Bound states of the Schrödinger equation for the Pöschl–Teller double-ring-shaped Coulomb potential*

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Pöschl–Teller double-ring-shaped Coulomb (PTDRSC) potential, the Coulomb potential surrounded by Pöschl–Teller and double-ring-shaped inversed square potential, is put forward. In spherical polar coordinates, PTDRSC potential has supersymmetry and shape invariance in φ, θ and r coordinates. By using the method of supersymmetry and shape invariance, exact bound state solutions of Schrödinger equation with PTDRSC potential are presented. The normalized φ, θ angular wave function expressed in terms of Jacobi polynomials and the normalized radial wave function expressed in terms of Laguerre polynomials are presented. Energy spectrum equations are obtained. Wave function and energy spectrum equations of the system are related to three quantum numbers and parameters of PTDRSC potential. The solutions of wave functions and corresponding eigenvalues are only suitable for the PTDRSC potential.

Keywords: Pöschl–Teller double-ring-shaped Coulomb potential, supersymmetry and shape invariance, Schrödinger equation, bound states

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1. Introduction

The supersymmetry approach originating from the factorization in non-relativistic quantum mechanics has been extensively applied to deal with the solvable Hamiltonian where the concept of shape invariance plays a fundamental role. It is well known that if a potential satisfies the shape-invariance condition, the corresponding Hamiltonian can be solved exactly. Exactly solvable potentials in non-relativistic quantum fall within distinct classes of shape invariant potentials.[1−5] With the ideas of supersymmetry and shape invariance, a number of potential problems can be solved exactly. Although most of these potentials are either one-dimensional or central, there are some non-central separable potentials. Those non-centrally potentials have supersymmetry and shape invariance and can be exactly solved in terms of the supersymmetric quantum mechanics and shape invariance.

Pöschl–Teller double-ring-shaped Coulomb (PTDRSC) potential, the Coulomb potential surrounded by Pöschl–Teller and double-ring-shaped inversed square potential, is put forward in the paper. In spherical coordinates, PTDRSC potential, one of the non-centrally potentials, is defined by the following expression

$$\begin{aligned} &V(r,\theta,\varphi) \\ &= -\frac{\beta}{r} + \frac{1}{r^2}\left[\frac{b}{\sin^2\theta} + \frac{A(A-1)}{\cos^2\theta}\right] \\ &\quad + \frac{1}{r^2\sin^2\theta}\left[\frac{\alpha^2 D(D-1)}{\sin^2\alpha\varphi} + \frac{\alpha^2 C(C-1)}{\cos^2\alpha\varphi}\right], \quad (1) \end{aligned}$$

where $A, C, D > 1, \beta > 0, b \geq 0, \alpha = 1, 2, 3\ldots$ are positive real parameters of the potential. When $A = C = D = 1$ and $b = 0$, the PTDRSC potential reduces to the Coulomb potential, which is one of the most important models in classical and quantum physics, and is widely used in many branches. When $A = C = D = 1$, the PTDRSC potential reduces to Hartmann potential, which was introduced in quantum chemistry to describe ring-shaped molecules like benzene.[6,7] Bound states of Schrödinger equation for Hartmann potential have been studied.[8,9] When $C = D = 1$, the PTDRSC potential reduces to the double-ring-shaped Coulomb potential. Bound states of Schrödinger and Klein–Gordon equation for double-ring-shaped oscillator potential have been investigated.[10−14] The expressions for the energy eigenvalues and the eigenfunctions of the other noncentral potential have also been given.[15] In this

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paper, we show that PTDRSC potentials have supersymmetry and shape invariance in φ, θ and r coordinates, and acquire the bound states of Schrödinger equation with PTDRSC potentials. The normalized wave function and the exact energy spectrum equations are also presented.

This paper is organized as follows. In Section 2, the separation of variables is carried out for the Schrödinger equation with PTDRSC potential in spherical coordinates. In Sections 3 and 4, we obtain the normalized φ, θ angular wave function expressed in terms of the Jacobi polynomials. The normalized radial wave function and exact energy spectrum equation of bound states are presented in Section 5.

2. Separation of variables for the Schrödinger equation

In spherical polar coordinates, the Schrödinger equation is given by ($\hbar = 2\mu = 1$)

$$-\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}+\cot\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+V\psi=E\psi. \tag{2}$$

The most general potential for which the Schrödinger equation is separable is given[5] by

$$V(r,\theta,\varphi)=V(r)+\frac{V(\theta)}{r^2}+\frac{V(\varphi)}{r^2\sin^2\theta}, \tag{3}$$

where $V(r)$, $V(\theta)$ and $V(\varphi)$ are arbitrary functions of certain arguments. For PTDRSC potential, there are

$$V(r)=-\frac{\beta}{r},\quad V(\theta)=\frac{b}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta},$$
$$V(\varphi)=\frac{\alpha^2D(D-1)}{\sin^2\alpha\varphi}+\frac{\alpha^2C(C-1)}{\cos^2\alpha\varphi}. \tag{4}$$

It is convenient to write the wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}\frac{H(\theta)}{(\sin\theta)^{1/2}}K(\varphi). \tag{5}$$

Substituting Eqs. (3), (4) and (5) into Eq. (2), and using the standard procedure of separation of variables, one obtains the following equations for $K(\varphi)$, $H(\theta)$ and $R(r)$

$$-\frac{\mathrm{d}^2K(\varphi)}{\mathrm{d}\varphi^2}+\left[\frac{\alpha^2D(D-1)}{\sin^2\alpha\varphi}+\frac{\alpha^2C(C-1)}{\cos^2\alpha\varphi}\right]K(\varphi) = m^2K(\varphi), \tag{6}$$

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2}+\left[\frac{b+m^2-1/4}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta}\right]H(\theta) = l^2H(\theta), \tag{7}$$

$$-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\left[-\frac{\beta}{r}+\frac{l'(l'+1)}{r^2}\right]R(r)=ER(r),\quad l'=l-\frac{1}{2}. \tag{8}$$

3. Supersymmetry and shape invariance for φ coordinate

In Eq. (6), whose ground state function and eigenvalue are given by

$$K_0(\varphi)=(\sin\alpha\varphi)^D(\cos\alpha\varphi)^C,\quad m_0^2=\alpha^2(C+D)^2. \tag{9}$$

The corresponding superpotential related to the ground state function is

$$W(x)=-K_0'(x)/K_0(x). \tag{10}$$

So the corresponding superpotential is given by

$$W(\varphi;C,D,\alpha)=\alpha C\tan\alpha\varphi-\alpha D\cot\alpha\varphi. \tag{11}$$

Then, we can obtain the partner potentials as

$$\begin{aligned}&V_+(\varphi,A,B)\\ &=W^2+W'\\ &=-\alpha^2(C+D)^2+\alpha^2C(C+1)\sec^2\alpha\varphi\\ &\quad+\alpha^2D(D+1)\csc^2\alpha\varphi,\end{aligned} \tag{12}$$

$$\begin{aligned}&V_-(\varphi,A,B)\\ &=W^2-W'\\ &=-\alpha^2(C+D)^2+\alpha^2C(C-1)\sec^2\alpha\varphi\\ &\quad+\alpha^2D(D-1)\csc^2\alpha\varphi.\end{aligned} \tag{13}$$

It can be deduced that

$$\begin{aligned}&V_+(\varphi;C,D,\alpha)\\ &=V_-(\varphi;C+1,D+1,\alpha)\\ &\quad+R(C+1,D+1,\alpha),\end{aligned} \tag{14}$$

where

$$\begin{aligned}&R(C+1,D+1,\alpha)\\ &=\alpha^2(C+D+2)^2-\alpha^2(C+D)^2\ ,\end{aligned} \tag{15}$$

hence, the shape invariance condition is satisfied for φ coordinate, and the complete eigenvalue spectrum is given by

$$m^{2-}=\sum_{k=1}^{t_0}R(C+k,D+k)$$

$$= \alpha^2(C+D+2t_0)^2 - \alpha^2(C+D)^2, \quad t_0 = 1,2,3\cdots \tag{16}$$

Thus, one can have the value of m in Eq. (6) as follows:

$$m^2 = \alpha^2(C+D+2t)^2, \quad t = 0,1,2,3\ldots . \tag{17}$$

Notice that in general there is no degradation in m. In principle, we can use supersymmetric quantum mechanics and shape invariance to find the eigenfunction of Eq. (6), here an alternative method is used. Letting

$$K(\varphi) = (\sin\alpha\varphi)^D(\cos\alpha\varphi)^C Y(\varphi), \tag{18}$$

and substituting Eqs. (17) and (18) into Eq. (6), we obtain

$$\frac{\mathrm{d}^2Y(\varphi)}{\mathrm{d}\varphi^2} + 2\alpha(D\cot\alpha\varphi - C\tan\alpha\varphi)\frac{\mathrm{d}Y(\varphi)}{\mathrm{d}\varphi} + 4t\alpha^2(C+D+t)Y(\varphi) = 0. \tag{19}$$

When a new variable $x = \sin^2\alpha\varphi$ is introduced, a Jacobi differential equation is obtained

$$x(1-x)\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} + \left[\left(D+\frac{1}{2}\right) - (C+D+1)x\right]\frac{\mathrm{d}Y}{\mathrm{d}x} + t(C+D+t)Y = 0. \tag{20}$$

The solution of Eq. (20) is

$$Y(x) = \mathrm{F}(-t, p_1+t, \gamma_1, x), \quad p_1 = C+D, \gamma_1 = D+1/2, \tag{21}$$

where $\mathrm{F}(-n,\beta,\gamma,z)$ is the Jacobi polynomials,[16] defined as

$$\mathrm{F}(-n,\beta,\gamma,z) = \sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{k!(\gamma)_k}z^k. \tag{22}$$

The φ angular wave function can be written as

$$K(\varphi) = N_t(\sin\alpha\varphi)^D(\cos\alpha\varphi)^C\mathrm{F}(-t, p_1+t, \gamma_1, x), \tag{23}$$

where N_t is the normalization constant of the angular wave function. According to the normalized condition

$$\int_0^{2\pi} K^2(\varphi)\,\mathrm{d}\varphi = 4\alpha\int_0^{\pi/(2\alpha)} K^2(\varphi)\,\mathrm{d}\varphi = 1, \tag{24}$$

and using the relation of the orthogonality of the Jacobi polynomials[16]

$$\int_0^1 z^{\gamma-1}(1-z)^{p-\gamma}[\mathrm{F}(-n,p+n,\gamma,z)]^2\,\mathrm{d}z = \frac{n!}{p+2n}\cdot\frac{\Gamma^2(\gamma)\Gamma(p+n-\gamma+1)}{\Gamma(p+n)\Gamma(\gamma+n)}, \tag{25}$$

we have

$$N_t = \sqrt{\frac{1}{2}\cdot\frac{p_1+2t}{t!}\cdot\frac{\Gamma(p_1+t)\Gamma(\gamma_1+t)}{\Gamma^2(\gamma_1)\Gamma(p_1-\gamma_1+t+1)}}. \tag{26}$$

So the normalized φ angular wave function is

$$K_t(\varphi) = \sqrt{\frac{1}{2}\cdot\frac{C+D+2t}{t!}\cdot\frac{\Gamma(C+D+t)\Gamma(D+t+1/2)}{\Gamma^2(D+1/2)\Gamma(C+t+1/2)}} \times (\sin\alpha\varphi)^D(\cos\alpha\varphi)^C\mathrm{F}(-t, C+D+t, D+1/2, \sin^2\alpha\varphi). \tag{27}$$

4. Supersymmetry and shape invariance for θ coordinate

Letting $B = \sqrt{b+m^2}+1/2 = \sqrt{b+(C+D+2t)^2}+1/2$, equation (7) becomes

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + \left[\frac{B(B-1)}{\sin^2\theta} + \frac{A(A-1)}{\cos^2\theta}\right]H(\theta) = l^2H(\theta), \tag{28}$$

whose ground state function and eigenvalue are given by

$$H_0(\theta) = (\sin\theta)^B(\cos\theta)^A, \quad l_0^2 = (A+B)^2. \tag{29}$$

The corresponding superpotential is given by

$$W(\theta; A, B) = A\tan\theta - B\cot\theta. \tag{30}$$

Then, we can obtain the partner potentials

$$V_+(\theta,A,B) = W^2 + W' = -(A+B)^2 + A(A+1)\sec^2\theta + B(B+1)\csc^2\theta, \tag{31}$$

$$V_-(\theta,A,B) = W^2 - W' = -(A+B)^2 + A(A-1)\sec^2\theta + B(B-1)\csc^2\theta. \tag{32}$$

It is deduced that

$$\begin{aligned} & V_+(\theta, A, B) \\ &= V_-(\theta, A+1, B+1) + R(A+1, B+1), \end{aligned} \quad (33)$$

where

$$R(A+1, B+1) = (A+B+2)^2 - (A+B)^2, \quad (34)$$

hence, the shape invariance condition is satisfied for θ coordinates, and the complete eigenvalue spectrum is given by

$$\begin{aligned} & l^{2-} \\ &= \sum_{k=1}^{s_0} R(A+k, B+k) \\ &= (A+B+2s_0)^2 - (A+B)^2, s_0 = 1, 2, 3 \ldots . \end{aligned} \quad (35)$$

Thus, one can have the value of l in Eq. (28) as follows:

$$\begin{aligned} & l^2 = (A+B+2s)^2, \quad l = A+B+2s, \\ & s = 0, 1, 2, 3 \ldots . \end{aligned} \quad (36)$$

Notice that in general there is no degradation in l except in the special case when $b = 0$. Letting

$$H(\theta) = (\sin\theta)^B (\cos\theta)^A Y(\theta), \quad (37)$$

when a new variable $x = \sin^2\theta$ is introduced, a Jacobi differential equation is obtained[16]

$$\begin{aligned} x(1-x)\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + [(B+1/2) - (A+B+1)x]\frac{\mathrm{d}Y}{\mathrm{d}x} \\ + \ s(A+B+s)Y = 0. \end{aligned} \quad (38)$$

The solution of Eq. (38) is Jacobi polynomials

$$\begin{aligned} & Y(x) = \mathrm{F}(-s, p_2+s, \gamma_2, x), \\ & p_2 = A+B, \quad \gamma_2 = B+1/2, \end{aligned} \quad (39)$$

where $\mathrm{F}(\alpha, \beta, \gamma, x)$ is the hypergeometric function. The angular wave function can be written as

$$H(\theta) = N_s (\sin\theta)^B (\cos\theta)^A \mathrm{F}(-s, p_2+s, \gamma_2, x), \quad (40)$$

where N_{sl} is the normalization constant of θ angular wave function. According to the normalized condition

$$\int_0^{\pi} H^2(\theta)\mathrm{d}\theta = 2\int_0^{\pi/2} H^2(\theta)\mathrm{d}\theta = 1, \quad (41)$$

and using Eq. (25), we have

$$N_s = \sqrt{\frac{p_2+2s}{s!} \cdot \frac{\Gamma(p_2+s)\Gamma(\gamma_2+s)}{\Gamma^2(\gamma_2)\Gamma(p_2-\gamma_2+s+1)}}. \quad (42)$$

So the normalized θ angular wave function is

$$H_s(\theta) = \sqrt{\frac{A+B+2s}{s!} \cdot \frac{\Gamma(A+B+s)\Gamma(B+s+1/2)}{\Gamma^2(B+1/2)\Gamma(A+s+1/2)}} (\sin\theta)^B (\cos\theta)^A \mathrm{F}(-s, A+B+s, B+1/2, \sin^2\theta). \quad (43)$$

5. Supersymmetry and shape invariance for r coordinate and the energy spectrum

The ground state function and eigenvalue of Eq. (8), the radial part of the Schrödinger equation can be written as

$$\begin{aligned} & R_0(r) = r^{l'+1} \exp\left[-\frac{\beta r}{2(l'+1)}\right], \\ & E_0 = -\frac{\beta^2}{4(l'+1)^2}. \end{aligned} \quad (44)$$

The corresponding superpotential is

$$W(r) = \frac{\beta}{2(l'+1)} - \frac{l'+1}{r}. \quad (45)$$

Then, the partner potential is obtained as follows:

$$\begin{aligned} V_+(r, l') &= W^2 + W' \\ &= -\frac{\beta}{r} + \frac{(l'+1)(l'+2)}{r^2} + \frac{\beta^2}{4(l'+1)^2}, \end{aligned} \quad (46)$$

$$\begin{aligned} V_-(r, l') &= W^2 - W' \\ &= -\frac{\beta}{r} + \frac{l'(l'+1)}{r^2} + \frac{\beta^2}{4(l'+1)^2}. \end{aligned} \quad (47)$$

It is deduced that

$$\begin{aligned} & V_+(r, l') = V_-(r, l'+1) + R(l'+1), \\ & R(l'+1) = \frac{\beta^2}{4}\left[\frac{1}{(l'+1)^2} - \frac{1}{(l'+2)^2}\right], \end{aligned} \quad (48)$$

hence, the shape invariance condition is satisfied for r coordinate, and the complete eigenvalue spectrum is given by

$$E_{n_r}^- = \sum_{k=1}^{n_{r0}} R(l'+k)$$

$$= \frac{\beta^2}{4}\left[\frac{1}{(l'+1)^2} - \frac{1}{(n_{r0}+l'+1)^2}\right], \quad n_{r0} = 1, 2, 3, \ldots \tag{49}$$

Thus, the energy spectrum equation is presented as follows:

$$E_{n_r} = -\frac{\beta^2}{4(n_r+l'+1)^2}, \quad n_r = 0, 1, 2, 3, \ldots \tag{50}$$

Introducing a new variable $y = \beta r/(n_r + l' + 1)$, and letting

$$R(r) = y^{l'+1} \cdot \exp(-y/2) \cdot \mathrm{L}_{n_r}(y), \tag{51}$$

substituting Eqs. (50) and (51) into Eq. (8), we obtain

$$y\frac{\mathrm{d}^2\mathrm{L}_{n_r}(y)}{\mathrm{d}y^2} + [(2l'+1)+1-y]\frac{\mathrm{d}\mathrm{L}_{n_r}(y)}{\mathrm{d}y} + n_r\mathrm{L}_{n_r}(y) = 0, \tag{52}$$

which is just the well-known differential equation satisfied by Laguerre polynomials; and the analytic solutions to Eq. (52) can be expressed as the Laguerre polynomials[16]

$$\mathrm{L}_{n_r}(y) = \mathrm{L}_{n_r}^{2l'+1}(y). \tag{53}$$

The radial wave function can be obtained as follows:

$$R_{n_r l'}(r) = N_{n_r l'} y^{l'+1} \cdot \exp(-y/2) \cdot \mathrm{L}_{n_r}^{2l'+1}(y), \tag{54}$$

where $N_{n_r l'}$ is normalization constant of the radial wave function. Using the relation of the orthogonality of generalized Laguerre polynomials[16]

$$\int_0^\infty z^{2l'+2}\,\mathrm{e}^{-z}\mathrm{L}_{n_r}^{2l'+1}(z)\mathrm{L}_{n_r}^{2l'+1}(z)\mathrm{d}z = \frac{2(n_r+l'+1)}{n_r!}\Gamma(n_r+2l'+2), \tag{55}$$

and the normalized condition

$$\int_0^\infty R^2(r)\mathrm{d}r = 1, \tag{56}$$

and Eq. (54), one derives the normalization constant

$$N_{n_r l'} = \frac{1}{n_r+l'+1}\sqrt{\frac{\beta}{2}\cdot\frac{n_r!}{\Gamma(n_r+2l'+2)}}. \tag{57}$$

So the normalized radial wave function of bound states of Schrödinger equation for PTDRSC potential is

$$R_{n_r l'}(r) = \frac{1}{n_r+l'+1}\sqrt{\frac{\beta}{2}\cdot\frac{n_r!}{\Gamma(n_r+2l'+2)}}y^{l'+1} \cdot \exp\left(\frac{-y}{2}\right)\cdot\mathrm{L}_{n_r}^{2l'+1}(y). \tag{58}$$

Thus the total eigenfunction for the noncentral potential is given by the form of Eq. (5), with $K(\varphi)$, $H(\theta)$, $R(r)$ given by Eqs. (27), (43) and (58) respectively.

6. Discussion

When $C = 1$, $D = 1$ and $\alpha = 1$, then from Eq. (4) $V(\varphi) = 0$. If particle is in the angular free potential, equation (6) becomes

$$-\frac{\mathrm{d}^2K(\varphi)}{\mathrm{d}\varphi^2} = m^2K(\varphi). \tag{59}$$

According to the periodic boundary condition, the φ angular wave function is

$$K_m(\varphi) = \frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}, \quad |m| = 0, 1, 2, \ldots \tag{60}$$

In the ground state, $m = 0$, there is

$$K_{m0}(\varphi) = \frac{1}{\sqrt{2\pi}}. \tag{61}$$

On the other hand, from Eq. (27), the φ angular wave function of Eq. (59) can be written as

$$K_t(\varphi) = \sqrt{\frac{1}{2}\cdot\frac{2+2t}{t!}\cdot\frac{\Gamma(2+t)}{\Gamma^2(1+1/2)}} \times \sin\varphi\cos\varphi\mathrm{F}(-t, 2+t, 3/2, \sin^2\varphi), \quad t = 0, 1, 2, 3\ldots. \tag{62}$$

In the ground state, $t = 0$, there is

$$K_{t0}(\varphi) = \frac{2\sin\varphi\cos\varphi}{\sqrt{\pi}}. \tag{63}$$

It is obvious that equation (61) is not the special case of Eq. (63), and equation (27) cannot reduce to Eq. (60). This is because equation (27) is given by the condition of $C, D > 1$. In the condition of $C, D > 1$, the corresponding potential $V(\varphi)$ is a four-infinite well (see Fig. 1) and the φ angular wave function must be equal to zero at $\varphi = 0, \pi/2, \pi, 3\pi/2$. In angular free particle potential, angular wave function must not equal zero at $\varphi \in [0, 2\pi]$, and this generates the difference of the wave function and eigenvalues. Based on the similar reason, $H_s(\theta)$ in Eq. (43) and $R_{n_r l'}(r)$ in Eq. (58) cannot degenerate when $A = 1$, $C = 1$ and $D = 1$.

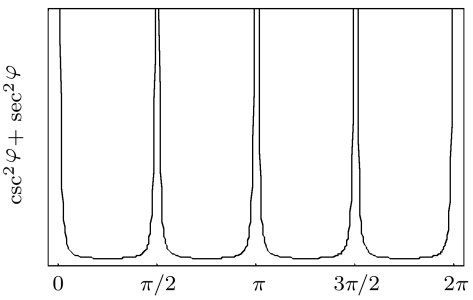


Fig. 1. The plot of a four-infinite well of $V(\varphi) = \csc^2\varphi + \sec^2\varphi$.

Notice that the wave functions (27), (43), (58) and the corresponding eigenvalues (17), (36) and (50) are only suitable for the PTDRSC potential, and cannot be applied to double ring-shaped Coulomb potential, Hartmann potential and Coulomb potential.

7. Conclusions

We have shown that in the spherical coordinates, some non-centrally separable potential have supersymmetry and shape invariance in φ, θ and r coordinates. We have also obtained the exact bound state solutions of Schrödinger equation for PTDRSC by using the method of supersymmetry and shape invariance. The normalized wave functions expressed in terms of the product of Jacobi polynomials and Laguerre polynomials are presented. Wave function and energy spectrum equations of a system are related to three quantum numbers (t, s and n_r) and parameters of PTDRSC potential (α, β, A, B, C and D). Unfortunately, the solutions of wave functions and corresponding eigenvalues are only suitable for the PTDRSC potential.

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