Two new integrable couplings of the soliton hierarchies with self-consistent sources*

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A kind of integrable coupling of soliton equations hierarchy with self-consistent sources associated with $\tilde{sl}(4)$ has been presented (Yu F J and Li L 2009 *Appl. Math. Comput.* **207** 171; Yu F J 2008 *Phys. Lett.* A **372** 6613). Based on this method, we construct two integrable couplings of the soliton hierarchy with self-consistent sources by using the loop algebra $\tilde{sl}(4)$. In this paper, we also point out that there are some errors in these references and we have corrected these errors and set up new formula. The method can be generalized to other soliton hierarchy with self-consistent sources.

Keywords: TC hierarchy, generalized Burgers hierarchy, self-consistent sources, integrable couplings, loop algebra

PACC: 0340K, 0220, 0365G

1. Introduction

Soliton equations with self-consistent sources (SESCS)[1−8] may have attracted considerable attention. Physically, the sources may result in solitary waves with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kinds of systems are usually used to describe interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, etc. For example, a nonlinear Schrödinger equation with self-consistent sources is relevant to some problems of plasma physics and solid state physics. Some of these kinds of nonlinear integrable systems were constructed by adding a new operator to the original Lax representation and by other methods in Refs. [1]–[8].

How to obtain an integrable coupling of the SESCS has also attracted attention.[9,10] Yu has derived an integrable coupling of integrable hierarchy with self-consistent sources such as the Dirac soliton with self-consistent sources by making use of the loop algebra $\tilde{sl}(4)$. But there are some errors in Refs. [9] and [10]. In this paper, we have corrected these errors and constructed a new integrable coupling with self-consistent sources by making use of the loop algebra $\tilde{sl}(4)$ and a new idea.

This paper is organized as follows. In Section 2, we will deduce a kind of integrable coupling of soliton equations with self-consistent sources associated with $\tilde{sl}(4)$. In Section 3, a new integrable coupling of the TC hierarchy with self-consistent sources is derived by using the loop algebra $\tilde{sl}(4)$. In Section 4, a new integrable coupling of the generalized Burgers soliton hierarchy with self-consistent sources is derived by using the loop algebra $\tilde{sl}(4)$. Finally, some conclusions are drawn.

2. A new formula on integrable couplings of soliton equation hierarchy with self-consistent sources

In the following, we consider a set of matrix Lie algebra $sl(4)$ based on Ref. [11]

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

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$$e_3=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad e_4=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$e_5=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad e_6=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{1}$$

Let

$$\begin{aligned}&sl(4)=\text{span}\{e_1,e_2,e_3,e_4,e_5,e_6\},\\&sl(4)_1=\text{span}\{e_1,e_2,e_3\},\\&sl(4)_2=\text{span}\{e_4,e_5,e_6\}.\end{aligned}$$

Therefore we construct three Lie algebras satisfying $sl(4)=sl(4)_1\oplus sl(4)_2, [sl(4)_1,sl(4)_2]=sl(4)_1sl(4)_2-sl(4)_2sl(4)_1\subseteq sl(4)_2$. In what follows, we introduce the loop algebra $\tilde{sl}(4)$

$$\begin{aligned}&\tilde{sl}(4)=\{A|A\in R[\lambda]\otimes sl(4)\},\\&\tilde{sl}(4)_1=\{A|A\in R[\lambda]\otimes sl(4)_1\},\\&\tilde{sl}(4)_2=\{A|A\in R[\lambda]\otimes sl(4)_2\},\end{aligned} \tag{2}$$

where the loop algebra $\tilde{sl}(4)$ is defined by span $\{\lambda^nA|n\in Z, A\in sl(4)\}$, and $R[\lambda]$ is a polynomial ring on λ.

Consider the auxiliary linear problem

$$\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}_x=U(u,\lambda)\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix},$$

$$U(u,\lambda)=\tilde{e}_1(\lambda)+\sum_{i=2}^{6}u_i\tilde{e}_i(\lambda),$$

$$\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}_{t_n}=V^{(n)}(u,\lambda)\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}, \tag{3}$$

where $u=(u_1,\ldots,u_s)^{\mathrm{T}}, u_i=u_i(x,t)$ $(i=1,2,\ldots,s), \phi_j=\phi_j(x,t)$ $(j=1,2,3,4)$ are field variables defined on $x\in R, t\in R, \tilde{e}_i(\lambda)\in\tilde{sl}(4), \tilde{sl}(4)$ denotes a finite-dimensional Lie algebra over R. The general scheme of searching for the consistent $V^{(n)}$ and generating a hierarchy of zero curvature equations has been proposed as follows.[8,11−18] We solve the equation

$$V_x=UV-VU,$$

$$V=\sum_{m=0}^{\infty}V_m(u)\lambda^{-m}=\sum_{m=0}^{\infty}\begin{pmatrix}a_m&b_m+c_m&d_m&e_m+f_m\\b_m-c_m&-a_m&e_m-f_m&-d_m\\0&0&a_m&b_m+c_m\\0&0&b_m-c_m&-a_m\end{pmatrix}\lambda^{-m}. \tag{4}$$

We search for $(\Delta_n\in\tilde{sl}(4))$ such that $V^{(n)}$ can be constructed by

$$V^{(n)}=\sum_{m=0}^{n}V_m(u)\lambda^{n-m}+\Delta_n(u,\lambda),\quad \Delta_n(u,\lambda)=\begin{pmatrix}\Delta_{n1}&\Delta_{n2}+\Delta_{n3}&\Delta_{n4}&\Delta_{n5}+\Delta_{n6}\\\Delta_{n2}-\Delta_{n3}&-\Delta_{n1}&\Delta_{n5}-\Delta_{n6}&-\Delta_{n4}\\0&0&\Delta_{n1}&\Delta_{n2}+\Delta_{n3}\\0&0&\Delta_{n2}-\Delta_{n3}&-\Delta_{n1}\end{pmatrix}, \tag{5}$$

where Δ_{ni} are linear functions of a_n, b_n, c_n, d_n, e_n and f_n.

Under certain conditions, there is a constant γ such that the trace identity holds[19]

$$\begin{aligned}&\frac{\delta}{\delta u_i}\text{Tr}\left(V\frac{\partial U}{\partial\lambda}\right)\\&=\left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\text{Tr}\left(V\frac{\partial U}{\partial u_i}\right),\end{aligned} \tag{6}$$

where Tr denotes the trace of a matrix. Define a scalar $H=H(u,\lambda)$ by the equation

$$\begin{aligned}&\left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)H=\text{Tr}\left(V\frac{\partial U}{\partial\lambda}\right),\\&H=\sum_{m=0}^{\infty}H_m(u)\lambda^{-m}.\end{aligned} \tag{7}$$

From the trace identity, we obtain that

$$\sum_{m=0}^{\infty}\frac{\delta H_m}{\delta u_i}\lambda^{-m} = \mathrm{Tr}\left(V\frac{\partial U}{\partial u_i}\right) = \sum_{m=0}^{\infty}\mathrm{Tr}(V_m e_i)\lambda^{-m}. \tag{8}$$

The set $\{H_m\}$ proves the conserved densities of Eq. (6). In Ref. [18], Ma proposed the generalized Tu scheme to find a Hamiltonian function, a recursion operator L and symplectic operator J of the hierarchy based on zero curvature equation. The Hamiltonian form with H_{n+1} can be written as

$$u_{t_n} = J\frac{\delta H_{n+1}}{\delta u}, \quad n = 1, 2, \dots, \tag{9}$$

$$\frac{\delta H_n}{\delta u} = L\frac{\delta H_{n-1}}{\delta u} = \cdots = L^n\frac{\delta H_0}{\delta u}, \quad n = 1, 2, \dots, \tag{10}$$

where $n_0 \geq 0$ is the minimum integer such that $\delta H_{n_0}/\delta u \neq 0$ and $\delta H_k/\delta u = 0$ for $k < n_0$,

$$\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_1}, \dots, \frac{\delta}{\delta u_s}\right)^{\mathrm{T}}.$$

According to the condition (8), we obtain a new auxiliary linear problem. For N distinct $\lambda_j, j = 1, \dots, N$, the following systems result from expression (3)

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}_x = \left(\tilde{e}_1(\lambda) + \sum_{i=2}^{6} u_i\tilde{e}_i(\lambda)\right)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}, \quad j = 1, \dots, N, \tag{11a}$$

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}_{t_n} = \left(\sum_{m=0}^{n} V_m(u)\lambda^{n-m} + \Delta_n(u,\lambda)\right)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}, \quad j = 1, \dots, N. \tag{11b}$$

The following equation was presented in Ref. [20]

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N}\alpha_j\frac{\delta\lambda_j}{\delta u} = 0, \tag{12}$$

where α_j are constants, $\delta H_k/\delta u$ determines a finite-dimensional invariant set for the flow (12).

For Eq. (11a), we have the following result based on Ref. [20]

$$\frac{\delta\lambda_j}{\delta u_i} = \alpha_j\mathrm{Tr}\left(\Psi_j\frac{\partial U(u,\lambda_j)}{\partial u_i}\right) = \alpha_j\mathrm{Tr}(\Psi_j e_i(\lambda_j)), \quad i = 1, 2, \tag{13}$$

where

$$\Psi_j = \begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{3j}\phi_{4j} & -\phi_{3j}^2\\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{4j}^2 & -\phi_{3j}\phi_{4j}\\ 0 & 0 & \phi_{1j}\phi_{2j} & -\phi_{1j}^2\\ 0 & 0 & \phi_{2j}^2 & -\phi_{1j}\phi_{2j}\end{pmatrix}, \quad j = 1, \dots, N.$$

And α_j is a constant.

For $i = 3, 4$ we define that

$$\frac{\delta\lambda_j}{\delta u_i} = \beta_j\mathrm{Tr}\left(\Psi_{jA}\frac{\partial U_0(u,\lambda_j)}{\partial u_i}\right), \tag{14}$$

where

$$U = \begin{pmatrix}U_1 & U_0\\ 0 & U_1\end{pmatrix}_{4\times 4}, \quad \Psi_{jA} = \begin{pmatrix}\phi_{3j}\phi_{4j} & -\phi_{3j}^2\\ \phi_{4j}^2 & -\phi_{3j}\phi_{4j}\end{pmatrix}$$

and β_j is a constant.

According to Eqs. (13) and (14), we obtain a kind of integrable couplings of Hamiltonian soliton equation hierarchy with self-consistent sources as follows:

$$u_{t_n} = J\frac{\delta H_{n+1}}{\delta u} + J\sum_{j=1}^{N}\frac{\delta \lambda_j}{\delta u} = J\left[\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} - \sum_{j=1}^{N}\frac{\delta \lambda_j}{\delta u}\right], \quad n = 1, 2, \ldots . \tag{15}$$

Remarks Here we point out that there are some errors in Refs. [9] and [10]. For example, equations (17) and (18) in Refs. [9] and [10] are not true. We see that the left-hand side of Eqs. (17) and (18) is a number entry. But the right side of Eqs. (17) and (18) is a 4×4 matrix. And equation (16) in Refs. [9] and [10] is also inaccurate.

3. Integrable couplings of TC hierarchy with self-consistent sources

In Ref. [21], Yan have obtained integrable couplings of the TC hierarchy. In order to set up integrable couplings of the TC hierarchy with self-consistent sources, we consider the following matrix spectral problem

$$\phi_x = U(u,\lambda)\phi, \quad U(u,\lambda) = \begin{pmatrix} 0 & 1+\dfrac{u_1+u_2}{2\lambda} & u_3 & u_3+u_4 \\ \lambda+\dfrac{u_1-u_2}{2} & 0 & -u_3+u_4 & -u_3 \\ 0 & 0 & 0 & 1+\dfrac{u_1+u_2}{2\lambda} \\ 0 & 0 & \lambda+\dfrac{u_1-u_2}{2} & 0 \end{pmatrix} = \begin{pmatrix} U_1 & U_0 \\ 0 & U_1 \end{pmatrix}, \tag{16}$$

where λ is a spectral parameter and U_1 satisfies $\varphi_x = U_1\varphi$ which is matrix spectral problem of the TC soliton hierarchy.[11] Therefore, by using the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{17}$$

we have the following integrable couplings of the TC soliton hierarchy:

$$\begin{aligned} u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t &= -\begin{pmatrix} \left(b_n - \dfrac{u_1}{u_2}c_n\right)_x \\ -\left(c_{nx} - \dfrac{u_1}{u_2}b_{nx}\right) \\ d_{nx} + (u_1+u_2)e_n - u_4\left(1+\dfrac{u_1}{u_2}\right)c_n \\ -e_{nx} + (u_1+u_2)d_n - u_3\left(1+\dfrac{u_1}{u_2}\right)c_n \end{pmatrix} \\ &= \begin{pmatrix} \partial & \partial\dfrac{u_1}{u_2} & 0 & 0 \\ \dfrac{u_1}{u_2}\partial & \partial & 0 & 0 \\ 0 & u_4+\dfrac{u_1u_4}{u_2} & \partial & -(u_1+u_2) \\ 0 & u_3+\dfrac{u_1u_3}{u_2} & -(u_1+u_2) & \partial \end{pmatrix} \begin{pmatrix} -b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} \end{aligned}$$

$$= J\begin{pmatrix} -b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} = JL^{n-1}\begin{pmatrix} -b_1 \\ c_1 \\ d_1 \\ e_1 \end{pmatrix} = JL^{n-1}\begin{pmatrix} 0 \\ \frac{1}{2}\beta u_2 \\ \frac{1}{2}\beta u_3 \\ \frac{1}{2}\beta u_4 \end{pmatrix}, \tag{18}$$

where

$$L = \begin{pmatrix} -\frac{1}{2}\partial^{-1}u_1\partial & -\frac{1}{2}\partial^{-1}u_2\partial & 0 & 0 \\ -\frac{1}{2}u_2 + \frac{1}{4}\partial\frac{u_1}{u_2}\partial & -\frac{1}{2}u_1 + \frac{1}{4}\partial^2 & 0 & 0 \\ -\frac{1}{2}u_3 & \frac{1}{2}u_3 & -\frac{1}{2}(u_1+u_2) & \frac{1}{2}\partial \\ -\frac{1}{2}u_4 & \frac{1}{2}u_4 & \frac{1}{2}\partial & \frac{1}{2}(u_1+u_2) \end{pmatrix}.$$

According to formulas (13) and (14), we have the following results by direct computations:

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \sum_{j=1}^{N}\begin{pmatrix} \frac{\delta\lambda_j}{\delta u_1} \\ \frac{\delta\lambda_j}{\delta u_2} \\ \frac{\delta\lambda_j}{\delta u_3} \\ \frac{\delta\lambda_j}{\delta u_4} \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 - \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}(\phi_{3j}^2 + \phi_{4j}^2 + 2\phi_{3j}\phi_{4j}) \\ \sum\limits_{j=1}^{N}(\phi_{4j}^2 - \phi_{3j}^2). \end{pmatrix}$$

by taking $\alpha_j = 1,\ \beta_j = 1$ in formulas (13) and (14). Therefore we have a novel integrable coupling system of the TC equations hierarchy with self-consistent sources as follows:

$$\begin{aligned} u_t &= J\begin{pmatrix} -b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} = JL^{n-1}\begin{pmatrix} -b_1 \\ c_1 \\ d_1 \\ e_1 \end{pmatrix} + J\begin{pmatrix} \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 - \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}(\phi_{3j}^2 + \phi_{4j}^2 + 2\phi_{3j}\phi_{4j}) \\ \sum\limits_{j=1}^{N}(\phi_{4j}^2 - \phi_{3j}^2) \end{pmatrix} \\ &= JL^{n-1}\begin{pmatrix} 0 \\ \frac{1}{2}\beta u_2 \\ \frac{1}{2}\beta u_3 \\ \frac{1}{2}\beta u_4 \end{pmatrix} + J\begin{pmatrix} \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 - \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}(\phi_{3j}^2 + \phi_{4j}^2 + 2\phi_{3j}\phi_{4j}) \\ \sum\limits_{j=1}^{N}(\phi_{4j}^2 - \phi_{3j}^2) \end{pmatrix}, \end{aligned} \tag{19}$$

where $\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^{\mathrm{T}},\ i = 1, 2, 3, 4.$ $\langle\cdot,\cdot\rangle$ is the standard inner product in R^N and $\beta \neq 0$.

It is obvious that the following equation is the TC soliton hierarchy with self-consistent sources:

$$u_{t1} = J'L'^{n-1}\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + J'\begin{pmatrix} \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 - \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum\limits_{j=1}^{N}\phi_{1j}^2 \end{pmatrix} = J'L'^{n-1}\begin{pmatrix} 0 \\ \frac{1}{2}\beta u_2 \end{pmatrix} + J'\begin{pmatrix} \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 - \sum\limits_{j=1}^{N}\phi_{1j}^2 \\ \sum\limits_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum\limits_{j=1}^{N}\phi_{1j}^2 \end{pmatrix},$$

where

$$J' = \begin{pmatrix} \partial & \partial\frac{u_1}{u_2} \\ \frac{u_1}{u_2}\partial & \partial \end{pmatrix}, \quad L' = \begin{pmatrix} -\frac{1}{2}\partial^{-1}u_1\partial & -\frac{1}{2}\partial^{-1}u_2\partial \\ -\frac{1}{2}u_2 + \frac{1}{4}\partial\frac{u_1}{u_2}\partial & -\frac{1}{2}u_1 + \frac{1}{4}\partial^2 \end{pmatrix}.$$

When $n = 1$, we have

$$\begin{aligned}
u_{1t} &= \frac{1}{2}\beta u_2\partial\frac{u_1}{u_2} + \partial\sum_{j=1}^{N}\left(\frac{1}{\lambda_j}\phi_{2j}^2 - \phi_{1j}^2\right) + \partial\frac{u_1}{u_2}\left(\sum_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \phi_{1j}^2\right), \\
u_{2t} &= \frac{1}{2}\beta u_{2x} + \frac{u_1}{u_2}\partial\left(\sum_{j=1}^{N}\left(\frac{1}{\lambda_j}\phi_{2j}^2 - \phi_{1j}^2\right)\right) + \partial\left(\sum_{j=1}^{N}\frac{1}{\lambda_j}\phi_{2j}^2 + \sum_{j=1}^{N}\phi_{1j}^2\right), \\
u_{3t} &= \frac{1}{2}\beta u_{3x} + \left(u_4 + \frac{u_1u_4}{u_2}\right)\left(\sum_{j=1}^{N}\left(\frac{1}{\lambda_j}\phi_{2j}^2 + \phi_{1j}^2\right)\right) + \partial\left(\sum_{j=1}^{N}\left(\phi_{3j}^2 + \phi_{4j}^2 + 2\phi_{3j}\phi_{4j}\right)\right) \\
&\quad + (u_1 + u_2)\left(\sum_{j=1}^{N}\left(\phi_{4j}^2 - \phi_{3j}^2\right)\right), \\
u_{4t} &= \frac{1}{2}\beta u_{4x} + \left(u_3 + \frac{u_1u_3}{u_2}\right)\left(\sum_{j=1}^{N}\left(\frac{1}{\lambda_j}\phi_{2j}^2 + \phi_{1j}^2\right)\right) - (u_1 + u_2)\sum_{j=1}^{N}\left(\phi_{3j}^2 + \phi_{4j}^2 + 2\phi_{3j}\phi_{4j}\right) \\
&\quad + \partial\left(\sum_{j=1}^{N}\left(\phi_{4j}^2 - \phi_{3j}^2\right)\right), \\
\phi_{1jx} &= \left(1 + \frac{u_1 + u_2}{2\lambda_j}\right)\phi_{2j} + u_3\phi_{3j} + u_4\phi_{4j}, \quad \phi_{2jx} = \left(\lambda_j + \frac{u_1 - u_2}{2}\right)\phi_{1j} + (u_4 - u_3)\,\phi_{3j} - u_3\phi_{4j}, \\
\phi_{3jx} &= \left(1 + \frac{u_1 + u_2}{2\lambda_j}\right)\phi_{4j}, \quad \phi_{4jx} = \left(\lambda_j + \frac{u_1 - u_2}{2}\right)\phi_{3j}, \quad j = 1, \ldots, N,
\end{aligned} \tag{20}$$

which are nonlinear evolution equations with self-consistent sources.

4. Integrable couplings of generalized Burgers hierarchy with self-consistent sources

Consider the following isospectral problem

$$\phi_x = \overline{U}\phi, \quad \lambda_t = 0, \quad \overline{U} = \begin{pmatrix} 1 & -\lambda + u_1 + u_2 & \frac{1}{2}u_3 & u_4 \\ \lambda + u_1 - u_2 & -1 & -u_3 & -\frac{1}{2}u_3 \\ 0 & 0 & 1 & -\lambda + u_1 + u_2 \\ 0 & 0 & -\lambda + u_1 - u_2 & -1 \end{pmatrix} = \begin{pmatrix} U_1 & U_0 \\ 0 & U_1 \end{pmatrix}, \tag{21}$$

where λ is a spectral parameter and U_1 satisfies $\varphi_x = U_1\varphi$ which is a matrix spectral problem of the generalized Burgers soliton hierarchy.[22] Therefore, by using the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0,$$

we have the following integrable couplings of the the generalized Burgers soliton hierarchy:

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4}\partial\frac{1}{r} \\ 0 & 0 & 1 & \frac{1}{2u_2} \\ 0 & -1 & 0 & \frac{u_4}{u_2} \\ \frac{1}{4u_2}\partial & -\frac{1}{2u_2} & -1-\frac{u_4}{u_2} & 0 \end{pmatrix} \begin{pmatrix} 4b_{n+1}+4f_{n+1} \\ -4c_{n+1}-4g_{n+1} \\ a_{n+1} \\ -4c_{n+1} \end{pmatrix} = JL^n \begin{pmatrix} 8\beta \\ 4\beta(r+u_4) \\ 0 \\ 4\beta r \end{pmatrix}, \tag{22}$$

where L is a recurrence operator and $\beta \neq 0$ is a constant. According to formulas (13) and (14), we have the following results by direct computations:

$$\sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u} = \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta\lambda_j}{\delta u_1} \\ \frac{\delta\lambda_j}{\delta u_2} \\ \frac{\delta\lambda_j}{\delta u_3} \\ \frac{\delta\lambda_j}{\delta u_4} \end{pmatrix} = \begin{pmatrix} 2\sum\limits_{j=1}^{N}(\phi_{2j}^2 - \phi_{1j}^2) \\ 2\sum\limits_{j=1}^{N}(\phi_{1j}^2 + \phi_{2j}^2) \\ \sum\limits_{j=1}^{N}\phi_{3j}\phi_{4j} \\ \sum\limits_{j=1}^{N}(\phi_{4j}^2 + \phi_{3j}^2) \end{pmatrix} = \begin{pmatrix} 2(\langle \Phi_2, \Phi_2\rangle - \langle \Phi_1, \Phi_1\rangle) \\ 2(\langle \Phi_2, \Phi_2\rangle + \langle \Phi_1, \Phi_1\rangle) \\ \langle \Phi_3, \Phi_4\rangle \\ \langle \Phi_3, \Phi_3\rangle + \langle \Phi_4, \Phi_4\rangle \end{pmatrix}. \tag{23}$$

Therefore we have a novel integrable coupling system of the generalized Burgers equations hierarchy with self-consistent sources as follows:

$$u_t = JL^n \begin{pmatrix} 8\beta \\ 4\beta(u_2+u_4) \\ 0 \\ 4\beta r \end{pmatrix} + J \begin{pmatrix} 2(\langle \Phi_2, \Phi_2\rangle - \langle \Phi_1, \Phi_1\rangle) \\ 2(\langle \Phi_2, \Phi_2\rangle + \langle \Phi_1, \Phi_1\rangle) \\ \langle \Phi_3, \Phi_4\rangle \\ \langle \Phi_3, \Phi_3\rangle + \langle \Phi_4, \Phi_4\rangle \end{pmatrix} \tag{24}$$

by taking $\alpha_j = 1, \beta_j = 1$ in formulas (13) and (14). When $n = 1$, we have

$$\begin{aligned}
u_{1t} &= \beta\left(\frac{u_1 u_{2x}}{u_2} - \frac{2u_{2x}}{u_2}\partial^{-1}u_2\right) - \frac{1}{4}\partial\frac{1}{u_2}(\langle \Phi_3, \Phi_3\rangle + \langle \Phi_4, \Phi_4\rangle), \\
u_{2t} &= -\beta(-u_{2x} + 2u_1) + \langle \Phi_3, \Phi_4\rangle + \frac{1}{2u_2}(\langle \Phi_3, \Phi_3\rangle + \langle \Phi_4, \Phi_4\rangle), \\
u_{3t} &= 4\beta[M_6 u_2 + (u_2+u_4)(u_1 - 2u_2\partial^{-1}u_1) - 2u_2] - 2(\langle \Phi_2, \Phi_2\rangle + \langle \Phi_1, \Phi_1\rangle) \\
&\quad + \left(1 + \frac{u_4}{u_2}\right)(\langle \Phi_3, \Phi_3\rangle + \langle \Phi_4, \Phi_4\rangle), \\
u_{4t} &= \beta\left[\left(-2\frac{1}{u_2}\partial M_1 - \frac{1}{u_2}\partial\frac{1}{u_2}\partial\right) - \left(\frac{1}{u_2}\partial M_2 - 2\frac{2u_1}{u_2} + \frac{u_{2x}}{u_2^3}\right)(u_2+u_4)\right. \\
&\quad \left. - \frac{1}{u_2}\partial M_4 - \frac{2}{u_2}\partial M_6 - \left(1 + \frac{u_6}{u_2}\right)(\partial^{-1}u_2 - u_{2x})\right] - \frac{1}{2u_2}\partial(\langle \Phi_2, \Phi_2\rangle - \langle \Phi_1, \Phi_1\rangle) \\
&\quad - \frac{1}{2u_2}(\langle \Phi_2, \Phi_2\rangle + \langle \Phi_1, \Phi_1\rangle)\left(-1 - \frac{u_4}{u_2}\right)\langle \Phi_3, \Phi_4\rangle, \\
\phi_{1jx} &= \phi_{1j} + (-\lambda_j + u_1 + u_2)\phi_{2j} + \frac{1}{2}u_3\phi_{3j} + u_4\phi_{4j}, \\
\phi_{2jx} &= (-\lambda_j + u_1 - u_2)\phi_{1j} - \phi_{2j} - u_4\phi_{3j} - \frac{1}{2}u_3\phi_{4j}, \\
\phi_{3jx} &= \phi_{3j} + (u_1 + u_2 - \lambda_j)\phi_{4j}, \\
\phi_{4jx} &= (u_1 - u_2 - \lambda_j)\phi_{3j} - \phi_{4j}, \quad j = 1, \ldots, N,
\end{aligned} \tag{25}$$

which are nonlinear evolution equations with self-consistent sources, where

$$M_1 = \partial^{-1}u_1\partial - \frac{1}{2u_2}\partial;$$

$$M_2 = -4\partial^{-1}u_1 + \partial^{-1}u_2\partial + \frac{1}{u_2};$$
$$M_4 = -\frac{u_3}{u_2} - \frac{2}{u_2} - 8\partial^{-1}u_4\partial^{-1} - 2\partial^{-1}u_2u_3 + \partial^{-1}u_3\partial + 2\partial^{-1}u_2 + 2\partial^{-1}u_3u_2\partial^{-1};$$
$$M_6 = \partial\frac{u_3+4}{4u_2} + 2u_4\partial^{-1}$$

and $\beta \neq 0$ is a constant.

5. Conclusions

In this paper, we have set up a set of new formulas (13) and (14) and two new integrable couplings of soliton equations hierarchy with self-consistent sources based on the new formulas. The method can be used to other soliton hierarchy with self-consistent sources. In the near future we will investigate exact solutions of soliton equations with self-consistent sources which derived by using our method.

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