Monogamy relations in tripartite quantum system*

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This paper presents monogamy relations for three-qubit quantum states by using convex-roof extended negativity, which inspires studying the difference between the three entanglement measures, concurrence, negativity and convex-roof extended negativity. It finds that the convex-roof extended negativity is a stronger entanglement measure than concurrence in multipartite higher-dimensional quantum system.

Keywords: concurrence, monogamy relation, convex-roof extended negativity

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1. Introduction

Entanglement of quantum states of composite systems is an essential feature of quantum mechanics and plays a crucial role in various applications in quantum information processing.[1−3] In order to apply entanglement to more useful quantum information processing, there are several important things we should consider. One is to quantify the degree of entanglement, the other is to know more about properties of entanglement. We know that concurrence and negativity are two remarkable entanglement measures for bipartite systems of qubit, and monogamy of entanglement is an important property of entanglement. In recent years a lot of work has shown that multipartite entanglement cannot be freely shared among the parties.

Monogamy of entanglement (MoE) is a key ingredient to make quantum cryptography secure,[4] and it also plays an important role in condensed-matter physics.[5] The first inequality for the quantification of the entanglement monogamy was achieved by Coffman, Kundu, and Wootters,[6] and its generalization to multi-qubit states was recently proved by Osborne and Verstraete[7] as well. Therefore, it is necessary to characterize MoE to understand the whole picture of quantum entanglement in multipartite systems, as well as its possible applications in quantum information theory. Although MoE is a typical property of multipartite quantum entanglement, there are only a few entanglement measures known so far,[8,9] which can show the monogamy properties of entanglement in multipartite quantum systems, and their results are restricted to multi-qubit systems. In other words, there exist quantum states in higher-dimensional systems,[10] which violate the monogamy properties in terms of concurrence, and this shows the importance of choosing a proper entanglement measure. Lee, Chi, Oh and Kim[11] have proposed convex-roof extended negativity (CREN) as an entanglement measure for bipartite quantum systems, which was proved more powerful than concurrence. It has been shown that any known examples violating monogamy inequality of entanglement for multi-qubit systems using concurrence can also be rephrased by CREN. Furthermore, the counter examples in higher-dimensional systems[12,13] can be proved true in terms of CREN as an entanglement measure.

The tripartite quantum entanglement states play important roles in quantum information processing, the typical examples are Greenberger–Horne–Zeilinger (GHZ) state and W state, and there are many researches on their properties.[14,15] From monogamy inequalities of entanglement, it has been defined three-tangle in terms of concurrence and three-π in terms of negativity,[16,17] respectively. Later, the monogamy equalities for three-qubit states have been proved,[18,19] which shows the MoE more explicitly than the Coffman–Kundu–Wootters (CKW) inequality. The monogamy equality in the tripartite quantum system reveals the difference between C_{AB}^2 and its

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dual $(C^a_{AB})^2$.[20] Thus, it is necessary to consider the dual of the MoE, which requires us to research into the optimal decomposition of a given pure state.[21] It is also important to investigate whether the monogamy equality would be possible in any higher-dimensional tripartite quantum systems, which could be helpful for us to understand multipartite entanglement. On this account, it is meaningful for us to study the monogamy equality in tripartite quantum system by using CREN.

In the end, the main results of this paper are to provide a monogamy equality in terms of CREN in the tripartite quantum system. In Section 2, we recall some basic concepts of entanglement measure, such as concurrence, negativity and CREN. In Section 3, we deduce the monogamy equality in terms of CREN, which reveals the relation between CREN and its duality. In Section 4, by generalizing the equality to higher-dimensional quantum system, we find that an inequality always holds, which can also illustrate that the CREN is quite useful as entanglement measure in higher-dimensional quantum system. The conclusions are given in Section 5.

2. Comparison of concurrence with negativity and CREN

In this section, we review the basic concepts of concurrence, negativity and CREN. After comparing the difference between negativity and CREN, we give the relations of concurrence, negativity, and CREN.

For any bipartite pure state $|\phi\rangle_{AB}$ in $d\otimes d'$ $(d\leq d')$ quantum system, its concurrence is defined as[8]

$$C(|\phi\rangle_{AB})=\sqrt{2(1-\mathrm{tr}\rho_A^2)}, \tag{1}$$

where $\rho_A=\mathrm{tr}_B(|\phi\rangle_{AB}\langle\phi|)$.

If $d=d'=2$, then

$$C(|\phi\rangle_{AB})=\max\left\{0,\lambda_1-\sum_{i=2}^{4}\lambda_i\right\}, \tag{2}$$

where λ_i are the square roots of the eigenvalues of $\rho\widetilde{\rho}$ in decreasing order with $\widetilde{\rho}=(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$, and σ_y is the Pauli matrix.

For a mixed state ρ_{AB}, the concurrence is defined as

$$C(\rho_{AB})=\min\sum_k p_k C(|\phi_k\rangle_{AB}), \tag{3}$$

where the minimum is taken over all possible pure state decompositions $\rho_{AB}=\sum_k p_k|\phi_k\rangle_{AB}\langle\phi_k|$.

For either pure or mixed states ϱ in the tensor product $H_A\otimes H_B$ of two Hilbert space H_A, H_B for two systems A and B, the negativity is defined as[9]

$$N(\rho)=\|\rho^{T_B}\|_1-1. \tag{4}$$

It is helpful to recall the appearing theorem,[22] which states that for any $m\otimes n$ $(m\leq n)$ mixed state ρ, $C(\rho)$ satisfies

$$\sqrt{\frac{2}{m(m-1)}}(\|\rho^{T_A}\|_1-1)\leq C(\rho). \tag{5}$$

Remark 1 In $2\otimes d$ $(d\geq 2)$ quantum systems, we have $N(\rho)\leq C(\rho)$.

Remark 2 The equality in inequality (5) always holds for pure two-qubit state ρ and some given mixed states.[23]

It is known that positive partial transposition (PPT) gives a separability criterion for the two-qubit system, and it is also a necessary and sufficient condition for non-distillability in the $2\otimes d$ system. However, in a higher-dimensional quantum system rather than $2\otimes 2$ and $2\otimes 3$ quantum systems, there are mixed entangled states with PPT.[16,24] For this case, negativity cannot distinguish PPT bound entangled states from separable states, and thus, negativity itself is not sufficient to be a good entanglement measure even in $2\otimes 2$ quantum system. Fortunately, a generalization of negativity to overcome its lack of separability criterion is CREN,[11] which gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system.

For a bipartite mixed state ρ_{AB}, CREN is defined as

$$N_c(\rho_{AB})\equiv\min\sum_k p_k N(|\phi_k\rangle), \tag{6}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB}=\sum_k p_k|\phi_k\rangle\langle\phi_k|$.

Next, we consider the relationship between CREN and concurrence.[12]

For any bipartite pure state $|\phi_{AB}\rangle$ with Schmidt rank 2 in $2\otimes d$ quantum system, we have

$$N(|\phi\rangle_{AB})=C(|\phi\rangle_{AB}). \tag{7}$$

For any two-qubit mixed state, we have $\rho_{AB}=\sum_k p_k|\phi_k\rangle\langle\phi_k|$,

$$\begin{aligned}N_c(\rho_{AB}) &= \min\sum_k p_k N(|\phi_k\rangle)\\ &= \min\sum_k p_k C(|\phi_k\rangle)\end{aligned}$$

$$= C(\rho_{AB}), \tag{8}$$

where the minima are taken over all pure state decompositions of ρ_{AB}. The first equality follows from the definition of CREN, the second equality follows from Eq. (7).

For the following use, we have to define a dual to CREN, namely CRENoA, similar to the duality between C and CoA. By taking the maximum value of average negativity over all possible pure state decomposition, we can obtain

$$N_c^a(\rho_{AB}) = \max \sum_k p_k N(|\phi\rangle_k), \tag{9}$$

where $\rho_{AB} = \sum_k p_k |\phi_k\rangle\langle\phi_k|$.

In particular, for a two-qubit quantum system,

$$N_c^a(\rho_{AB}) = C^a(\rho_{AB}). \tag{10}$$

Hence, no matter ρ is a pure or mixed state, C and N_c, CoA and CRENoA are identical respectively in two-qubit quantum system. But the situation is not so simple in higher-dimensional quantum systems, we find that there are examples which illustrate that CREN is a stronger entanglement measure than concurrence.

3. Entanglement monogamy of three-qubit quantum pure states

In this section, first, we prove the monogamy equality in terms of CREN for pure states in a three-qubit quantum system. Then, we give another way to explain the monogamy equality, and furthermore compare CREN with original negativity. Finally, we conjecture a monogamy inequality in terms of CREN for pure states in tripartite quantum system.

The tripartite quantum pure states of qubits defined in the Hilbert space $H_A \otimes H_B \otimes H_C$ can be written in standard bases by

$$\begin{aligned} |\Psi\rangle_{AB} &= \sum_{i,j,k=0}^{1} a_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C \\ &= \sum_{i,j,k=0}^{1} a_{ijk} |ijk\rangle_{ABC}. \end{aligned} \tag{11}$$

It is well known that the states defined in Eq. (11) can be divided into six inequivalent classes under stochastic local operations and classical communication (SLOCC). The last two classes are GHZ class state and W class state, which play a special role in quantum information. Taking GHZ state as an example, Zhang *et al.* proposed schemes for concentrating unknown non-maximally tripartite GHZ entangled state,[14] and Lin *et al.* presented schemes for realizing the remote preparation of a GHZ state.[15] Similar to the CKW inequality, we give another inequality for three-qubit quantum system in terms of CREN

$$N_c^2(\rho_{A(BC)}) \geq N_c^2(\rho_{AB}) + N_c^2(\rho_{AC}). \tag{12}$$

Furthermore, we can define an entanglement monotone τ as

$$\tau = N_c^2(\rho_{A(BC)}) - N_c^2(\rho_{AB}) - N_c^2(\rho_{AC}), \tag{13}$$

and it is easy to know that this definition of τ can also distinguish the GHZ class from the W class.

The above results are obvious for CREN in a three-qubit system, and CREN is a better entanglement measure in multi-qubit systems than concurrence. Next, we consider a pure state $|\psi_{ABC}\rangle$ in the three-qubit quantum system.

Theorem 1 For a pure state $|\psi_{ABC}\rangle$ in $2\otimes 2\otimes 2$-dimensional quantum system,

$$N_{ci(jk)}^2 = N_{cij}^2 + (N_{cik}^a)^2, \tag{14}$$

where $i, j \in \{A, B, C\}$, $N_{ci(jk)} = N_c(|\psi_{i(jk)}\rangle\langle\psi_{i(jk)}|)$, $N_{cij} = N_c(\rho_{ij})$, $N_{cij}^a = N_c^a(\rho_{ij})$ and $\rho_{ij} = \mathrm{tr}_k(|\psi\rangle_{ABC}\langle\psi|)$.

Proof For $|\psi_{ABC}\rangle$ is a three-qubit pure state, $N(|\psi\rangle_{AB}) = N_c(|\psi\rangle_{AB})$, and the monogamy equality in terms of concurrence holds,

$$C_{cA(BC)}^2 = C_{cAB}^2 + (C_{cAC}^a)^2. \tag{15}$$

Then, from Eqs. (7) and (10) we obtain $N_{cA(BC)}^2 = N_{cAB}^2 + (N_{cAC}^a)^2$.

Since the dimensions of three subsystems A, B, C are the same, the position of the three subsystems are identical. It is easy to obtain the rest monogamy equalities in terms of CREN. Hence,

$$N_{ci(jk)}^2 = N_{cij}^2 + (N_{cik}^a)^2.$$

In order to expound the monogamy equality of Theorem 1 in detail, we use the similar method in Ref. [19] to define partial tangle. Noting the partial tangle as τ_{ij}, $i, j \in \{A, B, C\}$, we obtain the following equalities

$$\tau_{AB}^2 = N_{cA(BC)}^2 - N_{cAC}^2 = \tau + N_{cAB}^2 = \tau_{BA}^2,$$

$$\tau_{BC}^2 = N_{cB(CA)}^2 - N_{cBA}^2 = \tau + N_{cBC}^2 = \tau_{CB}^2,$$

$$\tau^2_{CA} = N^2_{cC(AB)} - N^2_{cCB} = \tau + N^2_{cCA} = \tau^2_{AC}.$$

Hence

$$\tau^2_{AB} + \tau^2_{BC} + \tau^2_{CA} = 3\tau + N^2_{cAB} + N^2_{cBC} + N^2_{cCA}.$$

Remark 3 From the above equalities, we find that $\tau_{ij} = N_{cij}$ if and only if $\tau = 0$, then $\tau_{ij} = N_{cij}$ can also be regarded as a property of the W class states.

From Theorem 1, it is easy to derive $\tau^2_{ij} = (N^a_{cij})^2$, which can be considered as another definition of N^a_{cij} in three-qubit quantum system.

Remark 4 For a three-qubit pure state, we derive N^a_{cij} as $N^a_{cij} = \tau_{ij}$, $i \in \{A, B, C\}$. Then we obtain

$$\begin{aligned}&(N^a_{cAB})^2 + (N^a_{cBC})^2 + (N^a_{cCA})^2\\ &= 3\tau + N^2_{cAB} + N^2_{cBC} + N^2_{cCA}.\end{aligned}$$

Ou and Fan gave a definition of residual entanglement, namely, three-π,[17] in order to distinguish from three tangle τ. We have known that for a pure state $|\psi\rangle_{ABC}$ in three-qubit quantum system $N_c(|\psi\rangle_{ABC}) = N(|\psi\rangle_{ABC})$, if we substitute CREN with the original negativity, we will obtain the different equalities

$$\pi^2_{AB} = N^2_{A(BC)} - N^2_{AC} = \pi_A + N^2_{AB},$$

$$\pi^2_{BC} = N^2_{B(CA)} - N^2_{BA} = \pi_B + N^2_{BC},$$

$$\pi^2_{CA} = N^2_{C(AB)} - N^2_{CB} = \pi_C + N^2_{CA},$$

where $\pi_A = N^2_{A(BC)} - N^2_{AB} - N^2_{AC}$, $\pi_B = N^2_{B(AC)} - N^2_{BA} - N^2_{BC}$, $\pi_C = N^2_{C(AB)} - N^2_{CA} - N^2_{CB}$.

But

$$\pi^2_{BA} = N^2_{B(AC)} - N^2_{BC} = \pi_B + N^2_{BA},$$

$$\pi^2_{CB} = N^2_{C(BA)} - N^2_{CA} = \pi_C + N^2_{CB},$$

$$\pi^2_{AC} = N^2_{A(CB)} - N^2_{CB} = \pi_A + N^2_{AC}.$$

From the above equalities, we can see

$$\pi^2_{AB} \neq \pi^2_{BA},\ \pi^2_{BC} \neq \pi^2_{CB},\ \pi^2_{CA} \neq \pi^2_{AC},$$

but we can still derive

$$\pi^2_{AB} + \pi^2_{BC} + \pi^2_{CA} = 3\pi + N^2_{AB} + N^2_{BC} + N^2_{CA},$$

$$\pi^2_{BA} + \pi^2_{CB} + \pi^2_{AC} = 3\pi + N^2_{BA} + N^2_{CB} + N^2_{AC}.$$

Therefore, $\pi^2_{AB} + \pi^2_{BC} + \pi^2_{CA} = \pi^2_{BA} + \pi^2_{CB} + \pi^2_{AC}$.

From the above results we find that the CREN can be used more widely than negativity, at least, the monogamy equality holds in terms of the CREN. In order to look into the CREN, we illustrate Theorem 1 in a different way.

Let $f(\rho)$ be the fully entangled fraction of a two-qubit state ρ, that is, $f(\rho) = \max\langle e|\rho|e\rangle$. For $i \in \{A, B, C\}$, let f_i be defined as

$$f_i = \max[\langle 0|U\rho_i U^\dagger|0\rangle f(\rho^\circ_{jk}) + \langle 1|U\rho_i U^\dagger|1\rangle f(\rho^1_{jk})].$$

Utilizing the results in Ref. [19], we can show that

$$\begin{aligned}f_i =\ & \frac{1}{2}\max_U[\langle 0|U\rho_i U^\dagger|0\rangle(1 + N_c(\rho^\circ_{jk}))\\ &+ \langle 1|U\rho_i U^\dagger|1\rangle(1 + N_c(\rho^1_{jk}))],\end{aligned}$$

and

$$N^2_{ci(jk)} - N^2_{c(ik)} = (2f_k - 1)^2 = \tau^2_{ij},$$

$$\begin{aligned}2f_C - 1 =\ & \max_U[\langle 0|U\rho_C U^\dagger|0\rangle N_c(\rho^\circ_{AB})\\ &+ \langle 1|U\rho_C U^\dagger|1\rangle N_c(\rho^1_{AB})].\end{aligned} \tag{16}$$

Since the quantity on the right-hand side of Eq. (16) is the maximal average of the concurrences of all pure-state ensembles on system AB induced by all corresponding local projective measurements on system C, we have $\tau_{AB} = 2f_C - 1 = N^a_{cAB}$. Hence we obtain the monogamy equality for three-qubit pure states, that is

$$N^2_{cA(BC)} = N^2_{cAC} + (N^a_{cAB})^2. \tag{17}$$

In the sequel we will show that equation (17) holds for a pure state in $2 \otimes 2 \otimes d$ quantum system as well.

Let $|\psi\rangle_{ABC}$ be a $2 \otimes 2 \otimes d$-dimensional quantum pure state, $\rho_{AB} = \mathrm{tr}_C(|\psi\rangle_{AB}\langle\psi|)$ is a $2 \otimes 2$ mixed state, then CRENoA is defined as

$$N^a_c(|\psi_{ABC}\rangle) = \max \sum p_i N(|\phi_i\rangle_{AB}) = N^a_c(\rho_{AB}).$$

With this definition, we can obtain the following theorem.

Theorem 2 For a pure state $|\psi\rangle_{ABC}$ in $2 \otimes 2 \otimes d$ quantum system,

$$N^2_{cA(BC)} = N^2_{cAC} + (N^a_{cAB})^2, \tag{18}$$

where $N_{cA(BC)} = N_c(|\psi_{A(BC)}\rangle\langle\psi_{A(BC)}|)$, $N_{cAC} = N_c(\rho_{AC})$, $N^a_{cAB} = N^a_c(\rho_{AB})$ and $\rho_{ij} = \mathrm{tr}_k(|\psi\rangle_{ABC}\langle\psi|)$, $i, j, k \in \{A, B, C\}$.

Proof For a pure state $|\psi\rangle_{ABC}$, Yu and Song[20] have shown that

$$(C^a(\rho_{AB}))^2 = C^2(\rho_{AB}) + \tau(\rho_{AB}). \tag{19}$$

From Eqs. (7), (10) and (19) we obtain

$$(N^a_c(\rho_{AB}))^2 = N^2_c(\rho_{AB}) + \tau(\rho_{AB}). \tag{20}$$

With the definition of $\tau(\rho_{AB}) = N_c^2(\rho_{A(BC)}) - N_c^2(\rho_{AB}) - N_c^2(\rho_{AC})$, then we derive

$$N_{cA(BC)}^2 = N_{cAC}^2 + (N_{cAB}^a)^2.$$

Remark 5 In the above theorem, the dimensions of subsystems B and C are different, so we can not obtain $N_{cA(BC)}^2 = N_{cAB}^2 + (N_{cAC}^a)^2$, which can be seen in the following example.

Let us consider a pure state $|\psi\rangle_{ABC}$ in $2 \otimes 2 \otimes 3$ quantum system[13]

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(\sqrt{2}|111\rangle + |123\rangle + |213\rangle + \sqrt{2}|222\rangle).$$

After calculation, we obtain $N_{cA(BC)} = N_{A(BC)} = 1$, $(N_{cAB}^a)^2 = 1/9$, $N_{cAB}^2 = 0$, $N_{cAC}^2 = (N_{cAC}^a)^2 = 8/9$.

So, we derive $N_{cA(BC)}^2 = (N_{cAB}^a)^2 + N_{cAC}^2$ and $N_{cA(BC)}^2 \geq N_{cAB}^2 + (N_{cAC}^a)^2$.

Remark 6 For a pure state in higher-dimensional three-partite quantum system, we conjecture that the monogamy inequality in terms of CREN and CRENoA holds, that is

$$N_{cA(BC)}^2 \geq N_{cAB}^2 + (N_{cAC}^a)^2. \tag{21}$$

4. Examples to illustrate monogamy relations in higher-dimensional system

In this section, we give some examples to show that the monogamy equality also holds for a qudit tripartite pure state, and the conjecture in Remark 6 is true for a pure state in a higher-dimensional quantum system.

Example 1 Let us review the 3-qudit generalized W-class state

$$|w_3^d\rangle_{ABC} = \sum_{i=1}^{d-1}(a_{1i}|i00\rangle + a_{2i}|0i0\rangle + a_{3i}|00i\rangle)$$

with $\sum_{i=1}^{d-1}(|a_{1i}|^2 + |a_{2i}|^2 + |a_{3i}|^2) = 1$.

Then we consider a partially coherent superposition of a generalized W-class state and $|0\rangle^{\otimes 3}$ is given as

$$\begin{aligned}\rho_{ABC} = {} & p|w_3^d\rangle\langle w_3^d| + (1-p)|000\rangle\langle 000| \\ & + \lambda\sqrt{p(1-p)}(|w_3^d\rangle\langle 000| + |000\rangle\langle w_3^d|),\end{aligned}$$

where λ is the degree of coherence with $0 \leq \lambda \leq 1$. It has been obtained that[12]

$$N_c(\rho_{A(BC)}) = \min \sum p_i N(|\phi_i\rangle) = 2p\sqrt{A(1-A)},$$

where the minimum is taken over all possible pure state decompositions of $\rho_{A_1(A_2\cdots A_n)} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, and $A = 1 - \sum_{j=1}^{d-1}|a_{1j}|^2$.

Similarly, we obtain

$$N_c(\rho_{AB}) = 2p\sqrt{(1-A)(A-A_2)} = N_c^a(\rho_{AB}),$$

$$N_c(\rho_{AC}) = 2p\sqrt{(1-A)(A-A_3)} = N_c^a(\rho_{AC}),$$

where $A_2 = 1 - \sum_{j=1}^{d-1}(|a_{1j}|^2 + |a_{2j}|^2)$ and $A_3 = 1 - \sum_{j=1}^{d-1}(|a_{1j}|^2 + |a_{3j}|^2)$.

Thus

$$\begin{aligned}& (N_c^a(\rho_{AB}))^2 + N_c^2(\rho_{AC}) \\ = {} & 4p^2(1-A)(A-A_2) + 4p^2(1-A)(A-A_3) \\ = {} & 4p^2(1-A)(2A-A_2-A_3) \\ = {} & 4p^2A(1-A) = N_c^2(\rho_{A(BC)}), \\ & (N_c^a(\rho_{AB}))^2 + N_c^2(\rho_{AC}) \\ = {} & N_c^2(\rho_{A(BC)}).\end{aligned}$$

Example 2 (Ref. [10]) Let $|\psi\rangle_{ABC}$ in a $3\otimes 3\otimes 3$ quantum system such that

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |312\rangle).$$

For $\rho = |\psi\rangle_{ABC}\langle\psi|$, then we obtain

$$N(\rho_{A(BC)}) = N_c(\rho_{A(BC)}) = (\mathrm{tr}\sqrt{\rho_A})^2 - 1 = 2. \tag{22}$$

To calculate $N_c(\rho_{AB})$ and $N_c^a(\rho_{AB})$, we consider the reduced density matrix ρ_{AB} whose spectral decomposition is

$$\rho_{AB} = \frac{1}{3}(|x\rangle_{AB}\langle x| + |y\rangle_{AB}\langle y| + |z\rangle_{AB}\langle z|),$$

where

$$|x\rangle_{AB} = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle),$$

$$|y\rangle_{AB} = \frac{1}{\sqrt{2}}(|31\rangle - |13\rangle),$$

$$|z\rangle_{AB} = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle).$$

By a straightforward calculation, it can be shown that for arbitrary pure state $|\phi\rangle_{AB} = c_1|x\rangle_{AB} + c_2|y\rangle_{AB} + c_3|z\rangle_{AB}$ with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, their reduced matrix $\rho_A = \mathrm{tr}_B|\phi\rangle_{AB}\langle\phi|$ has the same spectrum $\{1/2, 1/2, 0\}$. Thus,

$$N(|\phi\rangle_{AB}) = (\mathrm{tr}\sqrt{\rho_A})^2 - 1 = 1,$$

for any $|\phi\rangle_{AB}$ which is a superposition of $|x\rangle_{AB}$, $|y\rangle_{AB}$ and $|z\rangle_{AB}$. By the Hughston–Jozsa–Wootters theorem,[21] any pure state ensemble of ρ_{AB} can be

realized as a superposition of $|x\rangle_{AB}$, $|y\rangle_{AB}$ and $|z\rangle_{AB}$, thus

$$\begin{aligned} N_c(\rho_{AB}) &= \min_{\sum_k p_k|\phi_k\rangle\langle\phi_k|=\rho_{AB}} \sum_k p_k N(\phi_k) \\ &= \frac{1}{3}(N(|x\rangle_{AB}) + N(|y\rangle_{AB}) + N(|z\rangle_{AB})) \\ &= 1, \end{aligned} \tag{23}$$

$$\begin{aligned} N_c^a(\rho_{AB}) &= \max_{\sum_k p_k|\phi_k\rangle\langle\phi_k|=\rho_{AB}} \sum_k p_k N(|\phi_k\rangle) \\ &= 1. \end{aligned} \tag{24}$$

Since $|\psi_{ABC}\rangle$ is an anti-symmetric pure state, we also have a similar result for N_{cAC} and N_{cAC}^a, that is

$$N_c(\rho_{AC}) = N_c^a(\rho_{AC}) = 1. \tag{25}$$

From Eq. (22) together with Eqs. (23)–(25) we have

$$N_c^2(\rho_{A(BC)}) \geq N_c^2(\rho_{AB}) + (N_c^a(\rho_{AC}))^2, \tag{26}$$

$$N_c^2(\rho_{A(BC)}) \geq (N_c^a(\rho_{AB}))^2 + N_c^2(\rho_{AC}). \tag{27}$$

This example shows that the monogamy equality becomes an inequality in three-qutrit quantum system. We conjecture that the inequality always holds in higher-dimensional quantum system.

5. Conclusions

In this paper, we reviewed the entanglement measure and compared the difference between negativity and CREN, then we obtained some relations among them. By taking advantage of CREN, we showed that the monogamy equality holds for tripartite quantum pure states. Furthermore, we see that CREN is a stronger entanglement measure than concurrence. We speculate that a monogamy inequality in terms of CREN holds in higher-dimensional quantum systems. We hope our findings will shed new light on the characterization of entanglement of tripartite pure states.

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