilized to its unstable equilibrium point (0, 0, 0, 0). As shown in Fig. 13, state variables x and z are closer to zero.

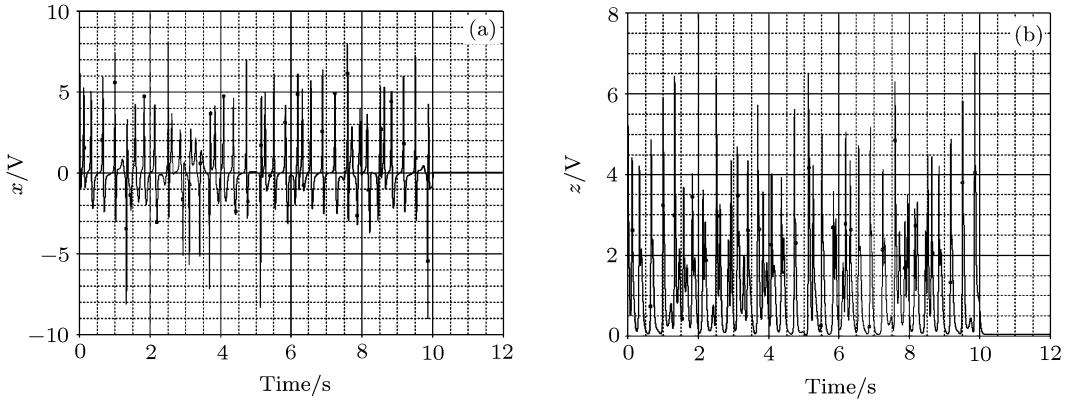


Fig. 13. Time evolution of stabilizing by using Pspice: (a) x phase plane; (b) z phase plane.

The experimental phase portraits of the new butterfly-shaped attractor are shown in Fig. 14. The results of the circuitry experiment are shown to be in agreement with the numerical simulation.

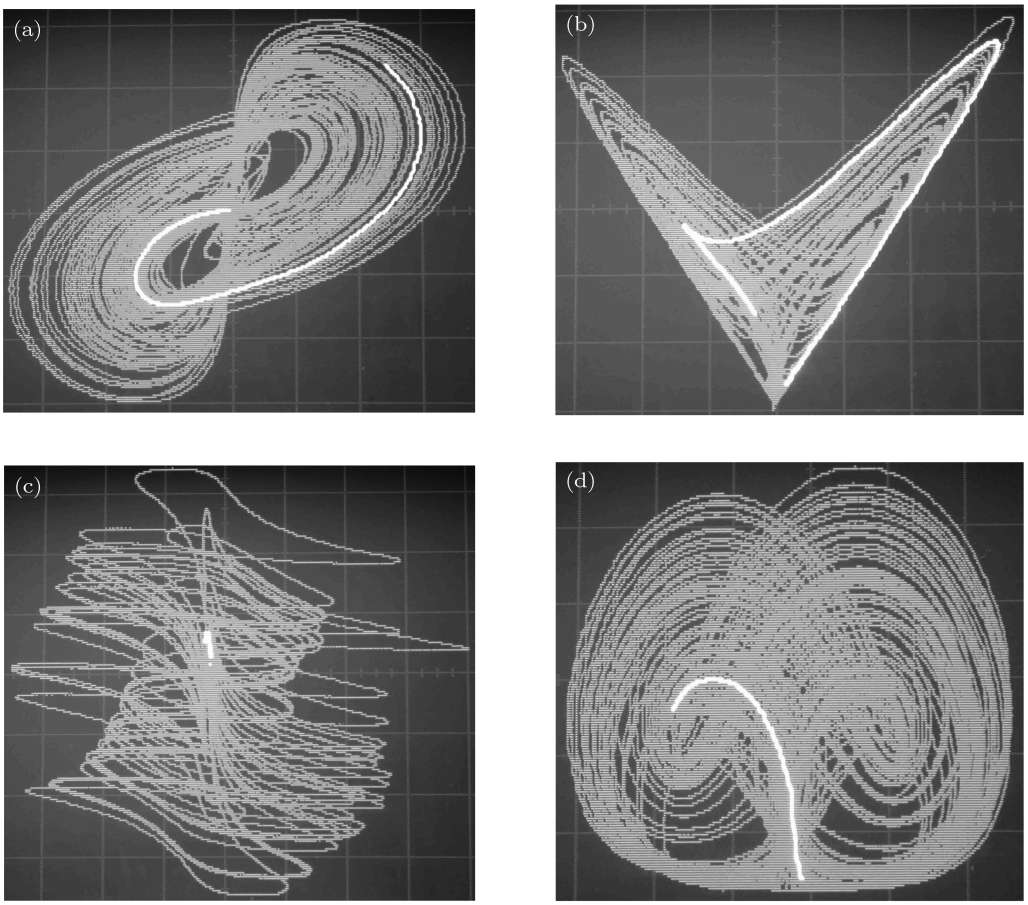


Fig. 14. Waveform of circuitry experiment: (a) x–y phase plane (x = 2 V/div, y = 1 V/div); (b) x–z phase plane (x = 2 V/div, z = 1 V/div); (c) x–v phase plane (x = 2 V/div, v = 1 V/div); (d) y–z phase plane (y = 1 V/div, z = 1 V/div).

When $b = 5.4$, the system is in periodic state. The results of circuitry experiment are shown in Fig. 15.

Though the outputs of the circuits are limited precisely because of being constructed by using low-cost components they are approximately in agreement with the numerical simulations.

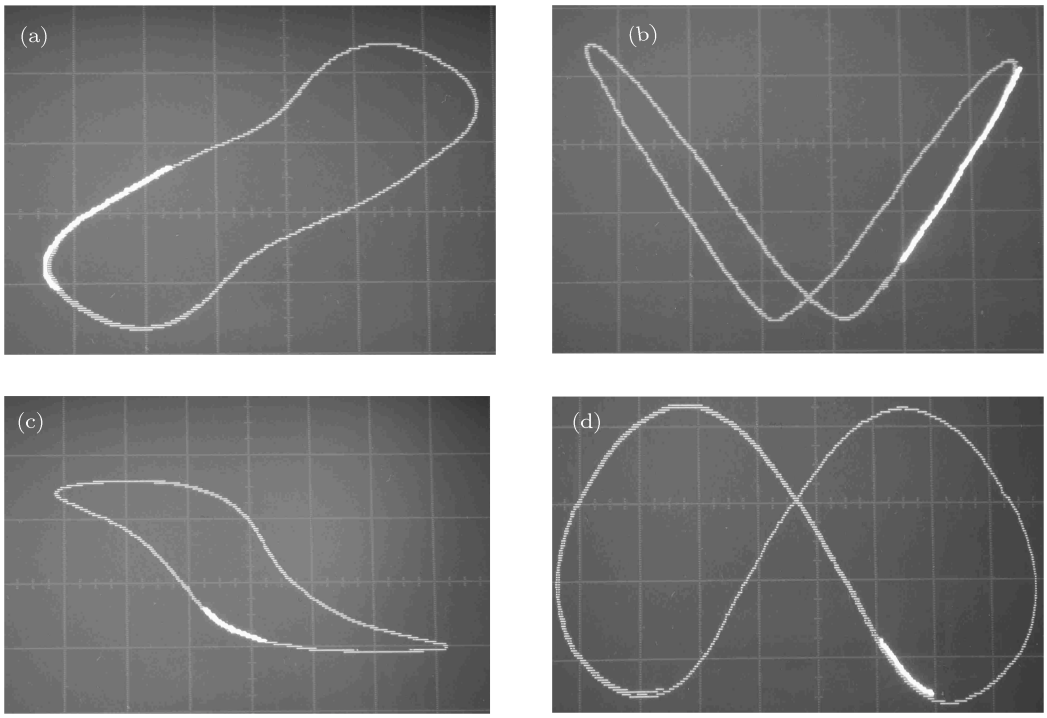


Fig. 15. Waveform of circuitry experiment for $b = 5.4$: (a) x–y phase plane ($b = 5.4$), ($x = 2$ V/div, $y = 1$ V/div); (b) x–z phase plane ($b = 5.4$), ($x = 2$ V/div, $z = 1$ V/div); (c) x–v phase plane ($b = 5.4$), ($x = 2$ V/div, $v = 0.5$ V/div); (d) y–z phase plane ($b = 5.4$), ($y = 0.5$ V/div, $z = 1$ V/div).

The waveforms of $x(t)$, $z(t)$ in time domain are shown in Fig. 16(a). In Fig. 16(b), the control signal is added to the hyperchaotic system. Here, it is easy to visualize and verify the satisfactory performance of the proposed control law. On the other hand, it has actually proved the robustness of the system to some extent.

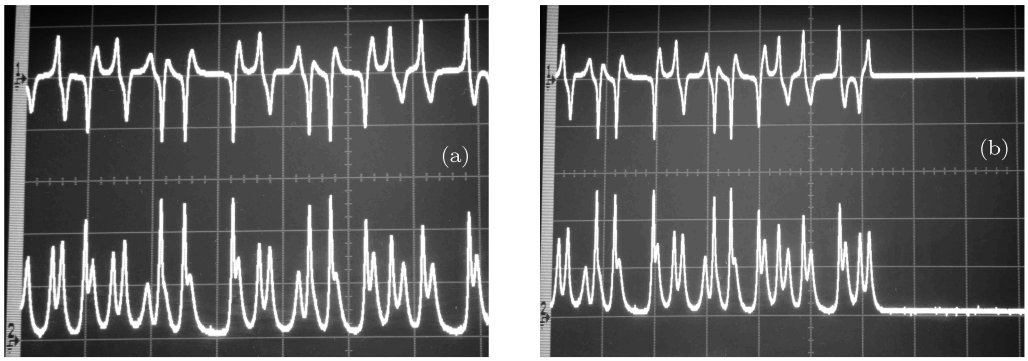


Fig. 16. Time evolution of circuitry experiment. (a) States x and z without controller, x: CH1 (5 V/div), z: CH2 (2 V/div); (b) stabilizing states x and z with controller, x: CH1 (5 V/div), z: CH2 (2 V/div).

5. Conclusions

This paper has stated an effective method for stabilizing the new hyperchaotic system, based on the inverse optimal control technique. Through simulation and circuit implementation, the method is verified to be feasible. It is remarkably simple as compared with other chaos control methods, like nonlinear state feedback, which may have more than one nonlinear feedback controller.

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