Tripartite states Bell-nonlocality sudden death in a quantum-critical environment*

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We demonstrate that multipartite Bell-inequalities violations can be fully destroyed in finite time in three-qubit states under a quantum-critical environment, which is an Ising model in a transverse field. We use the Mermin–Ardehali–Belinksii–Klyshko (MABK) inequality to detect the degree of nonlocality as measured by the extent of their violations. The effects of system-environment couplings, the size of degrees of freedom of environment and the strength of transverse field on the Bell-inequality violations are given for two different initial states, namely, the W class and GHZ class states. The results indicate that the Bell-inequality violations of the tripartite states will be completely destroyed by the decoherence under certain conditions.

Keywords: Bell-type inequality, Bell-nonlocality sudden death, quantum-critical environment

PACC: 0365, 0367

1. Introduction

It has been increasingly recognized that the rates of loss of joint-state coherence and of entanglement, which are two fundamental characteristics of quantum states, may differ within a composite system subject to the action of environment.[1−7] Recently, the dynamics of entanglement has received much attention since the work of Yu and Eberly,[2,3] in which the entanglement between the two particles coupled with two independent environments became completely vanishing in a finite time, called entanglement sudden death (ESD). However, the study of ESD is limited to bipartite systems previously because properly quantifying multipartite entanglement, especially for mixed states, is not clear for systems of more than two qubits.[8,9] Very recently, Jaeger and Ann demonstrated the Bell-nonlocality sudden death (BNSD) of the tripartite system by discussing Bell-inequality violations.[10] It is shown that multipartite Bell-inequality violations can be fully destroyed in finite time in three-qubit systems subject to the multilocal noise environment. The extension of the consideration of the sudden death of nonlocal properties is significant because tripartite systems can exhibit fundamental characteristics which do not exist in bipartite systems. Due to its intrinsic and practical interests, BNSD has attracted much attention.[11−13]

In the real world, quantum information processing will inevitably be affected by the decoherence induced by the external environment.[14,15] In most of the previous studies, uncorrelated environments were usually considered, and modeled by a reservoir consisting of harmonic oscillators or spin chains. However, particles of the environment always interact with each other. Consequently, a problem arises: what is the effect of decoherence induced by a correlated environment on quantum entanglement? To answer this question, researchers have discussed the effect of decoherence induced by a correlated environment on quantum entanglement.[16,17] In Ref. [16], the authors investigated the disentanglement of a bipartite system in which the two parties couple to the environment homogeneously; the researchers in Ref. [17] analysed the entanglement evolution of three-qubit states in a quantum-critical environment. They chose a correlated spin chain, the Ising model in a transverse field, as the surrounding system. However, for the three-qubit system in a correlated environment, the nonlocal behaviour which can be measured by the extent of the violation of the Bell-inequality is still an open issue.

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2. Theoretical model

We consider a three-qubit system coupled to an environment consisting a transverse Ising model which exhibits a quantum transition. The corresponding Hamiltonian is given by[17]

$$H = \sum_{i=-N_0}^{N_0} \sigma_i^x \sigma_{i+1}^x + \left(\omega + \sum_{j=1}^{3} g_j s_j^z\right) \sum_{i=-N_0}^{N_0} \frac{\sigma_i^z}{2}, \quad (1)$$

where ω quantifies the strength of the transverse field, g_j $(j = 1, 2, 3)$ indicates the coupling constants between the environment and the three qubits s_j $(j = 1, 2, 3)$. σ_i^α $(\alpha = x, y, z)$ are the Pauli operators representing the i-th site in the lattice. Obviously, the total site of the spin chain is $N_t = 2N_0 + 1$.

In order to obtain the time evolution of the density matrix of the system, we should follow the standard procedure of Hamiltonian diagonalization by employing the Jordan–Wigner transformation and the Fourier transformation to the momentum space. By using the following pseudospin operators $\sigma_{k\alpha}$ $(\alpha = x, y, z)$[16]

$$\begin{aligned}
&\sigma_{kx} = \gamma_k^\dagger \gamma_{-k}^\dagger + \mathrm{i}\,\gamma_{-k}\gamma_k, \quad (k = 1, 2, \ldots, N_0),\\
&\sigma_{0z} = 2\gamma_0^\dagger \gamma_0 - 1,\\
&\sigma_{ky} = -\,\mathrm{i}\,\gamma_k^\dagger \gamma_{-k}^\dagger + \mathrm{i}\,\gamma_{-k}\gamma_k,\\
&\sigma_{kz} = \gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k} - 1,
\end{aligned} \quad (2)$$

where $\gamma_k^\dagger$, γ_k $\{k = 0, 1, 2, \ldots\}$ denote the creation and annihilation operators of new fermions in the momentum space, the new Hamiltonian can be written as

$$\begin{aligned}
H = &\sum_{k>0} \mathrm{e}^{\mathrm{i}(\theta_k/2)\sigma_{kx}} (E_k \sigma_{kz}) \mathrm{e}^{-\,\mathrm{i}(\theta_k/2)\sigma_{kx}}\\
&+ \left(-\frac{\Delta}{2} + 1\right)\sigma_{0z},
\end{aligned} \quad (3)$$

where $\Delta = \omega + \sum_{j=1}^{3} g_j s_j^z$ and parameters E_k, θ_k take the following expressions, respectively

$$\begin{aligned}
&E_k = \sqrt{4 + \Delta^2 - 4\Delta\cos(ka)},\\
&\theta_k = \arctan\left[\frac{2\sin(ka)}{\Delta - 2\cos(ka)}\right].
\end{aligned} \quad (4)$$

In Eq. (4), a denotes the lattice spacing and Δ is used as a c number.

With these analytical expressions, we can go straightforwardly to obtaining the reduced density matrix of the system as follows:

$$\rho_s(t) = \mathrm{Tr}_E(\mathrm{e}^{-\,\mathrm{i}\,Ht}\rho_s(0) \otimes |\psi_E\rangle\langle\psi_E| \mathrm{e}^{-\,\mathrm{i}\,Ht}). \quad (5)$$

Here, the initial state of the system is separable from the initial state of the environment, and the initial state of the environment is assumed as the vacuum state $|\psi_E\rangle = |0\rangle_{k=0} \otimes |_{k\geq 0}|0\rangle_k|0\rangle_{-k}$, which satisfies the relation $\gamma_k|0\rangle_k = 0$.

In this paper, we assume that the system is initially in a general three-qubit pure state

$$\begin{aligned}
|\psi_s(0)\rangle = &\, a_1|000\rangle + a_2|001\rangle + a_3|010\rangle\\
&+ a_4|011\rangle + a_5|100\rangle\\
&+ a_6|101\rangle + a_7|110\rangle + a_8|111\rangle,
\end{aligned} \quad (6)$$

where the coefficients satisfy the relation for normalization $|a_1|^2 + |a_2|^2 + \cdots + |a_8|^2 = 1$. After a careful calculation, we obtain the elements $M_{a\beta}$ of the reduced density matrix of the system, $\alpha, \beta = 1, 2, \ldots, 8$. Here, we write the $M_{a\beta}$ in the following equations:

$$M_{a\beta} = a_\alpha a_\beta^* F_{\alpha\beta}, \quad (7)$$

where

$$\begin{aligned}
F_{\alpha\beta} = &\prod_{k>0} \{\sin(E_k^\alpha t)\sin(E_k^\beta t)\cos(\theta_k^{\{\alpha\}} - \theta_k^{\{\beta\}})\\
&+ \cos(E_k^\alpha t)\cos(E_k^\beta t)\\
&+ \mathrm{i}[\cos\theta_k^{\{\alpha\}}\cos(E_k^{\{\beta\}} t)\sin(E_k^{\{\alpha\}} t)\\
&- \cos(\theta_k^{\{\beta\}})\sin(E_k^{\{\beta\}} t)\cos(E_k^{\{\alpha\}} t)]\}
\end{aligned} \quad (8)$$

is the decoherence factor, and $E_k^{\{\alpha\}}$, $\theta_k^{\{\alpha\}}$ can be calculated by replacing Δ with Δ_α in Eq. (8). Δ_α, $\alpha = 1, 2, \ldots, 8$ take the following expressions:

$$\begin{aligned}
&\Delta_1 = \omega + \frac{g_1 + g_2 + g_3}{2},\\
&\Delta_2 = \omega + \frac{g_1 + g_2 - g_3}{2},\\
&\Delta_3 = \omega + \frac{g_1 - g_2 + g_3}{2},\\
&\Delta_4 = \omega + \frac{g_1 - g_2 - g_3}{2},\\
&\Delta_5 = \omega + \frac{-g_1 + g_2 + g_3}{2},\\
&\Delta_6 = \omega + \frac{-g_1 + g_2 - g_3}{2},\\
&\Delta_7 = \omega + \frac{-g_1 - g_2 + g_3}{2},\\
&\Delta_8 = \omega + \frac{-g_1 - g_2 - g_3}{2}.
\end{aligned} \quad (9)$$

3. The violation of the MABK inequality

For N-qubit systems, there are two kinds of multipartite Bell inequalities used to detect the degree of nonlocality as measured by the extent of their violations. The first is the inequality put forward by Zukowski and Brukner.[18] The second is the Mermin–Ardehali–Belinksii–Klyshko (MABK)

inequality.[19−21] Although it is not clear whether these two kinds of calculations are valid for $N > 3$, the nonlocal correlation for $N = 3$ is equivalent for these two inequalities. Here we use the MABK inequality as a criterion for Bell locality, because it is more easily computed than the first inequality. A quantum state ρ violates the MABK inequality if

$$|\langle \mathrm{B}_N \rangle_\rho| > 1. \tag{10}$$

In the simplest case of two qubits, $N = 2$, the operator B_N on the left-hand side of the MABK inequality is just the Clauser–Home–Shimony–Holt (CHSH) operator,[22] namely

$$B_2 = \frac{1}{2}[M_A M_B + M_A M'_B + M'_A M_B - M'_A M'_B]. \tag{11}$$

In the case we are interested, namely that of three qubits, the specific form of operator B_N is expressed as

$$\begin{aligned} B_3 = \frac{1}{2}[&M_A M_B M'_C + M_A M'_B M_C \\ &+ M'_A M_B M_C - M'_A M'_B M'_C] \end{aligned} \tag{12}$$

with the primed and unprimed terms denoting the two different directions in which the corresponding party measures. The measurement operators M_K and M'_K, denoting the operators corresponding to measurements on the qubit K (here labeled $K = A,\ B$, or C), where the second corresponds to a measurement performed along a direction differing by θ relative to that performed on the first qubit, are

$$\begin{pmatrix} M_K \\ M'_K \end{pmatrix} = R(\theta_K) \begin{pmatrix} M_A \\ M'_A \end{pmatrix}, \tag{13}$$

where

$$R(\theta_K) = \begin{pmatrix} \cos\theta_K & -\sin\theta_K \\ \sin\theta_K & \cos\theta_K \end{pmatrix}. \tag{14}$$

(i) If we set the parameters $a_1 = a_4 = a_6 = a_7 = a_8 = 0$, the initial Wclass states can be expressed as

$$|\psi_s(0)\rangle = a_2|001\rangle + a_3|010\rangle + a_5|100\rangle, \tag{15}$$

then the time evolution of the system density matrix is obtained as

$$\rho_s(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |a_2|^2 & a_2 a_3^* F_{23} & 0 & a_2 a_5^* F_{25} & 0 & 0 & 0 \\ 0 & a_3 a_2^* F_{32} & |a_3|^2 & 0 & a_3 a_5^* F_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 a_2^* F_{52} & a_5 a_3^* F_{53} & 0 & |a_5|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

In the case of W class states, the relative angles of measurement are $\theta_B = \pi/6$ and $\theta_C = \pi/3$. Thus, the corresponding measurement operators for qubits A, B, and C are written in terms of the usual Pauli operators σ_z and σ_x, as

$$M_A = \sigma_y \otimes I \otimes I, \tag{17}$$

$$M'_A = \sigma_x \otimes I \otimes I, \tag{18}$$

$$M_B = I \otimes \left[\cos\left(\frac{\pi}{6}\right)\sigma_y - \sin\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes I, \tag{19}$$

$$M'_B = I \otimes \left[\sin\left(\frac{\pi}{6}\right)\sigma_y + \cos\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes I, \tag{20}$$

$$M_C = I \otimes I \otimes \left[\cos\left(\frac{\pi}{6}\right)\sigma_y - \sin\left(\frac{\pi}{6}\right)\sigma_x\right], \tag{21}$$

$$M'_C = I \otimes I \otimes \left[\sin\left(\frac{\pi}{6}\right)\sigma_y + \cos\left(\frac{\pi}{6}\right)\sigma_x\right]. \tag{22}$$

The expectation value of the B_3 operator for the W state under the quantum-critical environment can be obtained as

$$\begin{aligned} |\langle B_3 \rangle_{\rho_w(t)}| = \frac{1}{2}|&1 + a_3 a_5^* F_{35} + a_3^* a_5 F_{53} + a_2 a_3^* F_{23} \\ &+ a_2^* a_3 F_{32} + a_2 a_5^* F_{25} + a_2^* a_5 F_{52}|. \end{aligned} \tag{23}$$

Particularly, we consider the case in which the initial state is a standard W state, namely,

$$|\psi_s(0)\rangle = \frac{\sqrt{3}}{3}|001\rangle + \frac{\sqrt{3}}{3}|010\rangle + \frac{\sqrt{3}}{3}|100\rangle, \tag{24}$$

we then have

$$|\langle B_3\rangle_{\rho_w(t)}| = \frac{|3 + F_{35} + F_{53} + F_{23} + F_{32} + F_{25} + F_{52}|}{6}. \tag{25}$$

When the time is short enough, we can have $|\langle B_3\rangle_{\rho_w(t)}| > 1$, that is, there exists Bell-nonlocality, and in the limit $t \to 0$ $|\langle B_3\rangle_{\rho_w(t)}| \to 1.5$. For the particular case, we consider the dependences on the total site of the spin chain N_t, the strength of the transverse field ω and the coupling constants g_i of the time evolution of Bell-nonlocality violations. In Fig. 1 we give a plot of the Bell-inequality violations as a function of time t for different values of N_t. Here and in the following numerical simulation, we have used the fact that the lattice spacing a takes the value of $2\pi/N_t$. It is clear that BNSD happens for definite time. The Bell-nonlocality violations as a function of time for different values of the transverse field are plotted in Fig. 2. From the figure we can observe that the BNSD can always take place at some definite time and the nonlocatity can revives under certain conditions.

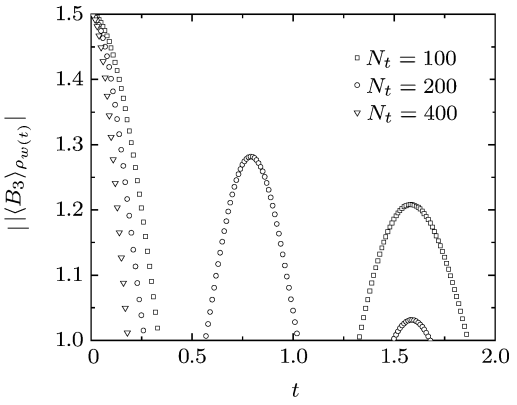


Fig. 1. $|\langle B_3\rangle_{\rho_w(t)}|$ versus time t when the initial state is a standard W state, where $g_1 = 0.05$, $g_2 = 0.1$, $g_3 = 0.2$ and $\omega = 0.5$.

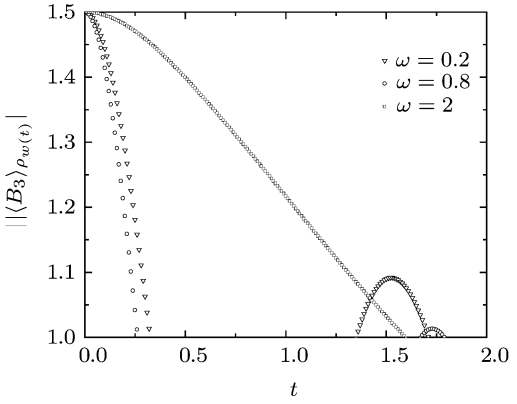


Fig. 2. $|\langle B_3\rangle_{\rho_w(t)}|$ versus time t when the initial state is a standard W state, where $g_1 = 0.1$, $g_2 = 0.2$, $g_3 = 0.3$ and $N_t = 100$.

(ii) Assume that the initial state of the system is a three-qubit GHZ state,

$$|\psi_s(0)\rangle = a_1|000\rangle + a_8|111\rangle, \tag{26}$$

then the time evolution of the system density matrix is obtained as

$$\rho_s(t) = \begin{bmatrix} |a_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 a_8^* F_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1^* a_8 F_{18}^* & 0 & 0 & 0 & 0 & 0 & 0 & |a_8|^2 \end{bmatrix}. \tag{27}$$

For the GHZ states, we use the measurement operators

$$M_A = \sigma_z \otimes I \otimes I, \tag{28}$$

$$M'_A = \sigma_x \otimes I \otimes I, \tag{29}$$

$$M_B = I \otimes \left[\cos\left(\frac{\pi}{6}\right)\sigma_z - \sin\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes I, \tag{30}$$

$$M'_B = I \otimes \left[\sin\left(\frac{\pi}{6}\right)\sigma_z + \cos\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes I, \tag{31}$$

$$M_C = I \otimes I \otimes \left[\cos\left(\frac{\pi}{6}\right)\sigma_z - \sin\left(\frac{\pi}{6}\right)\sigma_x\right], \tag{32}$$

$$M'_C = I \otimes I \otimes \left[\sin\left(\frac{\pi}{6}\right)\sigma_z + \cos\left(\frac{\pi}{6}\right)\sigma_x\right]. \tag{33}$$

If we consider that the initial state is a standard GHZ

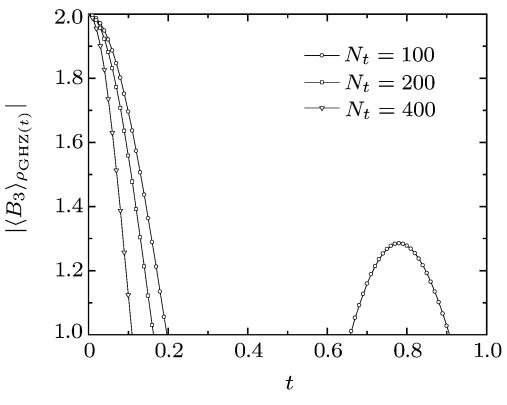


Fig. 3. $|\langle B_3\rangle_{\rho_{\rm GHZ}(t)}|$ versus time t when the initial state is a standard GHZ state, where $g = g_1 + g_2 + g_3 = 1$, $\omega = 1$.

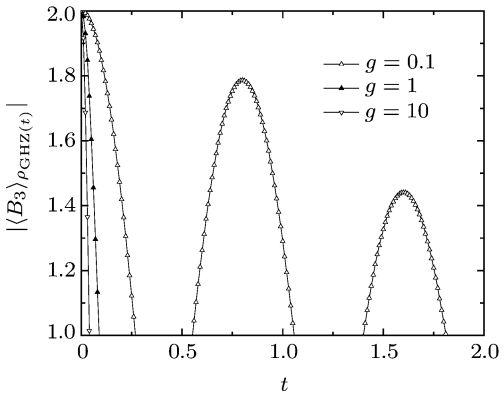


Fig. 4. $|\langle B_3\rangle_{\rho_{\rm GHZ(t)}}|$ versus time t when the initial state is a standard GHZ state, where $N_t = 100$, $\omega = 0.5$ and $g = g_1 + g_2 + g_3$.

state ($a_1 = a_8 = \sqrt{2}/2$), then the expectation value of the operator for the state under the quantum-critical environment can be obtained as

$$|\langle B_3\rangle_{\rho_{\rm GHZ(t)}}| = |F_{18} + F_{81}|. \tag{34}$$

When the time is short enough, it can have $|\langle B_3\rangle_{\rho_{\rm GHZ(t)}}| > 1$, and in the limit $t \to 0$ $|\langle \mathrm{B}_3\rangle_{\rho_{\rm GHZ(t)}}| \to 2$. In Fig. 3 we give a plot of the Bell-inequality violations as a function of time t for different values of N_t. It is clear that BNSD happens for definite time. This process will persist for some periods determined by the values of the total site of the spin chain. The Bell-nonlocality violations as a function of time for different values of coupling constants is plotted in Fig. 4, from which we can observe that the BNSD can always take place at some definite time.

4. Conclusion

In summary, we have studied the nonlocal behaviour of the tripartite system under the decoherence induced by the quantum-critical environment by calculating the extent of the violation of Bell inequality. It is demonstrated that there exists the destruction of Bell-nonlocality behaviour due to the decoherence induced by the correlated environment. The investigation of the sudden death of nonlocal property is important for tripartite systems because the essential characteristics of multipartite nonlocality which are beyond the bipartite case can be exhibited. It is important to study the BNSD of tripartite states under the decoherence induced by the quantum-critical environment.

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