Preliminary analysis of resonance effect by Helmholtz–Schrödinger method

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The Helmholtz–Schrödinger method is employed to study the electric field standing wave caused by coupling through a simple slot. There is a good agreement between the numerical results and the resonant conditions presented by the Helmholtz–Schrödinger method. Thus, it can be used in similar cases where the amplitude of the electric field is the important quantity or eigenfunctions of the Schrödinger equation are needed for complicated quantum structures with hard wall boundary conditions.

Keywords: Helmholtz–Schrödinger method, standing wave, resonance, coupling

PACC: 0365, 0540

1. Introduction

Quantum-mechanical phenomena occur in many different fields. However, as this paper presents, electromagnetic analog demonstrations may be designed using the well-established microwave techniques. This is likely to enhance the understanding of quantum-mechanical phenomena by researchers of suitable background specialized in other areas of physics.

Quantitative mapping of the electromagnetic coupling of the slot is very important for various technological and scientific reasons. Microwave coupling through thin slots is focused on, particularly those housed inside metallic containers which are susceptible to strongly concentrated electromagnetic fields. The degree of concentration of the field depends sensitively on the operating frequency and shape of the slots. Of greater scientific interest, electromagnetic resonators can be used to represent quantum mechanical potential wells. In this case, the standing wave patterns can reveal the probability density of the corresponding solution of the Schrödinger equation for an infinite square potential well of arbitrary shape. A great deal of interesting physical results can be explored by means of numerical understanding of the wave function behaviour in thin slots. To investigate the wave functions, one needs a simple and reliable method of imaging electromagnetic waves in coupling through thin slot. Some researchers have studied the coupling hole on RF cavity,[1] coupled quantum wells,[2] optical transmission enhancement of two-dimensional subwavelength hole arrays in metallic films,[3] and so on. In this paper we present numerical simulation to image the standing wave eigenfunctions of thin slots and their corresponding quantum eigenfunctions.

Imaging of electromagnetic standing wave in microwave resonators was pioneered by Slater and coworkers.[4] Other researchers have refined this kind of technique by modulating the perturbation to measure both electric and magnetic fields. More recently, derivatives of Slater's work have examined the standing wave patterns of electromagnetic resonators which simulate wave chaotic systems in two dimensions.[5−8] To investigate the fundamental wave (or quantum) mechanical behaviour, we exploit the electromagnetic analog of the Schrödinger equation in two dimensions (2D). Briefly, a two-dimensional microwave cavity with walls defining a nonintegrable infinite square potential well is used to simulate the solutions of the two-dimensional Schrödinger equation in the semiclassical limit. Some results of the macroscopical

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simulation present a macroscopical analogy for the understanding of the Schrödinger equation.[9] Statistics of impedance and scattering matrices in chaotic microwave cavities were discussed using Helmholtz–Schrödinger method.[10]

2. Review of theory

In this paper, we describe the method along with our numerical results obtained from coupling through thin slots. We can image the probability density $|\psi(x)|^2$ by numerically evaluating the electric fields in the standing wave pattern of the resonator and using the analogy between the Helmholtz and Schrödinger equations in one dimension.[11]

Both the time-independent Schrödinger equation and the Helmholtz equation are linear second-order differential equations respecting to space with constant coefficients:

$$\nabla^2\psi_n + 2m(\epsilon_n - V)/\hbar^2\psi_n = 0, \tag{1}$$

$$\nabla^2 E_{yn} + k^2 E_{yn} = 0. \tag{2}$$

The Schrödinger equation describes the n-th excited eigenfunction ψ_n of a particle of mass m in a potential V, where ϵ_n is the energy of the n-th excited state, and $\hbar$ is Planck's constant divided by 2π. In the Helmholtz equation, E_{yn} is the y component of the electric field, and k_n is the wave vector in the propagation direction of the i-th mode (it is assumed that the resonator is much smaller in the y direction than that in the x direction). In general $k_n^2 = \omega_n^2\mu\varepsilon$, where ω_n is the i-th resonant frequency of the cavity with uniform permittivity ε and permeability μ. The evolution of the one-point statistical property of eigenfunction distribution, $P(|\psi_n|^2)$, quantifies the degree of probability density, $|\psi_n|^2 = \psi_n\psi_n^*$. The probability density $|\psi_n|^2$ in the Schrödinger equation for the quantum mechanical problem is analogous to $|E_{yn}|^2$ in the Helmholtz equation for the electromagnetic problem, as long as the $2m(\epsilon_n - V)/\hbar^2$ and k_n^2 in these equations are constant. In a thin slot where E_y is uniform in the y direction, the solutions to the Helmholtz equation with perfectly conducting walls ($E_y = 0$ at the boundary) are equivalent to the solutions of the one-dimensional Schrödinger equation with hard wall boundaries ($\psi_n = 0$ at the boundary) of the same geometry.

3. Model setup

In the field of high-power microwave (HPM) effects and electromagnetic compatibility (EMC) study, researchers are much interested in the problem of electromagnetic wave coupling into the large complicated enclosures. Scattering and transmission happen when electromagnetic wave couples to external of enclosures by the mesopore and slot of encapsulation surface. Based on the analysis given above in Section 2, phenomenon of slot coupling was discussed by the Helmholtz–Schrödinger method under special condition. We can use this method to study the properties of any one-dimensional (1D) quantum well. A model (Fig. 1) is designed in which the dimension in y direction is significantly smaller than that in the x direction, where a is the slot length. Let us suppose that the metal making the wall is a perfect conductor ($E_y = 0$ at $x = 0$ and $x = a$). This case can be considered as a 1D quantum mechanical infinite square potential well. ψ_n is zero for $x = 0$ and $x = a$.

$$V(x) = \begin{cases} 0, & 0 < x < a, \\ \infty, & x < 0, \quad x > a. \end{cases} \tag{3}$$

The probability of the particle staying out of the potential well is zero. The solution of the Schrödinger equation is the wave function ψ_n which describes the state of a particle moving in the potential well.

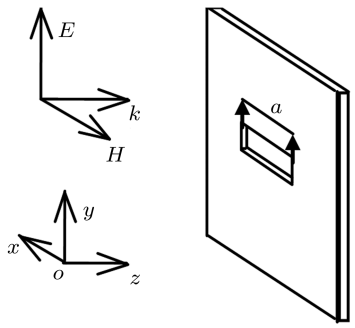


Fig. 1. Schematic diagram of the slot and the incident microwave.

We want to consider the solutions of the stationary state Schrödinger equation in this case. For

$0 < x < a$, the stationary state Schrödinger equation satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi + \frac{2m}{\hbar^2}E\psi = 0. \tag{4}$$

We can obtain the eigenfunctions and eigenvalues of energy

$$E = E_n = \hbar^2\pi^2 n^2/2ma^2, \quad n = 1, 2, 3, \ldots . \tag{5}$$

In terms of the energy level (E_n), the corresponding wave function is

$$\psi_n(x) = A_n \sin((n\pi/a)x). \tag{6}$$

The degree of probability density of the particle is

$$|\psi_n|^2 = |A_n \sin((n\pi/a)x)|\,. \tag{7}$$

At the centre position ($x = a/2$) of the potential well

$$|\psi_n| = \begin{cases} |\psi_n|_{\max}\,, & \text{for } n \text{ odd}, \\ |\psi_n|_{\min}\,, & \text{for } n \text{ even}, \end{cases} \tag{8}$$

where n is an integer, the wave function is standing waves of corresponding De Broglie's wavelength in the potential well for every energy eigenstate. If n is very large, i.e., the width of potential well (a) is much larger than the period of the standing wave, the degree of probability density of the particle is unity at any position x.

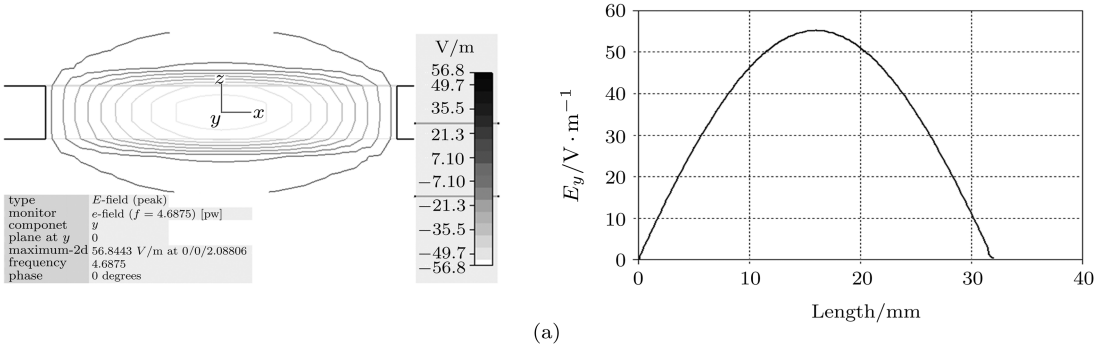
4. Numerical results

If we leave out the cavity, the coupling of electromagnetic energy into the slot will depend on its geometry and dimension of slot. The characters of the slot coupling are reflected by the characters of the slot centre coupling. Based upon analysis about the 1D quantum mechanical infinite square potential well above, we can predict that the characters of coupling at points near the centre of the slot should accord with the feature of standing wave in an infinite square potential well, when the direction of the incident electric field is perpendicular to x direction of the slot.

To test our prediction, we computed the problem of the narrow slot coupling of electromagnetic radiation as shown in Fig. 1. The slotted metal wall was chosen as a primary numerical simulation configuration because of its simplicity. In this paper, the coupling through slot in a wall is discussed. We focus on the character of coupling through the centre of slot in the wall.

Some results are presented below about coupling of a wall with simple wide and thin slots, as shown in Fig. 1. The slot is 32 mm long (denoted by a), with width (W) of 1 mm and thickness (T) of 5 mm. When the direction of the incident electric field is perpendicular to the long side of the slot, we can set up the 1D quantum mechanical infinite square potential well.

Numerical results are presented for coupling through different frequencies and slot thickness. In the case of a slot, the electric field intensity along the y direction (E_y) alters with different frequencies in the slot as shown in Fig. 2. The images are obtained from the slot at 4.6875, 9.375, 14.06, and 18.7 GHz in Figs. 2(a), 2(b), 2(c) and 2(d), respectively. In Fig. 2, for different frequencies, imaging of electromagnetic standing waves appears in the slot. Probability amplitude $\psi_n L$ images made by resonator at different frequencies are presented. Here L is the length of the slot.



(a)

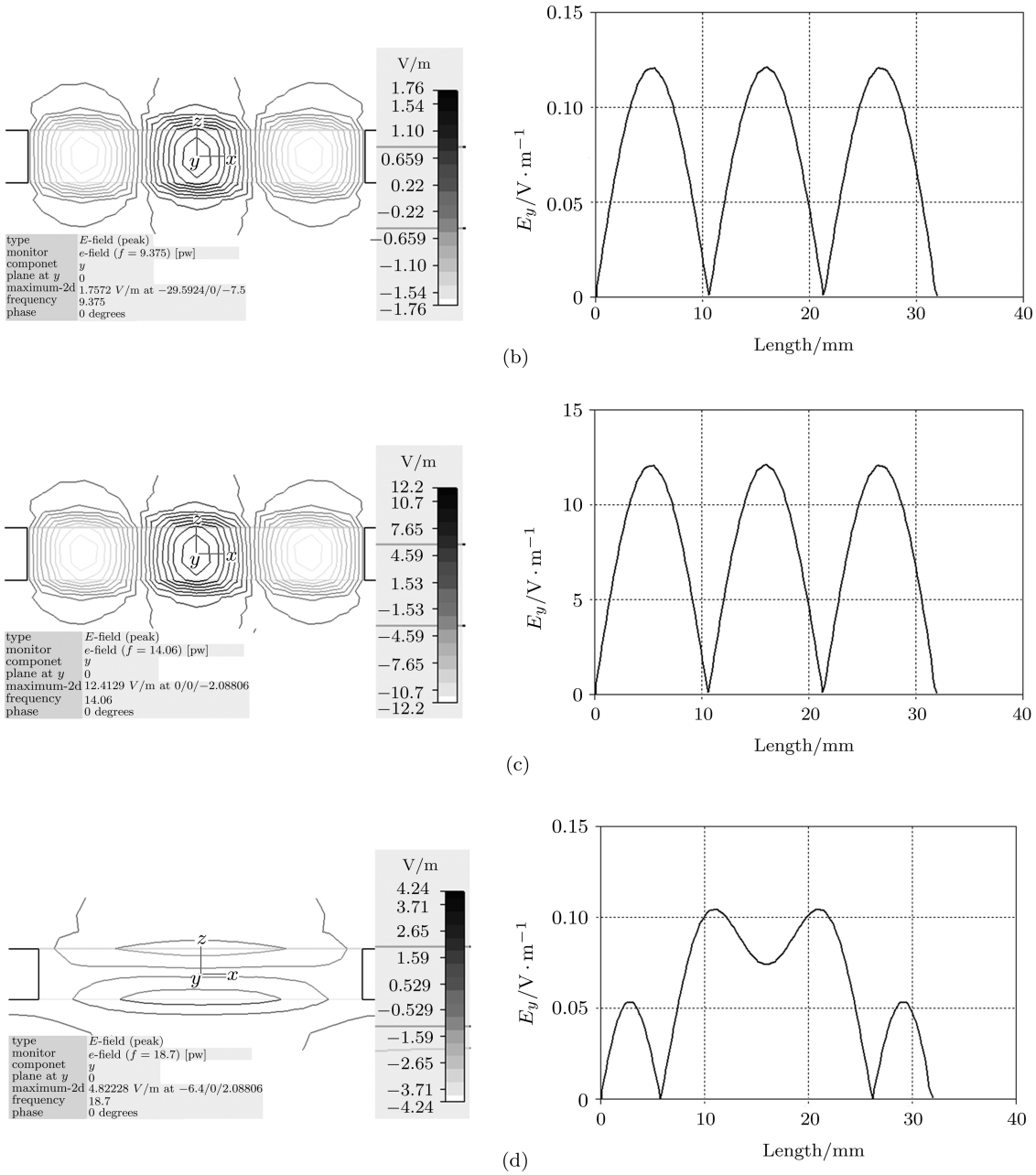


Fig. 2. The E_y alters with different frequencies in the slot: (a) $f = 4.6875$ GHz, (b) $f = 9.375$ GHz, (c) $f = 14.06$ GHz, (d) $f = 18.7$ GHz. The images are obtained with $T = 5$ mm. The insets show profile gauges of E_y corresponding to the frequencies.

The numerical results show that resonance effect takes place at 4.6875 and 14.06 GHz. There is a strong coupling band near the resonant frequencies, around which the enhancement effect can take place and condition for image of electromagnetic standing waves is well satisfied. Meanwhile the electric field intensity (E_y) is much smaller at 9.375 and 18.7 GHz than that at 4.6875 and 14.06 GHz. Besides the fundamental resonance, additional resonances having maximum electric field intensity are observed at points near the centre of the slot for frequencies which are odd multiples of the fundamental resonance. This can be explained by the standing waves of magnetic current along the slot which synchronize with the electric field across the slot.

In order to test the validity of the analysis, we change the slotted wall thickness to study the coupling characters. As shown in Fig. 3, the slot is 32 mm long (presented by a), with width (W) of 1 mm and thickness (T) of 1 mm. The direction of the incident electric field is the same as stated above.

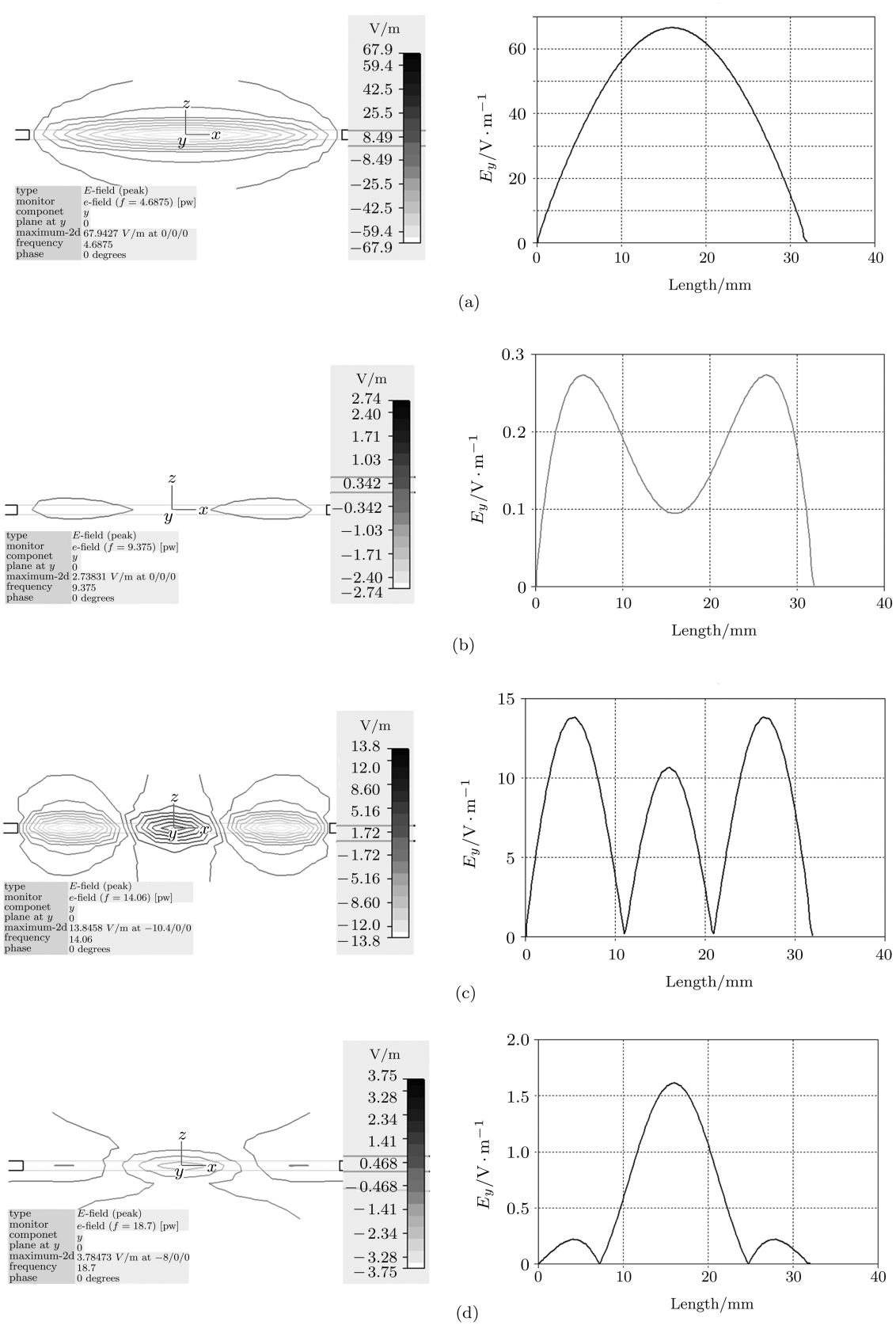


Fig. 3. The E_y alters with the different frequencies in the slot: (a) $f = 4.6875$ GHz, (b) $f = 9.375$ GHz, (c) $f = 14.06$ GHz, (d) $f = 18.7$ GHz. The images are obtained with $T = 1$ mm. The insets show profile gauges of E_y corresponding to the frequencies.

It is revealing to compare the E_y images in the slot for different frequencies and thickness (Fig. 3). Some interesting phenomena were discovered. As the thickness increases, probability amplitude $\psi_n L$ images (intensity of E_y) will take place variously. However, approximate enhancement frequency has approximately no change. Strong coupling band near the resonant frequency has almost no change when the length which the direction of the incident electric field is perpendicular to its invariableness under the condition of $a \gg W$.

It is obvious that, as the frequency of incident wave increases, the frequency tends towards gradually smoothness (Fig. 4). That is to say the energy has the uniform distribution in the slot when the length of slot is much longer than incident wavelength.

Studying of the potential well in quantum mechanics, when the wavelength of the quantum mechanical particle is much less than the characteristic size of the potential well, the probability of the particle is known to exist, is unity. As frequency increases, the electric field intensity (E_y) in the centre of the slot goes gradually homogeneous.

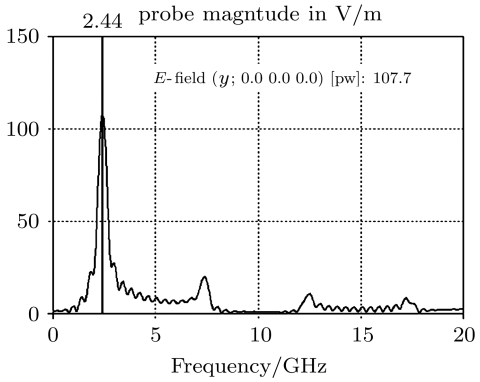


Fig. 4. The E_y changes with frequency in the slot (a = 60 mm, T = 5 mm and W = 1 mm).

We can study variation of the E_y along the $(0, 0, z)$ direction with the different frequencies (Fig. 5). Electric field intensity (E_y) approximates constant along the direction $(0, 0, z)$ with the incident frequency. It is obvious, that E_y is much greater in a small neighbourhood of 4.6875 GHz than that for other frequencies. Hence, the enhancement effect is outstanding at the lowest resonant frequency. According to the conditions of standing waves in a potential well, at points near the centre of the potential well, the particle density has the maximum for frequencies which are odd multiples of corresponding De Broglie's half-wavelength in the potential well for every energy eigenstate. Hence, using these mathematical analogies, we can examine the behaviour of electromagnetic wave coupling through a simple slotted wall.

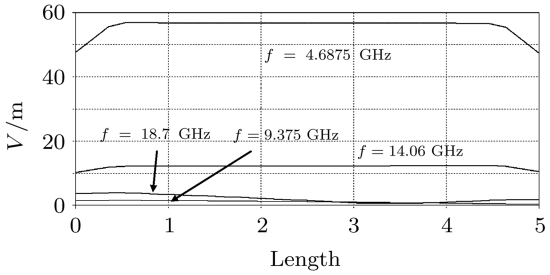


Fig. 5. The E_y along $(0, 0, z)$ for different frequencies.

5. Discussion

The results of our numerical tests and the comparison with this theory are presented in this paper. Resonance effect of the coupling through a simple slotted wall is discussed by using the Helmholtz–Schrödinger method. This method can be used in similar cases where the amplitude of the electric field is the important quantity or eigenfunctions of the Schrödinger equation are needed for complicated quantum structures with hard wall boundary conditions.

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