External electric field control in the charge qubits including three Josephson junctions*

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Based on the standard canonical quantization principle, this paper gives the quantization scheme for the charge qubits mesoscopic circuit including three Josephson junctions coupled capacitively. By virtue of the Heisenberg equation, the time evolution of the phase difference operators across the polar plates and the number operators of the Cooper-pairs on the island are investigated and the modification of the Josephson equation is discussed. The time evolution of the phase difference operators is analysed when the Josephson junctions are irradiated by the external electrical field, which is referred to as also the obtainable controlling parameter.

Keywords: control, charge qubit, Josephson junction

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1. Introduction

With the development of quantum computation and information, it has been recognized that the solid-state units such as quantum dots and Josephson tunnel junctions can be easily integrated into large-scale electronic circuits.[1−4] Thereby, they are promising for realizing a practical quantum computer. Mesoscopic Josephson junctions will play an important role in realizing the solid-state quantum computation.[5−8] The simplest model of Josephson junction qubit was proposed earlier,[9,10] which employed an isolated superconducting island to construct qubits. Such a qubit is called the charge qubit. By coupling two or multiple qubits, the quantum logic operations can be realized.[11,12] Recently, we proposed the Cooper-pair number-phase quantization scheme for the double-Josephson-junction charge qubit system and studied the phase difference evolution when the Josephson junction is irradiated by the external electrical field.[13,14] Based on this work, we couple three Josephson junctions together and try to search for the phase difference evolving mechanism when more charge qubits are coupled together so as to provide some help for the quantum control design. Our procedure is arranged as follows: first, we give the quantization for the charge qubits including three Josephson junctions; secondly, we obtain modification of the Josephson equation and finally, we analyse the external electric field's control on the Josephson junction charge qubits.

2. Quantization for the mesoscopic capacitance coupling circuit including three Josephson junctions

Consider a mesoscopic capacitance coupling circuit including three Josephson junctions (see Fig. 1).

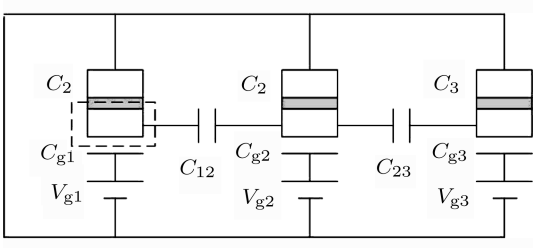


Fig. 1. Mesoscopic capacitance coupling circuit including three Josephson junctions.

It consists of five different elementary sections: three charge qubits structures and two coupling capacitance

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C_{12} and C_{23}. Here, C_{gl} $(l = 1, 2, 3)$ is called gate capacitance, C_l Josephson junction capacitance and V_{gl} gate voltages.

From classical Hamilton dynamics, we deduce the Hamiltonian for the mesoscopic capacitance coupling circuit including three Josephson junctions. A single Josephson junction is characterized by the following two equations

$$I_l = I_c \sin\varphi_l, \quad (l = 1, 2, 3), \tag{1}$$

$$\dot{\varphi}_l = \frac{2e}{\hbar} V_l, \tag{2}$$

where I_l is the superconductor tunnelling current, I_c is the critical electric current of the Josephson junction, V_l is the voltage drop across the l-th Josephson junction, and φ_l is the phase difference between the two wave functions across the l-th Josephson junction. The corresponding voltage of the equivalent capacitors C_l is

$$V_l = \frac{\hbar}{2e}\dot{\varphi}_l. \tag{3}$$

If we consider φ_l as the generalized coordinate, the potential energy related to φ_l of the system is

$$\mathscr{V} = \sum_{l=1}^{3} E_l(3 - \cos\varphi_l), \tag{4}$$

where $E_l = (\hbar/2e)I_c$ is coupling energy. And the kinetic energy related to $\dot{\varphi}_l$ is

$$\mathscr{T} = \frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 \sum_{l=1}^{3}\left(C_{\Sigma l}\dot{\varphi}_l^2 - 2C_{12}\dot{\varphi}_1\dot{\varphi}_2 - 2C_{23}\dot{\varphi}_2\dot{\varphi}_3 - \frac{4e}{\hbar}C_{gl}\dot{\varphi}_l\mathscr{V}_{gl} + 2e^2\frac{N_{gl}^2}{C_{gl}}\right), \tag{5}$$

where

$$\begin{aligned} C_{\Sigma 1} &= C_1 + C_{g1} + C_{12}, \\ C_{\Sigma 2} &= C_2 + C_{g2} + C_{23}, \\ C_{\Sigma 3} &= C_3 + C_{g3} + C_{23}, \end{aligned} \tag{6}$$

and $N_{gl} = -C_{gl}V_{gl}/2e$ is named gate charge number. Then the Lagrangian of the system can be obtained as

$$\mathscr{L} = \mathscr{T} - \mathscr{V}. \tag{7}$$

Considering that Cooper-pairs can tunnel from or onto the island, by virtue of Kirhhoff's law, we have

$$\begin{aligned} 2en_1 &= \frac{\hbar}{2e}\left(C_{\Sigma 1}\dot{\varphi}_1 - C_{12}\dot{\varphi}_2\right) + 2eN_{g1}, \\ 2en_2 &= \frac{\hbar}{2e}\left(C_{\Sigma 2}\dot{\varphi}_2 - C_{12}\dot{\varphi}_1 - C_{23}\dot{\varphi}_3\right) + 2eN_{g2}, \\ 2en_3 &= \frac{\hbar}{2e}\left(C_{\Sigma 3}\dot{\varphi}_3 - C_{23}\dot{\varphi}_2\right) + 2eN_{g3}, \end{aligned} \tag{8}$$

where n_l is the excess Cooper-pairs number on the l-th island. From Eqs. (7) and (8), we can obtain the generalized momentum p_l related to $\dot{\varphi}_l$ as

$$\begin{aligned} p_1 &= \frac{\partial\mathscr{L}}{\partial\dot{\varphi}_1} \\ &= \frac{\hbar^2}{4e^2}\left(C_{\Sigma 1}\dot{\varphi}_1 - C_{12}\dot{\varphi}_2 + \frac{4e^2}{\hbar}N_{g1}\right) \\ &= n_1\hbar, \\ p_2 &= \frac{\partial\mathscr{L}}{\partial\dot{\varphi}_2} \\ &= \frac{\hbar^2}{4e^2}\left(C_{\Sigma 2}\dot{\varphi}_2 - C_{12}\dot{\varphi}_1 - C_{23}\dot{\varphi}_3 + \frac{4e^2}{\hbar}N_{g2}\right) \\ &= n_2\hbar, \\ p_3 &= \frac{\partial\mathscr{L}}{\partial\dot{\varphi}_3} \\ &= \frac{\hbar^2}{4e^2}\left(C_{\Sigma 3}\dot{\varphi}_3 - C_{23}\dot{\varphi}_2 + \frac{4e^2}{\hbar}N_{g3}\right) \\ &= n_3\hbar, \end{aligned} \tag{9}$$

which are followed by

$$\begin{aligned} \dot{\varphi}_1 &= M[A(n_1 - N_{g1}) + B(n_2 - N_{g2}) + C(n_3 - N_{g3})], \\ \dot{\varphi}_2 &= M[B(n_1 - N_{g1}) + D(n_2 - N_{g2}) + E(n_3 - N_{g3})], \\ \dot{\varphi}_3 &= M[C(n_1 - N_{g1}) + E(n_2 - N_{g2}) + F(n_3 - N_{g3})], \end{aligned} \tag{10}$$

where

$$\begin{aligned} M &= \frac{4e^2}{\hbar\left(C_{\Sigma 1}C_{\Sigma 2}C_{\Sigma 3} - C_{\Sigma 1}C_{23}^2 - C_{\Sigma 3}C_{12}^2\right)}, \\ A &= C_{\Sigma 2}C_{\Sigma 3} - C_{23}^2, \quad B = C_{\Sigma 3}C_{12}, \\ C &= C_{12}C_{23}, \quad D = C_{\Sigma 1}C_{\Sigma 3}, \\ E &= C_{\Sigma 1}C_{23}, \quad F = C_{\Sigma 1}C_{\Sigma 2} - C_{12}^2. \end{aligned} \tag{11}$$

Thus, combining Eqs. (7), (9) and (10), we obtain the classical Hamiltonian of the system as

$$\mathscr{H} = \sum_{l}^{3}\mathscr{H}_l + \mathscr{H}_{int}, \tag{12}$$

where

$$\mathscr{H}_l = E_c^{(l)}\left(n_l - N_{gl}\right)^2 + E_l\left(1 - \cos\varphi_l\right) - 2\frac{N_{gl}^2 e^2}{C_{gl}} \tag{13}$$

is the effective Hamiltonian of a single qubit structure,

$$\mathscr{H}_{int} = \sum_{(l,k=1,l\neq k)}^{3} E_{cc}^{(lk)}\left(n_l - N_{gl}\right)\left(n_k - N_{gk}\right) \tag{14}$$

is the interacting term,

$$
\begin{aligned}
E_{\rm c}^{(1)} &= M^2\frac{\hbar^2}{8e^2}(C_{\Sigma 1}A^2 + C_{\Sigma 2}B^2 + C_{\Sigma 3}C^2 - 2C_{12}AB - 2C_{23}BC),\\
E_{\rm c}^{(2)} &= M^2\frac{\hbar^2}{8e^2}(C_{\Sigma 1}B^2 + C_{\Sigma 2}D^2 + C_{\Sigma 3}E^2 - 2C_{12}BD - 2C_{23}DE),\\
E_{\rm c}^{(3)} &= M^2\frac{\hbar^2}{8e^2}(C_{\Sigma 1}C^2 + C_{\Sigma 2}E^2 + C_{\Sigma 3}F^2 - 2C_{12}CE - 2C_{23}EF)
\end{aligned}
\tag{15}
$$

are respectively single Cooper-pair charging energies on the corresponding island, and

$$
\begin{aligned}
E_{\rm cc}^{(12)} &= M^2\frac{\hbar^2}{4e^2}[(C_{\Sigma 1}AB + C_{\Sigma 2}BD + C_{\Sigma 3}CE - C_{12}(AD + B^2) - C_{23}(BE + CD)],\\
E_{\rm cc}^{(13)} &= M^2\frac{\hbar^2}{4e^2}[(C_{\Sigma 1}AC + C_{\Sigma 2}BE + C_{\Sigma 3}CF - C_{12}(AE + BC) - C_{23}(BF + CE)],\\
E_{\rm cc}^{(23)} &= M^2\frac{\hbar^2}{4e^2}[(C_{\Sigma 1}BC + C_{\Sigma 2}DE + C_{\Sigma 3}EF - C_{12}(BE + CD) - C_{23}(DF + E^2)]
\end{aligned}
\tag{16}
$$

are respectively coupling energies between corresponding qubits.

According to the standard canonical quantization principle,[15] the canonical conjugate quantities n_l and φ_l are, respectively, associated with the Hermitian operators $\hat{n}_l$ and $\hat{\varphi}_l$, which are assigned by the following commutation relation

$$
[\hat{\varphi}_l\,, \hat{n}_l\,] = {\rm i}. \tag{17}
$$

Thus the Hamiltonian operator of the system is

$$
\hat{\mathscr{H}} = \sum_{l=1}^{3}\hat{\mathscr{H}}_l + \hat{\mathscr{H}}_{\rm int}. \tag{18}
$$

3. Modification of the Josephson equation

Commutation relation given by Eq. (17) implies the following: relations

$$
\begin{aligned}
&[\hat{n}_l, \exp({\rm i}\hat{\varphi}_l)] = \exp({\rm i}\hat{\varphi}_l)\,,\\
&[\hat{n}_l, \exp(-{\rm i}\hat{\varphi}_l)] = -\exp(-{\rm i}\hat{\varphi})\,,
\end{aligned}
\tag{19}
$$

$$
[\hat{n}_l, \cos\hat{\varphi}_l] = {\rm i}\sin\hat{\varphi},\ \ [\hat{n}_l, \sin\hat{\varphi}] = -{\rm i}\cos\hat{\varphi}_l. \tag{20}
$$

From Eqs. (19) and (20), by virtue of the Heisenberg equation, we can obtain

$$
-2e\frac{\rm d}{{\rm d}t}\hat{n}_l = -\frac{2e}{{\rm i}\hbar}\left[\hat{n}_l, \hat{\mathscr{H}}\right] = \frac{2eE_l}{\hbar}\sin\hat{\varphi}_l = \hat{I}_l, \tag{21}
$$

which is Josephson current operator through the single Josephson junction and it shows that capacitance coupled three Josephson junctions do not effect its current expression style. That is to say, in a capacitance coupled three-Josephson junction mesoscopic circuit, for one of the three Josephson junctions, its current expression style is similar to the current expression style when it is a single Josephson junction.

Similarly, by virtue of the Heisenberg equation, we can obtain

$$
\begin{aligned}
\frac{\rm d}{{\rm d}t}\hat{\varphi}_l &= \frac{1}{{\rm i}\hbar}\left[\hat{\varphi}_l, \hat{\mathscr{H}}\right]\\
&= \frac{1}{{\rm i}\hbar}\left[\hat{\varphi}_l, \hat{\mathscr{H}}_l + \hat{\mathscr{H}}_{\rm int}\right]\\
&= \frac{1}{\hbar}\left[2E_{\rm c}^{(l)}\,(\hat{n}_l - N_{{\rm g}l}) + E_{\rm cc}^{(lj)}\,(\hat{n}_j - N_{{\rm g}j})\right.\\
&\quad \left. + E_{\rm cc}^{(lk)}\,(\hat{n}_k - N_{{\rm g}k})\right]\\
&\equiv \frac{2e}{\hbar}\hat{\mathscr{V}}_l, \quad (l, j, k = 1, 2, 3;\ l \neq j \neq k)\,.
\end{aligned}
\tag{22}
$$

Comparing Eq. (2) with Eq. (22), we can deduce the effective voltage of the i-th Josephson junction as

$$
\begin{aligned}
\hat{\mathscr{V}}_l = \frac{1}{2e}&\left[2E_{\rm c}^{(l)}\,(\hat{n}_l - N_{{\rm g}l}) + E_{\rm cc}^{(lj)}\,(\hat{n}_j - N_{{\rm g}j})\right.\\
&\left. + E_{\rm cc}^{(lk)}\,(\hat{n}_k - N_{{\rm g}k})\right]\\
&(l, j, k = 1, 2, 3;\ l \neq j \neq k)\,.
\end{aligned}
\tag{23}
$$

From Eq. (23) we can see that the voltage of the l-th Josephson junction is affected by the number of excess Cooper-pairs on the other two islands, this is because we coupled the circuit with two capacitors. So we must consider the circuit as an integer.

4. External electric field control on the system

In this section we shall analyse the external electric field's (say light radiation) control on the Josephson junction charge qubits. For this aim, we suppose

that the l-th Josephson junction is irradiated by external light field, and the light field causes extra Josephson current across the l-th Josephson junction, and consider the work performed by the first junction current in a short time interval Δt,

$$\begin{aligned}&\hat{\mathscr{V}}_1\hat{\mathscr{I}}_1\Delta t\\ =\ &\Delta t\frac{E_1}{\hbar}\sin\hat{\varphi}_1\Big[2E_{\rm C}^{(1)}\left(\hat{n}_1\ -N_{\rm g1}\right)\\ &+E_{\rm cc}^{(12)}\left(\hat{n}_2-N_{\rm g2}\right)+E_{\rm cc}^{(13)}\left(\hat{n}_3\ -N_{\rm g3}\right)\Big],\end{aligned}\tag{24}$$

thus the corresponding interacting Hamiltonian operator can be expressed in the following form:

$$\begin{aligned}\hat{\mathscr{H}}_{\rm int}\ =\ &\sin\hat{\varphi}_1[\lambda_1\left(\hat{n}_1\ -N_{\rm g1}\right)+\lambda_2\left(\hat{n}_2\ -N_{\rm g2}\right)\\ &+\lambda_3\left(\hat{n}_3\ -N_{\rm g3}\right)],\end{aligned}\tag{25}$$

where λ_l is the coupling constant, the coupling term $E_l/\hbar$ is written in the constant term λ. So the Hamiltonian operator of the whole system is $\hat{\mathscr{H}}=\hat{\mathscr{H}}_0+\hat{\mathscr{H}}_{\rm int}$, and $\hat{\mathscr{H}}_{\rm int}$ can be regarded to as an infinitesimal term. Taking $\hat{\mathscr{H}}_{\rm int}$ in the interaction picture[16] and according to the equation of motion in this picture, we derive

$$\frac{\rm d}{{\rm d}t}\sin\hat{\varphi}_1=\frac{1}{{\rm i}\hbar}[\sin\hat{\varphi}_1,\hat{\mathscr{H}}_{\rm int}]=\frac{\lambda_1E_{\rm C}^{(1)}}{\hbar}\sin 2\hat{\varphi}_1,\tag{26}$$

$$\frac{\rm d}{{\rm d}t}\cos\hat{\varphi}_1=\frac{1}{{\rm i}\hbar}[\cos\hat{\varphi}_1,\hat{\mathscr{H}}_{\rm int}]=\frac{2\lambda_1E_{\rm C}^{(1)}}{\hbar}\sin^2\hat{\varphi}_1.\tag{27}$$

It then follows that

$$\frac{\rm d}{{\rm d}t}\tan\frac{\hat{\varphi}_1}{2}=\frac{2\lambda_1E_{\rm C}^{(1)}}{\hbar}\tan\frac{\hat{\varphi}_1}{2},\tag{28}$$

which has the solution

$$\tan\frac{\hat{\varphi}_1(t)}{2}=\exp\left[\frac{2\lambda_1E_{\rm C}^{(1)}}{\hbar}t\right]\tan\frac{\hat{\varphi}_1(0)}{2},\tag{29}$$

where $\hat{\varphi}_1(0)$ is the initial phase difference at $t=0$, this solution shows the evolution behaviour of the phase difference. Since the Josephson current is caused by the phase difference between two superconductors, the evolution of the phase difference certainly accompanies the variation of the extra number of Cooper-pairs on the first island, which is bound to affect the state on the qubit by the coupling constant λ_1. Due to the capacitively coupling, the three qubits should be considered in its totality. For the second junction, though it is not directly radiated, we have

$$\frac{\rm d}{{\rm d}t}\hat{\varphi}_2=\frac{1}{{\rm i}\hbar}[\hat{\varphi}_2,\hat{\mathscr{H}}_{\rm int}]=-\frac{\lambda_2}{\hbar}\sin\hat{\varphi}_1.\tag{30}$$

Jointing Eqs. (29) and (30), we find that the solution of Eq. (30) is

$$\hat{\varphi}_2(t)=\frac{2c\arctan(b\,{\rm e}^{at})}{a}+\hat{\varphi}_2(0).\tag{31}$$

Similarly, for the third junction, we have

$$\hat{\varphi}_3(t)=\frac{2d\arctan(b\,{\rm e}^{at})}{a}+\hat{\varphi}_3(0),\tag{32}$$

where $\hat{\varphi}_n(0)$ is the initial phase difference of the n-th Josephson junction, and

$$a=\frac{\lambda_1}{\hbar},\quad b=\tan\frac{\hat{\varphi}_1(0)}{2},\quad c=\frac{\lambda_2}{\hbar},\quad d=\frac{\lambda_3}{\hbar},\tag{33}$$

equations (29), (31) and (32) imply that once the coupling constants λ_1, λ_2 and λ_3 are known we can control the qubits indirectly by the external electrical field, which will provide some help for the quantum control design.

5. Conclusions

The scheme of number-phase quantization of the three qubits structure is proposed. By virtue of the Heisenberg equation, we investigated the time evolutions of the phase difference operators across the polar plates and the number operators of the Cooper-pairs on the island and discussed the modification of the Josephson equation because of the coupled capacitors. Then the time evolutions of the phase difference operators and number operators were analysed when the Josephson junctions are irradiated by external electrical field. We have found that changing the gate voltages can control qubits. It is very important for us to integrate a large-scale circuit.

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